# 5.10 Stability

## Consistency and Convergence

**Definition 5.18** A one-step difference equation with local truncation error $\tau_i(h)$ is said to be ***consistent*** if

$$\lim_{h\to 0} \max_{1\le i\le N} |\tau_i(h)| = 0$$

**Definition 5.19** A one-step difference equation is said to be ***convergent*** if

$$\lim_{h\to 0} \max_{1\le i\le N} |w_i - y(t_i)| = 0$$

where $y(t_i)$ is the exact solution and $w_i$ is the approximate solution.

**Example 1**. Consider to solve $y' = f(t, y), \quad a \leq t \leq b, \; y(a) = \alpha$. Let $|y''(t)| \leq M$, an $f(t, y)$ be continuous and satisfy a Lipschitz condition with Lipschitz constant $L$. Show that Euler's method is consistent and convergent.

Solution:

$$|\tau_{i+1}(h)| = |\frac{h}{2} y''(\xi_i)| \leq \frac{h}{2} M$$

$$\lim_{h \to 0} \max_{1 \leq i \leq N} |\tau_i(h)| \leq \lim_{h \to 0} \frac{h}{2} M = 0$$

Thus Euler's method is consistent.

By Theorem 5.9,

$$\max_{1 \leq i \leq N} |w_i - y(t_i)| \leq \frac{Mh}{2L} [e^{L(b-a)} - 1]$$

$$\lim_{h \to 0} \max_{1 \leq i \leq N} |w_i - y(t_i)| \leq \lim_{h \to 0} \frac{Mh}{2L} \left[e^{L(b-a)} - 1\right] = 0$$

Thus Euler's method is convergent.

The rate of convergence of Euler's method is $O(h)$.

**Stability:** small changes in the initial conditions produce correspondingly small changes in the subsequent approximations. The one-step method is **stable** if there is a constant $K$ and a step size $h_0 > 0$ such that the difference between two solutions $w_i$ and $\widetilde{w}_i$ with initial values $\alpha$ and $\tilde{\alpha}$ respectively, satisfies $|w_i - \widetilde{w}_i| < K|\alpha - \tilde{\alpha}|$ whenever $h < h_0$
and $nh \leq b - a$.

**Theorem 5.20** Suppose the IVP $y' = f(t, y), \quad a \leq t \leq b, \; y(a) = \alpha$ is approximated by a one-step difference method in the form

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + h\phi(t_i, w_i, h) \qquad \text{where } i = 0, 2, \ldots N.$$

Suppose also that $h_0 > 0$ exists and $\phi(t, w, h)$ is continuous with a Lipschitz condition in $w$ with constant $L$ on $D$,
$D = \{(t, w, h) |\; a \leq t \leq b, -\infty < w < \infty, 0 \leq h \leq h_0\}$. Then:

(1) The method is ***stable***;

(2) The method is ***convergent*** if and only if it is ***consistent***, which is equivalent to

$$\phi(t, w, 0) = f(t, y), \qquad \text{for all } a \leq t \leq b$$

(3) If a function $\tau$ exists s.t. $|\tau_i(h)| \leq \tau(h)$ when $0 \leq h \leq h_0$, then

$$|w_i - y(t_i)| \leq \frac{\tau(h)}{L} e^{L(t_i - a)}.$$

**Example 2.** Show modified Euler method $w_{i+1} = w_i + \frac{h}{2}\big(f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))\big)$ is stable and convergent. Suppose $f(t, y)$ satisfied a Lipschitz condition on $\{(t, w)\ |a \leq t \leq b,$ and $-\infty < w < \infty\}$ for $y$ variable with Lipschitz constant $L$, $f(t, y)$ is also continuous.

