5.3 High-Order Taylor Methods

Consider the IVP

$$y' = f(t, y), \quad a \leq t \leq b, \; y(a) = \beta.$$

Definition: The difference method

$$\begin{cases} w_0 = \beta \\ w_{i+1} = w_i + h\phi(t_i, w_i) \quad \text{for each } i = 0, 1, 2, \cdots, N-1 \end{cases}$$

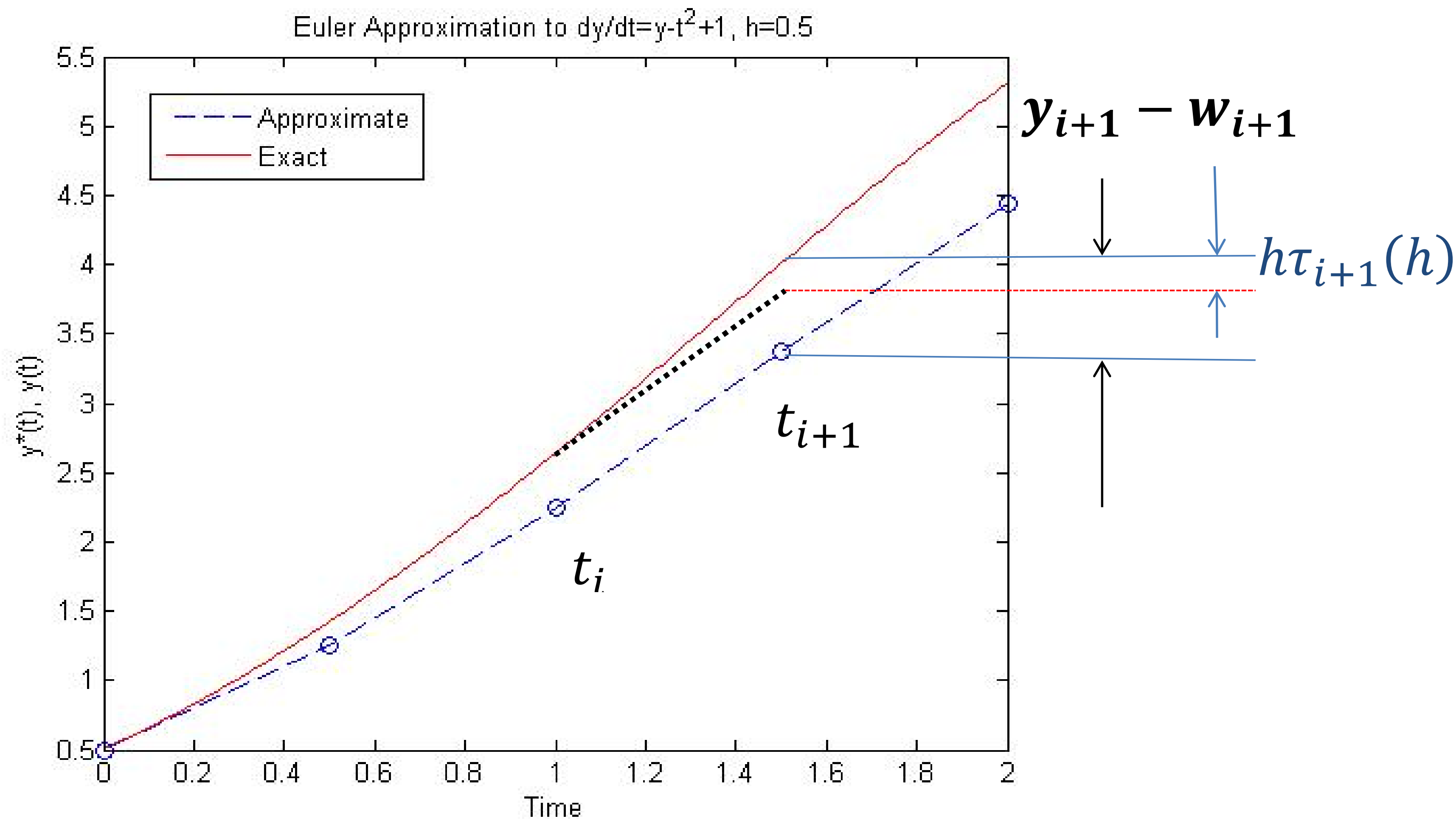
with step size $h = \frac{b-a}{N}$ has **Local Truncation Error**

$$\tau_{i+1}(h) = \frac{y_{i+1} - \big(y_i + h\phi(t_i, y_i)\big)}{h}$$

$$= \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i) \quad \text{for each } i = 0, 1, 2, \cdots, N-1.$$

Note: $y_i \coloneqq y(t_i)$ and $y_{i+1} \coloneqq y(t_{i+1})$.

