

4.4 Composite Numerical Integration

Motivation: 1) on large interval, use low order Newton-Cotes formulas are not accurate.

2) on large interval, interpolation using high degree polynomial is unsuitable because of oscillatory nature of high degree polynomials.

Main idea: divide integration interval $[a, b]$ into subintervals and use simple integration rule for each subinterval.

Example 1. **a)** Use Simpson's rule to approximate $\int_0^4 e^x dx$. The exact value is 53.59819. **b)** Divide $[0,4]$ into $[0,1] + [1,2] + [2,3] + [3,4]$. Use Simpson's rule to approximate $\int_0^1 e^x dx$, $\int_1^2 e^x dx$, $\int_2^3 e^x dx$ and $\int_3^4 e^x dx$. Then approximate $\int_0^4 e^x dx$ by adding approximations for $\int_0^1 e^x dx$, $\int_1^2 e^x dx$, $\int_2^3 e^x dx$ and $\int_3^4 e^x dx$. Compare with accurate value.

Solution:

$$\mathbf{a)} \quad h = \frac{4-0}{2}. \quad \int_0^4 e^x dx \approx \frac{2}{3}(e^0 + 4e^2 + e^4) = 56.76958.$$

$$\text{Error} = |53.59819 - 56.76958| = 3.17143$$

$$\mathbf{b)} \quad \int_0^4 e^x dx = \int_0^1 e^x dx + \int_1^2 e^x dx + \int_2^3 e^x dx + \int_3^4 e^x dx \approx \frac{0.5}{3}(e^0 + 4e^{0.5} + e^1) + \frac{0.5}{3}(e^1 + 4e^{1.5} + e^2) + \frac{0.5}{3}(e^2 + 4e^{2.5} + e^3) + \frac{0.5}{3}(e^3 + 4e^{3.5} + e^4) = 53.61622$$

$$\text{Error} = |53.59819 - 53.61622| = 0.01807$$

b) is much more accurate than **a)**.

Composite Trapezoidal rule

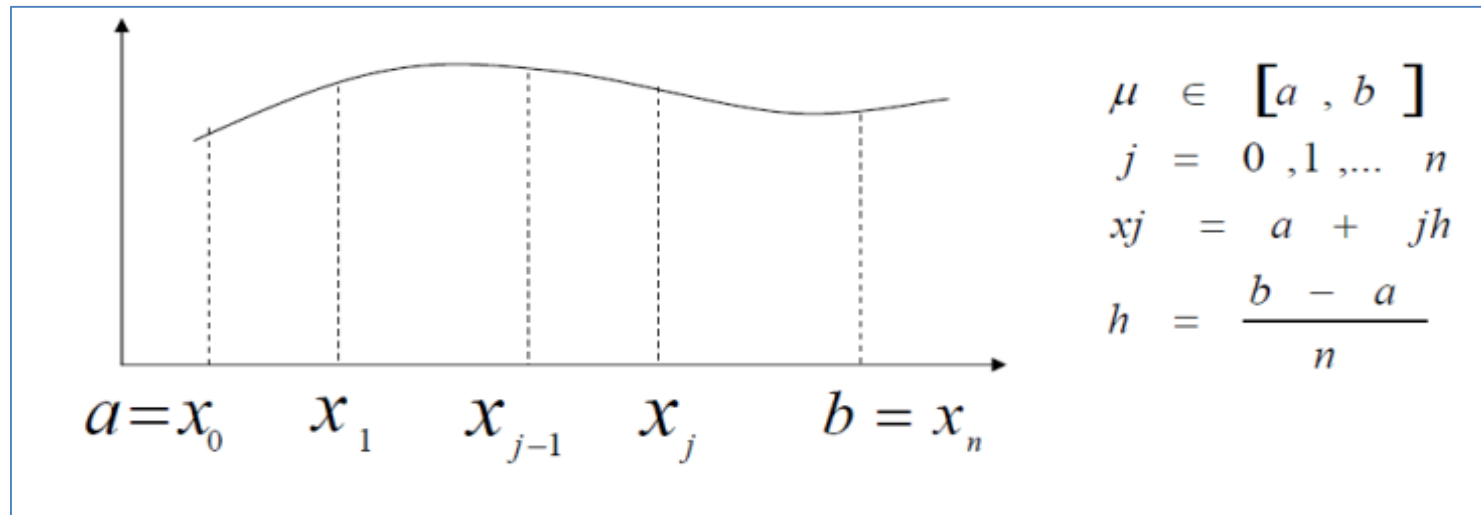


Figure 1 Composite Trapezoidal Rule

Let $f \in C^2[a, b]$, $h = \frac{b-a}{n}$, and $x_j = a + jh$ for $j = 0, \dots, n$.

