# 4.7 Gaussian Quadrature

**Motivation:** When approximate $\int_a^b f(x)dx$, nodes $x_0, x_1, \cdots, x_n$ in $[a, b]$ do not need to be equally spaced. This can lead to the greatest degree of precision (accuracy).

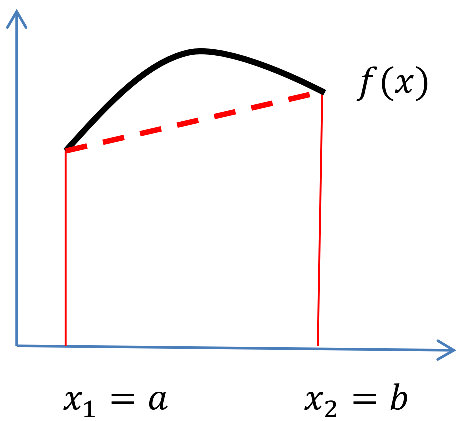


**Figure 1. Trapezoidal rule**

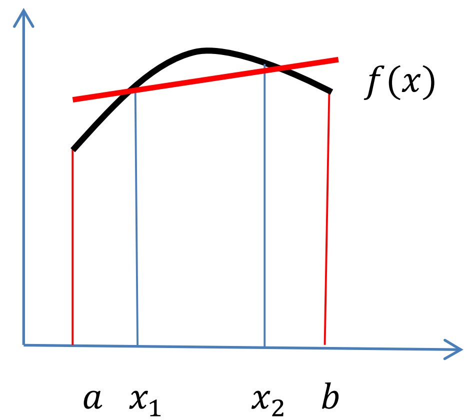


**Figure 2. Gaussian quadrature**

Consider $\int_a^b f(x)dx \approx \sum_{i=1}^n c_i f(x_i)$. Here $c_1, \cdots, c_n$ and $x_1, \cdots, x_n$ are $2n$ parameters. We therefore determine a class of polynomials of degree at most $2n-1$ for which the quadrature formulas have the degree of precision less than or equal to $2n-1$.

**Example** Consider $n = 2$ and $[a, b] = [-1,1]$. We want to determine $x_1, x_2$, $c_1$ and $c_2$ so that quadrature formula $\int_{-1}^1 f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$ has degree of precision 3.

**Solution**: Let $f(x) = 1$. $c_1 + c_2 = \int_{-1}^1 1dx = 2$ **(Eq. 1)**

Let $f(x) = x$. $c_1 x_1 + c_2 x_2 = \int_{-1}^1 xdx = 0$ **(Eq. 2)**

Let $f(x) = x^2$. $c_1 x_1^2 + c_2 x_2^2 = \int_{-1}^1 x^2 dx = \frac{2}{3}$ **(Eq. 3)**

Let $f(x) = x^3$. $c_1 x_1^3 + c_2 x_2^3 = \int_{-1}^1 x^3 dx = 1$ **(Eq. 4)**

Use equations **(1)-(4)** to solve for $x_1, x_2$, $c_1$ and $c_2$. We obtain:

$$\int_{-1}^{1} f(x)dx \approx f\left(\frac{-\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$$

**Remark**: Quadrature formula $\int_{-1}^{1} f(x)dx \approx f\left(\frac{-\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$ has degree of precision 3. Trapezoidal rule has degree of precision 1.

## Legendre Polynomials

Legendre polynomials $P_n(x)$ satisfy:

**1)** For each $n$, $P_n(x)$ is a monic polynomial of degree $n$.

**2)** $\int_{-1}^{1} P(x)P_n(x)dx = 0$ whenever $P(x)$ is a polynomial of degree less than $n$

Remark: Property **2)** is usually referred to as $P(x)$ and $P_n(x)$ are orthogonal.

**Examples.** First five Legendre polynomials: $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = x^2 - 1/3$, $P_3(x) = x^3 - \frac{3}{5}x$, $P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$.

**Theorem 4.7** Suppose that $x_1, \cdots, x_n$ are the roots of the nth Legendre polynomial $P_n(x)$ and that for each $i = 1,2, \cdots n$, the numbers $c_i$ are defined by

$$c_i = \int_{-1}^{1} \prod_{\substack{j=1;\\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j} dx$$

If $P(x)$ is any polynomial of degree less than $2n$, then

$$\int_{-1}^{1} P(x)dx = \sum_{i=1}^{n} c_i P(x_i)$$

**Remark**: Gaussian quadrature formula (more in Table 4.12)

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} c_i f(x_i)$$

| $n$ | Abscissae ($x_i$) | Weights ($c_i$) | Degree of Precision |
|---|---|---|---|
| 2 | $\sqrt{3}/3$ | 1.0 | 3 |
| | $-\sqrt{3}/3$ | 1.0 | |
| 3 | 0. 7745966692 | 0.5555555556 | 5 |
| | 0.0 | 0.8888888889 | |
| | -0.7745966692 | 0.5555555556 | |

**Example 1** Approximate $\int_{-1}^{1} e^{x}\cos(x)dx$ using Gaussian quadrature with n = 3.

## Gaussian quadrature on arbitrary intervals

Use substitution or transformation to transform $\int_a^b f(x)dx$ into an integral defined over $[-1,1]$.

Let $x = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)t$, with $t \in [-1,1]$

Then

$$\int_a^b f(x)dx = \int_{-1}^{1} f\left(\frac{1}{2}(a+b) + \frac{1}{2}(b-a)t\right)\left(\frac{b-a}{2}\right)dt$$

**Example 2.** Consider $\int_1^3 (x^6 - x^2 \sin(2x))dx = 317.3442466$. Compare results from the closed Newton-Cotes formula with n=1, the open Newton-Cotes formula with n =1 and Gaussian quadrature when n = 2.

Solution:

(a) n = 1 closed Newton-Cotes formula (Trapezoidal rule):

$$\int_1^3 x^6 - x^2 \sin(2x)\, dx \approx \frac{2}{2}[f(1) + f(3)] = 731.605$$

(b) n = 1 open Newton-Cotes formula:

$h = \frac{3-1}{1+2} = \frac{2}{3}$. Nodes are: $x_{-1} = 1, x_0 = \frac{5}{3}, x_1 = \frac{7}{3}, x_2 = 3$.

$$\int_1^3 x^6 - x^2 \sin(2x)\, dx \approx \frac{3}{2} h \left[f\left(\frac{5}{3}\right) + f\left(\frac{7}{3}\right)\right] = 188.786$$

(c) n = 2 Gaussian quadrature: