5.11 Stiff Differential Equation

Example. The initial-value problem $y' = -30y, \quad 0 \leq t \leq 1.5, \; y(0) = \frac{1}{3}$ has exact solution $y(t) = \frac{1}{3}e^{-30t}$. Use Euler's method and 4-stage Runge-Kutta method to solve with step size $h = 0.1$ respectively.

Solution: Euler's method

$$w_{i+1} = (1 - 30h)w_i = (1 - 30h)^2 w_{i-1} = \cdots = (1 - 30h)^{i+1} w_0$$

If $h > \frac{1}{15}$, then $|1 - 30h| > 1$, and $(1 - 30h)^{i+1}$ grows geometrically, in contrast to the true solution.

Facts:

1) A stiff differential equation is numerically unstable unless the step size is extremely small.

2) Stiff differential equations are characterized as those whose exact solution has a term of the form e^{-ct}, where c is a large positive constant.

3) Large derivatives of e^{-ct} give error terms that are dominating the solution.

Definition. The *test equation* is said to be

$$y' = \lambda y, \qquad y(0) = \alpha, \text{ where } \lambda < 0$$

The test equation has exact solution $y(t) = \alpha e^{\lambda t}$.

Euler's Method for solving the test equation

$$w_0 = \alpha$$

$$w_{j+1} = w_j + h(\lambda w_j) = (1 + h\lambda)w_j = (1 + h\lambda)(1 + h\lambda)w_{j-1} = \cdots$$
$$= (1 + h\lambda)^{j+1}\alpha \qquad \text{for } j = 0,1, \dots, N - 1$$

The absolute error is $|y(t_j) - w_j| = |e^{jh\lambda} - (1 + h\lambda)^j||\alpha| = |(e^{h\lambda})^j - (1 + h\lambda)^j||\alpha|$

So 1) the accuracy is determined by how well $(1 + h\lambda)$ approximate $e^{h\lambda}$.

2) $(e^{h\lambda})^j$ decays to zero as j increases. $(1 + h\lambda)^j$ will decay to zero only if $|1 + h\lambda| < 1$. This implies that

$$-2 < h\lambda < 0 \text{ or } h < 2/|\lambda|.$$

Note: Euler's method is expected to be stable for the test equation only if the step size $h < 2/|\lambda|$.

Also, **define** $Q(h\lambda) = 1 + h\lambda$ for Euler's method, then $w_{j+1} = Q(h\lambda)w_j$

Now suppose a round-off error δ_0 is introduced in the initial condition for Euler's method

$$w_0 = \alpha + \delta_0$$

$$w_j = (1 + h\lambda)^j(\alpha + \delta_0)$$

At the jth step, the round-off error is $\delta_j = (1 + h\lambda)^j \delta_0$.

So with $\lambda < 0$, the condition for control of the growth of round-off error is the same as the condition for controlling the absolute error $|1 + h\lambda| < 1$.

Nth-order Taylor Method for solving test euation

Applying the nth-order Taylor method to the test equation leads to

$$\left|1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \cdots + \frac{1}{n!}(h\lambda)^n\right| < 1$$

to have stability. Also, define $Q(h\lambda) = 1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \cdots + \frac{1}{n!}(h\lambda)^n$ for a nth-order Taylor method, i.e., $w_{j+1} = Q(h\lambda)w_j$.

Multistep Method for solving test equation

Apply a multistep method to the test equation:

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m}$$
$$h\lambda[b_m w_{i+1} + b_{m-1}w_i + \cdots + b_0 w_{i+1-m}],$$

This leads to:

$$(1 - h\lambda b_m)w_{i+1} - (a_{m-1} - h\lambda b_{m-1})w_i - \cdots - (a_0 - h\lambda b_0)w_{i+1-m} = 0$$

Define the associated **characteristic polynomial** to this difference equation

$$Q(z, h\lambda) = (1 - h\lambda b_m)z^m - (a_{m-1} - h\lambda b_{m-1})z^{m-1} - \cdots - (a_0 - h\lambda b_0).$$

Let $\beta_1, \beta_2, \dots, \beta_m$ be the zeros of the **characteristic polynomial** to the difference equation. Then $c_1, c_2, \dots, c_m$ exist with

$$w_i = \sum_{k=1}^{m} c_k (\beta_k)^i, \qquad \text{for} \quad i = 0, \dots N$$

and $|\beta_k| < 1$ is required for stability.

Region of Stability

Definition 5.25 The **region *R* of absolute stability** for a one-step method is $R = \{h\lambda \in C \mid |Q(h\lambda)| < 1\}$, and for a multistep method, it is $R = \{h\lambda \in C \mid |\beta_k| < 1,$ for all zeros β_k $of\ Q(z, h\lambda)\}$.

Example. Draw the stability region of the Euler method.

Remark: The technique commonly used for stiff systems is implicit methods.

Definition A numerical method is said to be A-stable if its region R of absolute stability contains the entire left half-plane.

Example. Show the **implicit Trapezoidal method** is A-stable.

$$w_0 = \alpha$$

$$w_{j+1} = w_j + \frac{h}{2}\left[f\left(t_j, w_j\right) + f\left(t_{j+1}, w_{j+1}\right)\right], \qquad \text{for} \quad 0 \le j \le N-1.$$

Remark: The only A-stable multistep method is the **implicit Trapezoidal method**

The A-stable **implicit backward Euler method**.

$$w_0 = \alpha$$

$$w_{j+1} = w_j + hf\left(t_{j+1}, w_{j+1}\right), \qquad \text{for} \quad 0 \le j \le N-1.$$

Example. Show Backward Euler method has $Q(h\lambda) = \frac{1}{1-h\lambda}$.

Solution: $w_{j+1} = w_j + h\lambda w_{j+1}$

$$w_{j+1} = \frac{1}{1-h\lambda} w_j = (\frac{1}{1-h\lambda})^2 w_{j-1} = \cdots = (\frac{1}{1-h\lambda})^{j+1} w_0$$

Stability implies $|\frac{1}{1-h\lambda}| < 1$.