On each subinterval $[x_{j-1}, x_j]$, for $j = 1, \dots, n$, apply Trapezoidal rule:

$$\int_a^b f(x) dx$$

$$= \left[\frac{h}{2} (f(x_0) + f(x_1)) - \frac{h^3}{12} f''(\xi_1) \right] + \left[\frac{h}{2} (f(x_1) + f(x_2)) - \frac{h^3}{12} f''(\xi_2) \right] + \dots + \left[\frac{h}{2} (f(x_{n-1}) + f(x_n)) - \frac{h^3}{12} f''(\xi_n) \right]$$

Error, which can be simplified

$$= \frac{h}{2} \left[f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right] - \frac{h^3}{12} \sum_{j=1}^n f''(\xi_j)$$

$$= \frac{h}{2} \left[f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right] - \frac{b-a}{12} h^2 f''(\mu)$$

Theorem 4.5 Let $f \in C^2[a, b]$, $h = \frac{b-a}{n}$, and $x_j = a + jh$ for each $j = 0, \dots, n$. There exists a $\mu \in (a, b)$ for which **Composite Trapezoidal rule** with its error term is

$$\int_a^b f(x)dx = \frac{h}{2} \left[f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right] - \frac{b-a}{12} h^2 f''(\mu)$$

Error term

Composite Simpson's rule

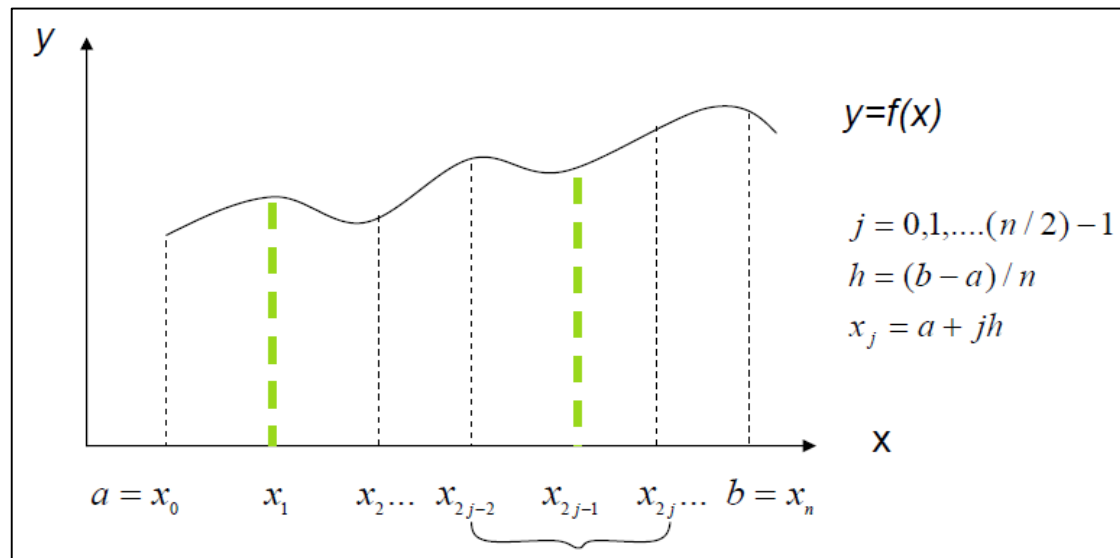


Figure 2 Composite Simpson's rule

Let $f \in C^2[a, b]$, n **be an even integer**, $h = \frac{b-a}{n}$, and $x_j = a + jh$ for $j = 0, \dots, n$.