Geometric view of local truncation error



Example. Analyze the local truncation error of Euler's method for solving
$y' = f(t, y), \quad a \le t \le b, \; y(a) = \beta$. Assume $|y''(t)| < M$ with $M > 0$ constant.

Consider the IVP

$$y' = f(t, y), \quad a \le t \le b, \; y(a) = \beta.$$

Compute $y'', y^{(3)} \cdots$ using $f(t, y)$ and its derivatives.

Derivation of higher-order Taylor methods

Consider the IVP

$$y' = f(t, y), \quad a \le t \le b, \; y(a) = \beta, \qquad \text{with step size } h = \frac{b-a}{N}, \quad t_{i+1} = a + ih.$$

Expand $y(t)$ in the *n*th Taylor polynomial about t_i, evaluated at t_{i+1}

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \cdots$$
$$+ \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i)$$

$$y(t_{i+1}) = y(t_i) + hf\big(t_i, y(t_i)\big) + \frac{h^2}{2}f'\big(t_i, y(t_i)\big) + \cdots$$
$$+ \frac{h^n}{n!}f^{(n-1)}\big(t_i, y(t_i)\big) + \frac{h^{n+1}}{(n+1)!}f^{(n)}(\xi_i, y(\xi_i))$$

for some $\xi_i \in (t_i, t_{i+1})$. Delete remainder term to obtain the *n*th Taylor method of order *n*.

Denote

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i)$$

Taylor method of order *n*

$w_0 = \beta$

$w_{i+1} = w_i + hT^{(n)}(t_i, w_i)$ for each $i = 0, 1, 2, \cdots, N-1$.

Remark: Euler's method is the Taylor method of order one.

Example 1. Use Taylor method of orders (a) two and (b) four with N = 10 to the IVP

$$y' = y - t^2 + 1, \qquad 0 \le t \le 2, \quad y(0) = 0.5.$$

Solution (b):

$$h = \frac{2-0}{N} = \frac{2-0}{10} = 0.$$

So $t_i = 0 + 0.2i = 0.2i$ for each $i = 0, 1, 2, \cdots, 10.$

$$f'(t, y(t)) = \frac{d}{dt}(y - t^2 + 1) = y' - 2t = y - t^2 + 1 - 2t$$

$$f''(t, y(t)) = \frac{d}{dt}(f') = (y - t^2 + 1 - 2t)' = y' - 2t - 2$$

$$= y - t^2 + 1 - 2t - 2 = y - t^2 - 2t - 1$$

$$f^{(3)}(t, y(t)) = \frac{d}{dt}(f'') = (y - t^2 - 2t - 1)' = y' - 2t - 2$$

$$= y - t^2 + 1 - 2t - 2 = y - t^2 - 2t - 1$$

$$T^{(4)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \frac{h^2}{3!} f''(t_i, w_i) + \frac{h^3}{4!} f^{(3)}(t_i, w_i)$$

$$= \left(w_i - t_i^2 + 1\right) + \frac{h}{2}\left(w_i - t_i^2 + 1 - 2t_i\right) +$$

$$\frac{h^2}{6}\left(w_i - t_i^2 - 2t_i - 1\right) + \frac{h^3}{24}\left(w_i - t_i^2 - 2t_i - 1\right)$$

$$= \left(1 + \frac{h}{2} + \frac{h^2}{6} + \frac{h^3}{24}\right)\left(w_i - t_i^2\right) -$$

$$\left(1 + \frac{h}{3} + \frac{h^2}{12}\right)(ht_i) + 1 + \frac{h}{2} - \frac{h^2}{6} - \frac{h^3}{24}$$

The 4th order Taylor method is

$$w_0 = 0.5$$

$$w_{i+1} = w_i + h\left[\left(1+\frac{h}{2}+\frac{h^2}{6}+\frac{h^3}{24}\right)(w_i - t_i^2) - \left(1+\frac{h}{3}+\frac{h^2}{12}\right)(ht_i) + 1 + \frac{h}{2} - \frac{h^2}{6} - \frac{h^3}{24}\right]$$

for each $i = 0, 1, 2, \cdots, 9$.

