

5.1 Elementary Theory of Initial-Value Problems

Definition 5.1 A function $f(t, y)$ is said to satisfy a **Lipschitz condition** in the variable y on a set $D \subset \mathbb{R}^2$ if a constant $L > 0$ exists with

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$$

whenever (t, y_1) and (t, y_2) are in D . The constant L is called a **Lipschitz constant** for f .

Exercise 5.1 Show that $f(t, y) = \frac{2}{t}y + t^2e^t$ satisfies a Lipschitz condition on the interval $D = \{(t, y) | 1 \leq t \leq 2 \text{ and } -2 \leq y \leq 5\}$.

Definition 5.2 A set $D \subset \mathbb{R}^2$ is said to be **convex** if whenever (t_1, y_1) and (t_2, y_2) belongs to D and $\lambda \in [0, 1]$, the point $((1 - \lambda)t_1 + \lambda t_2, (1 - \lambda)y_1 + \lambda y_2)$ also belongs to D .

Remark:

1. **Convex** means that line segment connecting (t_1, y_1) and (t_2, y_2) is in D whenever (t_1, y_1) and (t_2, y_2) belongs to D .
2. The set $D = \{(x, y) \mid a \leq t \leq b \text{ and } -\infty \leq y \leq \infty\}$ is convex.

Theorem 5.3 Suppose $f(t, y)$ is defined on a convex set $D \subset \mathbb{R}^2$. If a constant $L > 0$ exists with

$$\left| \frac{\partial f}{\partial y}(t, y) \right| \leq L$$

for all $(t, y) \in D$, then f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L .

Theorem 5.4 (existence & uniqueness) Suppose that $D = \{(x, y) \mid a \leq t \leq b \text{ and } -\infty < y < \infty\}$ and that $f(t, y)$ is continuous on D . If f satisfies a Lipschitz condition on D in the variable y , then the initial-value problem (IVP)

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \beta,$$

has a unique solution $y(t)$ for $a \leq t \leq b$.

Example 2. Show that there is a unique solution to the IVP

$$y' = 1 + t \sin(ty), \quad 0 \leq t \leq 2, \quad y(0) = 0.$$

Definition 5.5 The IVP $\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \beta$ is said to be a **well-posed problem** if:

1. A unique solution $y(t)$, to the problem exists, and
2. There exist constant $\varepsilon_0 > 0$ and $k > 0$ such that for any ε with $\varepsilon_0 > \varepsilon > 0$, whenever $\delta(t)$ is continuous with $|\delta(t)| < \varepsilon$ for all t in $[a, b]$, and when $|\delta_0| < \varepsilon$, the IVP (*a perturbed problem associated with original* $\frac{dy}{dt} = f(t, y)$)

$$\frac{dz}{dt} = f(t, z) + \delta(t), \quad a \leq t \leq b, \quad z(a) = \beta + \delta_0$$

has a unique solution $z(t)$ that satisfies

$$|z(t) - y(t)| < k\varepsilon \quad \text{for all } t \text{ in } [a, b].$$

Why well-posedness? Numerical methods always solve perturbed problem because of round-off errors.

Example 3. Show the IVP $y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5$ is well-posed on $D = \{(t, y) | 0 \leq t \leq 2 \text{ and } -\infty \leq y \leq \infty\}$

Theorem 5.6 Suppose $D = \{(x, y) \mid a \leq t \leq b \text{ and } -\infty < y < \infty\}$ and that $f(t, y)$ is continuous on D and satisfies a Lipschitz condition on D in the variable y , then IVP

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \beta,$$

is well-posed.

Example. Show the IVP $y' = y - t^2 + 1$, $0 \leq t \leq 2$, $y(0) = 0.5$ is well-posed on $D = \{(x, y) \mid 0 \leq t \leq 2 \text{ and } -\infty < y < \infty\}$

Solution: $\left| \frac{\partial}{\partial y} (y - t^2 + 1) \right| = |1| = 1$.