On **each consecutive pair** of subintervals (for example $[x_0, x_2]$, $[x_2, x_4]$, and $[x_{2j-2}, x_{2j}]$) for each $j = 1, \dots, n/2$, apply a Simpson's rule:

$$\begin{aligned}
\int_a^b f(x) dx &= \sum_{j=1}^{n/2} \int_{x_{2j-2}}^{x_{2j}} f(x) dx \\
&= \sum_{j=1}^{n/2} \frac{h}{3} \left(f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j}) - \frac{h^5}{90} f^{(4)}(\xi_j) \right) \\
&= \frac{h}{3} \left(f(x_0) + 2 \sum_{j=1}^{\left(\frac{n}{2}\right)-1} f(x_{2j}) + 4 \sum_{j=1}^{\left(\frac{n}{2}\right)} f(x_{2j-1}) + f(x_n) \right) \\
&\quad - \frac{h^5}{90} \sum_{j=1}^{\left(\frac{n}{2}\right)} f^{(4)}(\xi_j)
\end{aligned}$$

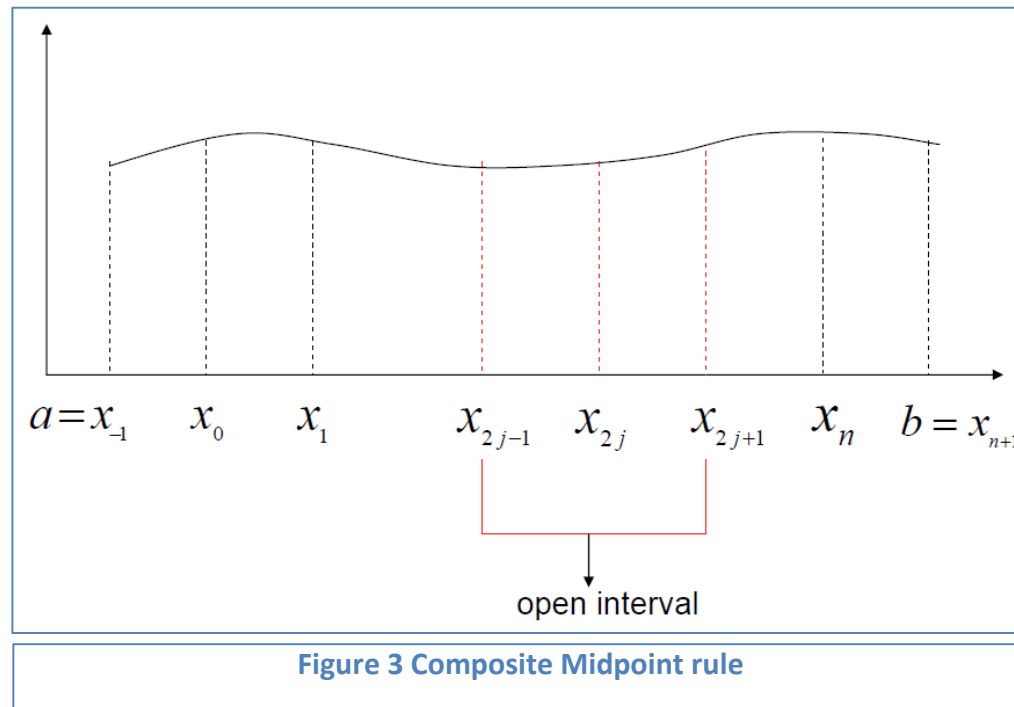

 Error, which can be simplified

Theorem 4.4 Let $f \in C^4[a, b]$, n be *even integer*, $h = \frac{b-a}{n}$, and $x_j = a + jh$ for each $j = 0, \dots, n$. There exists a $\mu \in (a, b)$ for which **Composite Simpson's rule** with its error term is

$$\int_a^b f(x) dx = \frac{h}{3} \left[f(a) + 2 \sum_{j=1}^{(\frac{n}{2})-1} f(x_{2j}) + 4 \sum_{j=1}^{(\frac{n}{2})} f(x_{2j-1}) + f(b) \right] - \frac{b-a}{180} h^4 f^{(4)}(\mu)$$

Error Term

Composite Midpoint rule



Theorem 4.6 Let $f \in C^2[a, b]$, n be **even**, $h = \frac{b-a}{n+2}$, and $x_j = a + (j + 1)h$ for each $j = -1, 0, \dots, n, n + 1$. There exists a $\mu \in (a, b)$ for which **Composite Midpoint rule** with its error term is

$$\int_a^b f(x) dx = 2h \sum_{j=0}^{\frac{n}{2}} f(x_{2j}) + \frac{b-a}{6} h^2 f''(\mu)$$

Exercise 13. Determine the values of n and h required to approximate $\int_0^2 \frac{1}{x+4} dx$ to within 10^{-5} and compute the approximation. Use

- a. Composite Trapezoidal rule.
- b. Composite Simpson's rule.
- c. Composite Midpoint rule.