Now compute approximate solutions at each time step:

$$w_1 = 0.5 + 0.2\left(\left(1+\frac{0.2}{2}+\frac{0.2^2}{6}+\frac{0.2^3}{24}\right)(0.5-0) - \left(1+\frac{0.2}{3}+\frac{0.2^2}{12}\right)(0) + 1 + \frac{0.2}{2} - \frac{0.2^2}{6} - \frac{0.2^3}{24}\right) = 0.8293$$

$$\text{abs. eror of 4th order Taylor at } t_1: \quad |w_1 - y_1| = 0.000001$$

$$
\begin{aligned}
w_2 &= 0.8293 \\
&\quad + 0.2\left(\left(1 + \frac{0.2}{2} + \frac{0.2^2}{6} + \frac{0.2^3}{24}\right)(0.8293 - 0.2^2)\right. \\
&\quad \left. - \left(1 + \frac{0.2}{3} + \frac{0.2^2}{12}\right)(0.2(0.2)) + 1 + \frac{0.2}{2} - \frac{0.2^2}{6} - \frac{0.2^3}{24}\right) \\
&\quad = 1.214091
\end{aligned}
$$

$$\text{abs. eror 4th order Taylor at } t_2: \quad |w_2 - y_2| = 0.000003$$

Finding approximations at time other than t_i

Example. (Table 5.4 on Page 259). Assume the IVP $y' = y - t^2 + 1$, $0 \leq t \leq 2$, $y(0) = 0.5$ is solved by the 4th order Taylors method with time step size $h = 0.2$. $w_6 = 3.1799640$ ($t_6 = 1.2$), $w_7 = 3.7324321$ ($t_7 = 1.4$). Find $y(1.25)$.

Solution:

Method 1: use linear Lagrange interpolation.

$$y(1.25) \approx \frac{1.25-1.4}{1.2-1.4} w_6 + \frac{1.25-1.2}{1.4-1.2} w_7 = 3.3180810$$

Method 2: use Hermite polynomial interpolation (more accurate than the result by linear Lagrange interpolation).

First use $y' = y - t^2 + 1$ to approximate $y'(1.2)$ and $y'(1.4)$.

$y'(1.2) = y(1.2) - (1.2)^2 + 1 \approx 3.1799640 - (1.2)^2 + 1 = 2.7399640$

$y'(1.4) = y(1.4) - (1.4)^2 + 1 \approx 3.7324321 - (1.4)^2 + 1 = 2.7724321$

Then use **Theorem 3.9** to construct Hermite polynomial $H_3(x)$.

$$y(1.25) \approx H_3(1.25).$$

Error analysis

Theorem 5.12 If Taylor method of order n is used to approximate the solution to the IVP

$$y' = f(t, y), \quad a \le t \le b, \; y(a) = \beta$$

with step size h and if $y \in C^{n+1}[a, b]$, then the **local truncation error** is $O(h^n)$.

Remark: $y(t_{i+1}) = y(t_i) + hf\big(t_i, y(t_i)\big) + \frac{h^2}{2} f'\big(t_i, y(t_i)\big) + \cdots + \frac{h^n}{n!} f^{(n-1)}\big(t_i, y(t_i)\big) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))$

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - T^{(n)}(t_i, y_i) = \frac{h^n}{(n+1)!} f^{(n)}\big(\xi_i, y(\xi_i)\big).$$

Assume $y^{(n+1)}(t) = f^{(n)}\big(t, y(t)\big)$ is bounded by $\left|y^{(n+1)}(t)\right| \le M$.

Thus $|\tau_{i+1}(h)| \le \frac{h^n}{(n+1)!} M$.

So the local truncation error in Euler's method is $O(h^n)$.