## Multi-Step Methods

**Definition.** The **local truncation error** $\tau_{i+1}(h)$ of a m-step method of the form:

$$w_0 = \alpha, w_1 = \alpha_1, \dots, w_{m-1} = \alpha_{m-1}$$
$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m}$$
$$+h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i)$$
$$+ \cdots + b_0 f(t_{i+1-m}, w_{i+1-m})]$$

is:
$$\tau_{i+1}(h) = \frac{y(t_{i+1}) - a_{m-1}y(t_i) - a_{m-2}y(t_{i-1}) - \cdots - a_0 y(t_{i+1-m})}{h}$$
$$-[b_m f(t_{i+1}, y_{i+1}) + b_{m-1} f(t_i, y_i) + \cdots + b_0 f(t_{i+1-m}, y_{i+1-m})]$$

**Definition.** A $m$-step multistep is **consistent** if $\lim_{h\to 0} |\tau_i(h)| = 0$, for all $i = m, m+1, \dots, N$ and $\lim_{h\to 0} |\alpha_i - y(t_i)| = 0$, for all $i = 1,2,\dots,m-1$. $\{\alpha_i\}$ are the starting values computed by some one-step method.

**Definition.** A $m$-step multistep is **convergent** if

$$\lim_{h\to 0} \max_{1\le i\le N} |w_i - y(t_i)| = 0$$

**Theorem 5.21** Suppose the IVP $y' = f(t, y), a \le t \le b,\ y(a) = \alpha$ is approximated by an explicit Adams predictor-corrector method with an *m*-step Adams-Bashforth predictor equation $w_{i+1} = w_i + h[b_{m-1} f(t_i, w_i) + \cdots + b_0 f(t_{i+1-m}, w_{i+1-m})]$ with local truncation error $\tau_{i+1}(h)$ and an (*m*-1)-step implicit Adams-Moulton corrector equation $w_{i+1} = w_i + h[\tilde{b}_{m-1} f(t_i, w_i) + \cdots + \tilde{b}_0 f(t_{i+2-m}, w_{i+2-m})]$ with local truncation error $\tilde{\tau}_{i+1}(h)$. In addition, suppose that $f(t, y)$ and $f_y(t, y)$ are continuous on $\{(t, y)\ |a \le t \le b, \text{and} -\infty < y < \infty\}$ and that $f_y(t, y)$ is bounded. Then the local truncation error $\sigma_{i+1}(h)$ of the predictor-corrector method is

$$\sigma_{i+1}(h) = \tilde{\tau}_{i+1}(h) + \tau_{i+1}(h)\tilde{b}_{m-1} f_y(t_{i+1}, \theta_{i+1})$$

where $\theta_{i+1}$ is a number between zero and $h\tau_{i+1}(h)$.
Moreover, there exist constant $k_1$ and $k_2$ such that

$$|w_i - y(t_i)| \le \left[\max_{0 \le j \le m-1} \left|w_j - y(t_j)\right| + k_1 \sigma(h)\right] e^{k_2(t_i - a)}$$

where $\sigma(h) = \max_{m \le j \le N} \left|\sigma_j(h)\right|$.

**Example.** Consider the IVP $y' = 0, \quad 0 \le t \le 10, \ y(0) = 1$, which is solved by $w_{i+1} = -4w_i + 5w_{i-1} + h(4f(t_i, w_i) + 2f(t_{i-1}, w_{i-1}))$. If in each step, there is a round-off error $\varepsilon$, and $w_1 = 1 + \varepsilon$. Find out how error propagates with respect to time.

**Solution**: $w_2 = -4(1+\varepsilon) + 5(1) = 1 - 4\varepsilon$

$$w_3 = -4(1-\varepsilon) + 5(1+\varepsilon) = 1 + 21\varepsilon$$

$$w_4 = -4(1+21\varepsilon) + 5(1-4\varepsilon) = 1 - 104\varepsilon.$$

**Definition.** Consider to solve the IVP: $y' = f(t, y), \quad a \le t \le b, \ y(a) = \alpha$. by an $m$-step multistep method

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m}$$
$$h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \cdots$$
$$+ b_0 f(t_{i+1-m}, w_{i+1-m})],$$

The **characteristic polynomial** of the method is given by

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \cdots - a_1\lambda - a_0.$$

**Remark:**

(1) The **characteristic polynomial** can be viewed as derived by solving $y' = 0,\ y(a) = \alpha$ using the $m$-step multistep method.