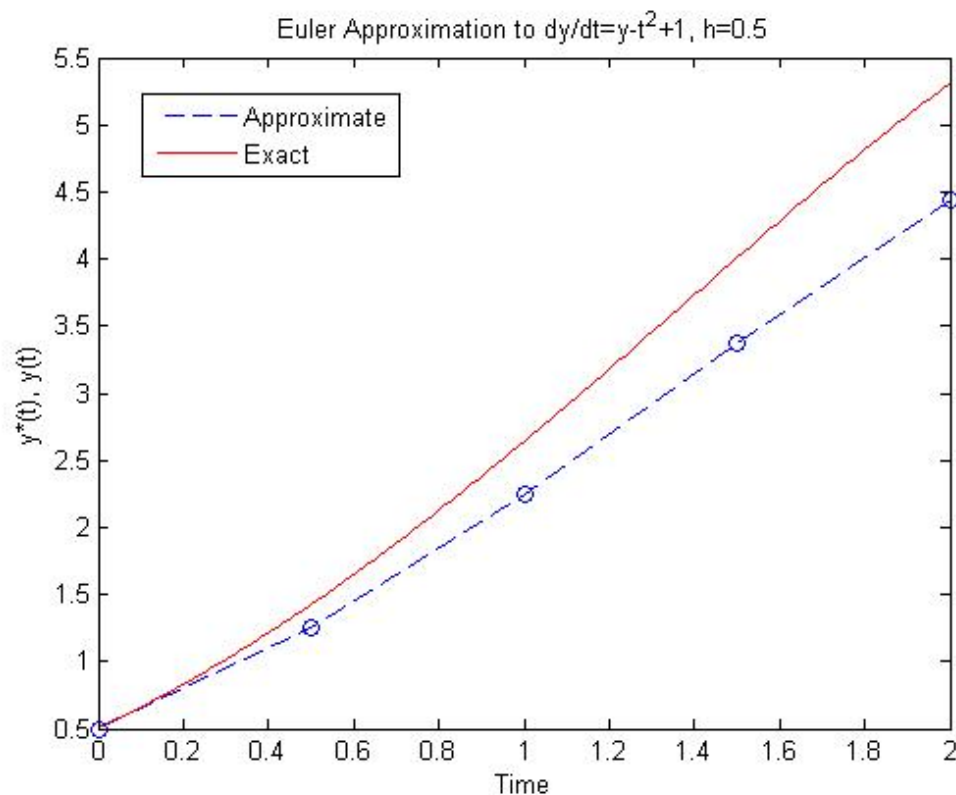
Function $(y - t^2 + 1)$ satisfies Lipschitz condition with $L=1$. So **Theorem 5.6** implies the IVP is well posed.

5.2 Euler's Method

Example 1. Solve $y' = y - t^2 + 1$, $0 \leq t \leq 2$, $y(0) = 0.5$ numerically with time step size $h = 0.2$.

Geometric interpretation of Euler's Method

$f(t_i, w_i) \approx y'(t_i) = f(t_i, y(t_i))$ implies $f(t_i, w_i)$ is an approximation to slope of $y(t)$ at t_i



Error bound

Theorem 5.9 Suppose $D = \{(x, y) \mid a \leq t \leq b \text{ and } -\infty < y < \infty\}$ and that $f(t, y)$ is continuous on D and satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L and that a constant M exists with

$$|y''(t)| \leq M, \quad \text{for all } t \in [a, b].$$

Let $y(t)$ denote the unique solution to the IVP

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \beta,$$

and w_0, w_1, \dots, w_n as in Euler's method. Then

$$|y(t_i) - w_i| \leq \frac{hM}{2L} [e^{L(t_i-a)} - 1].$$

Example 2 The solution to the IVP $y' = y - t^2 + 1, 0 \leq t \leq 2, y(0) = 0.5$ was approximated by Euler's method with $h = 0.2$. Find the bound for approximation.

Effects of round-off error

Theorem 5.10 Let $y(t)$ be the unique solution to the IVP

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

and u_0, u_1, \dots, u_N be approximate solution obtained using

$$\begin{cases} u_0 = \alpha + \delta_0 \\ u_{i+1} = u_i + hf(t_i, u_i) + \delta_{i+1}, \quad \text{for each } i = 0, 1, \dots, N-1 \end{cases}$$

If $|\delta_i| < \delta$ for each $i = 0, 1, \dots, N$ and the hypotheses of Theorem 5.9 holds for $y' = f(t, y)$, then

$$|y(t_i) - u_i| \leq \frac{1}{L} \left(\frac{hM}{2} + \frac{\delta}{h} \right) [e^{L(t_i-a)} - 1] + |\delta_0| e^{L(t_i-a)}$$

for each $i = 0, 1, \dots, N$.

Let $E(h) = \frac{hM}{2} + \frac{\delta}{h}$. Minimal value of $E(h)$ occurs when $h = \sqrt{\frac{2\delta}{M}}$.