

5.4 Runge-Kutta Methods

Motivation: Obtain high-order accuracy of Taylor method without knowledge of derivatives of $f(t, y)$.

Theorem 5.13(Taylor's Theorem in Two Variables) Suppose $f(t, y)$ and partial derivative up to order $n + 1$ continuous on $D = \{(t, y) | a \leq t \leq b, c \leq y \leq d\}$, let $(t_0, y_0) \in D$. For $(t, y) \in D$, there is $\xi \in [t, t_0]$ and $\mu \in [y, y_0]$ with

$$f(t, y) = P_n(t, y) + R_n(t, y).$$

Here

$$\begin{aligned} P_n(t, y) = & f(t_0, y_0) + \left[(t - t_0) \frac{\partial f}{\partial t}(t_0, y_0) + (y - y_0) \frac{\partial f}{\partial y}(t_0, y_0) \right] \\ & + \left[\frac{(t - t_0)^2}{2} \frac{\partial^2 f}{\partial t^2}(t_0, y_0) + (t - t_0)(y - y_0) \frac{\partial^2 f}{\partial t \partial y}(t_0, y_0) \right. \\ & \quad \left. + \frac{(y - y_0)^2}{2} \frac{\partial^2 f}{\partial y^2}(t_0, y_0) \right] + \\ & \left[\frac{1}{n!} \sum_{j=0}^n \binom{n}{j} (t - t_0)^{n-j} (y - y_0)^j \frac{\partial^n f}{\partial t^{n-j} \partial y^j}(t_0, y_0) \right] \end{aligned}$$

$$R_n(t, y) = \frac{1}{(n+1)!} \sum_{j=0}^{n+1} \binom{n+1}{j} (t - t_0)^{n+1-j} (y - y_0)^j \frac{\partial^{n+1} f}{\partial t^{n+1-j} \partial y^j}(\xi, \mu)$$

$P_n(t, y)$ is the n th Taylor polynomial in two variables. $R_n(t, y)$ is the remainder term associated with $P_n(t, y)$.

$$\xi \in (t, t_0), \quad \mu \in (y, y_0)$$

Derivation of Runge-Kutta method of order two

Determine a_1, α_1, β_1 such that

$$a_1 f(t + \alpha_1, y + \beta_1) \approx f(t, y) + \frac{h}{2} f'(t, y) \text{ with } O(h^2) \text{ error.}$$

Notice $f'(t, y) = \frac{df(t, y(t))}{dt} = \frac{\partial f}{\partial t}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t)) \cdot y'(t) =$
 $\frac{\partial f}{\partial t}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t)) \cdot f(t, y(t))$

Define:

$$T^{(2)}(t, y) = f(t, y) + \frac{h}{2} \frac{\partial f}{\partial t}(t, y(t)) + \frac{h}{2} \frac{\partial f}{\partial y}(t, y(t)) \cdot f(t, y(t)) \quad (1)$$

1. Expand $a_1 f(t + \alpha_1, y + \beta_1)$ in 1st degree Taylor polynomial:

$$\begin{aligned} & a_1 f(t + \alpha_1, y + \beta_1) \\ &= a_1 f(t, y) + a_1 \alpha_1 \frac{\partial f}{\partial t}(t, y) + a_1 \beta_1 \frac{\partial f}{\partial y}(t, y) \\ &+ a_1 R_1(t + \alpha_1, y + \beta_1) \end{aligned} \quad (2)$$

2. Match coefficients of equation (1) and (2) gives

$$a_1 = 1, \quad a_1 \alpha_1 = \frac{h}{2}, \quad a_1 \beta_1 = \frac{h}{2} f(t, y(t))$$

with unique solution

$$a_1 = 1, \quad \alpha_1 = \frac{h}{2}, \quad \beta_1 = \frac{h}{2} f(t, y(t))$$

$$T^{(2)}(t, y) = f\left(t + \frac{h}{2}, y + \frac{h}{2}f(t, y(t))\right)$$

3. This gives:

$$-R_1\left(t + \frac{h}{2}, y + \frac{h}{2}f(t, y(t))\right)$$

$$\text{with } R_1\left(t + \frac{h}{2}, y + \frac{h}{2}f(t, y(t))\right) = O(h^2)$$

Midpoint Method (one of Runge-Kutta methods of order two)

Consider to solve the IVP

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \beta.$$

with step size $h = \frac{b-a}{N}$.

$$w_0 = \beta$$

$$w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i)\right),$$

for each $i = 0, 1, 2, \dots, N - 1$.

Local truncation error is $O(h^2)$.

Two stage formula of Runge-Kutta method of order two:

$$\begin{aligned} w_0 &= \beta \\ \left\{ \begin{array}{l} k_1 = f(t_i, w_i) \\ k_2 = f\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} k_1\right) \end{array} \right. \\ w_{i+1} &= w_i + h k_2 \\ \text{for each } i &= 0, 1, 2, \dots, N-1. \end{aligned}$$

Example 2. Use the Midpoint method with $N = 10$, $h = 0.2$, $t_i = 0.2i$ and $w_0 = 0.5$ to solve the IVP

$$y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5.$$

Modified Euler Method (Another Runge-Kutta method of order two)

Consider to solve the IVP

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \beta. \text{ with step size } h = \frac{b-a}{N}.$$

$$w_0 = \beta$$

$$w_{i+1} = w_i + \frac{h}{2} [f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))]$$

for each $i = 0, 1, 2, \dots, N - 1$.

Local truncation error is $O(h^2)$.

Two stage formula of the Modified Euler Method:

$$w_0 = \beta$$

$$k_1 = f(t_i, w_i)$$

$$k_2 = f(t_{i+1}, w_i + hk_1)$$

$$w_{i+1} = w_i + \frac{h}{2} [k_1 + k_2]$$

for each $i = 0, 1, 2, \dots, N - 1$.

Example. Use the Modified Euler method with $N = 10, h = 0.2$, $t_i = 0.2i$ and $w_0 = 0.5$ to solve the IVP

$$y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5.$$

Heun's Method (Runge-Kutta Method of order three)

Idea: Approximate $T^3(t, y)$ with $O(h^3)$ error by $f(t + \alpha_1, y + \delta_1 f(t + \alpha_2, y + \delta_2 f(t, y)))$

$$w_0 = \beta$$

$$w_{i+1} = w_i + \frac{h}{4} \left[f(t_i, w_i) + 3f\left(t_i + \frac{2h}{3}, w_i + \frac{2h}{3} f\left(t_i + \frac{h}{3}, w_i + \frac{h}{3} f(t_i, w_i)\right)\right) \right]$$

for each $i = 0, 1, 2, \dots, N - 1$.

Runge-Kutta Method of order four

$$\begin{aligned}w_0 &= \beta \\k_1 &= hf(t_i, w_i) \\k_2 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1\right) \\k_3 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2\right) \\k_4 &= hf(t_{i+1}, w_i + k_3)\end{aligned}$$

$$\begin{aligned}w_{i+1} &= w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \\&\text{for each } i = 0, 1, 2, \dots, N - 1.\end{aligned}$$

Example 3. Use the Runge-Kutta method of order 4 with $N = 10$, $h = 0.2$, $t_i = 0.2i$ and $w_0 = 0.5$ to solve the IVP
$$y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5.$$