(2) If $\lambda$ is a root of the characteristic polynomial, then $w_i = (\lambda)^i$ for each $i$ is a solution to $w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m}$.
This is because $\lambda^{i+1} - a_{m-1}\lambda^i - a_{m-2}\lambda^{i-1} - \cdots - a_0\lambda^{i+1-m} = \lambda^{i+1-m}(\lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \cdots - a_1\lambda - a_0) = 0$

(3) If $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_m$ are distinct zeros of the **characteristic polynomial,** solution to $w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m}$ can be represented by $w_i = \sum_{j=1}^{m} c_j \lambda_j^i$ for some unique constants $c_1, \dots, c_m$.

(4) $w_i = \alpha$ is a solution to $w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m}$, this is because $y(t) = \alpha$ is the exact solution to $y' = 0$, $y(a) = \alpha$.

(5) From (4), $0 = \alpha - a_{m-1}\alpha - a_{m-2}\alpha - \cdots - a_0\alpha = \alpha[1 - a_{m-1} - a_{m-2} - \cdots - a_0]$. Compare this with definition of characteristic polynomial, this shows that $\lambda = 1$ is one of the zeros of the characteristic polynomial.

(6) Let $\lambda_1 = 1$ and $c_1 = \alpha$, solution to $y' = 0, \quad y(0) = \alpha$ is expressed as $w_i = \alpha + \sum_{j=2}^{m} c_j \lambda_j^i$. This means that $c_2, \ldots, c_m$ would be zero if all the calculations were exact. However, $c_2, \ldots, c_m$ are not zero in practice due to round-off error.

**(*)** The stability of a multistep method with respect to round-off error is dictated by magnitudes of zeros of the characteristic polynomial. If $|\lambda_j| > 1$ for any of $\lambda_2, \lambda_3, \ldots, \lambda_m$, the round-off error grows exponentially.

**Example.** Analyze stability of $w_{i+1} = -4w_i + 5w_{i-1} + h(4f(t_i, w_i) + 2f(t_{i-1}, w_{i-1}))$ for solving $y' = 0, \quad 0 \le t \le 10, \; y(0) = 1$ , with initial condition $w_0 = 1, w_1 = 1 + \delta$. $\delta$ is due to round-off error.

**Definition 5.22** Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the roots of the **characteristic equation** $P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \cdots - a_1\lambda - a_0 = 0$ associated with the $m$-step multistep method

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m}$$
$$h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \cdots$$
$$+ b_0 f(t_{i+1-m}, w_{i+1-m})],$$

If $|\lambda_i| \leq 1$ and all roots with absolute value 1 are simple roots, then the difference equation is said to satisfy the **root condition**.

**Stability of multistep method**

**Definition 5.23**

1) Methods that satisfy the root condition and have $\lambda = 1$ as the only root of the characteristic equation with magnitude one are called **strongly stable**.
2) Methods that satisfy the root condition and have more than one distinct roots with magnitude one are called **weakly stable**.
3) Methods that do not satisfy the root condition are called **unstable**.

**Example**. Show 4th order Adams-Bashforth method

$$w_{i+1} = w_i + \frac{h}{24}[55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})]$$

is strongly stable.

**Solution**: The characteristic equation of the 4th order Adams-Bashforth method is

$$P(\lambda) = \lambda^4 - \lambda^3 = 0$$

$$0 = \lambda^4 - \lambda^3 = \lambda^3(\lambda - 1)$$

$P(\lambda)$ has roots $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0$.

Therefore $P(\lambda)$ satisfies root condition and the method is strongly stable.

**Example**. Show 4th order Miline's method

$$w_{i+1} = w_{i-3} + \frac{4h}{3}[2f(t_i, w_i) - f(t_{i-1}, w_{i-1}) + 2f(t_{i-2}, w_{i-2})]$$

is weakly stable.

**Solution:** The characteristic equation $P(\lambda) = \lambda^4 - 1 = 0$

$$0 = \lambda^4 - 1 = (\lambda^2 - 1)(\lambda^2 + 1)$$

$P(\lambda)$ has roots $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = i, \lambda_4 = -i$.

All roots have magnitude one. So the method is weakly stable.

**Theorem 5.24** A multistep method

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m}$$
$$h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \cdots$$
$$+ b_0 f(t_{i+1-m}, w_{i+1-m})],$$

is stable **if and only if** it satisfies the root condition. If it is also consistent, then it is stable **if and only if** it is convergent.