# 6.1 Linear Systems of Equations

To solve a system of linear equations

$$\begin{aligned} E_1&: \; a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1 \\ E_2&: \; a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2 \\ &\vdots \\ E_n&: \; a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n = b_n \end{aligned}$$

for $x_1, x_2, \ldots, x_n$ by **Gaussian elimination with backward substitution**.

Matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

**Three elementary row operations.**

**1.**Multiply one row by a nonzero number: $(\lambda E_i) \rightarrow (E_i)$

**2.**Interchange two rows: : $(E_j) \leftrightarrow (E_i)$

**3.**Add a multiple of one row to a different row: $\left(E_i + \lambda E_j\right) \rightarrow (E_i)$

**Backward substitution**

**Example 1.** To solve $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & -5/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ -3 \end{bmatrix}$

**Solution**: From $-\frac{5}{2}x_3 = -3$

$$x_3 = \frac{6}{5}.$$

Then from $2x_2 + x_3 = 4$

$$2x_2 + \frac{6}{5} = 4$$

$$x_2 = \frac{7}{5}.$$

Lastly from $x_1 + x_2 + 2x_3 = 6$

$$x_1 + \frac{7}{5} + 2\left(\frac{6}{5}\right) = 6$$

$$x_1 = \frac{11}{5}$$

## Gaussian Elimination with Backward Substitution

1. Write the system of linear equations as an **augmented matrix** $[A \mid b]$.
2. Perform elementary row operations to put the augmented matrix in the upper triangular form
3. Solve the echelon form using backward substitution

**Example 2**. Solve the system of linear equations

$$\begin{aligned} 2x_2 + x_3 &= 4 \\ x_1 + x_2 + 2x_3 &= 6 \\ 2x_1 + x_2 + x_3 &= 7 \end{aligned}$$

**Solution**:

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & 4 \\ 1 & 1 & 2 & 6 \\ 2 & 1 & 1 & 7 \end{array}\right] \underset{(E_1) \leftrightarrow (E_2)}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 2 & 1 & 4 \\ 2 & 1 & 1 & 7 \end{array}\right]$$

$$\underset{(E_3 - 2 * E_1) \rightarrow (E_3)}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 2 & 1 & 4 \\ 0 & -1 & -3 & -5 \end{array}\right] \underset{(E_3 + 0.5 * E_2) \rightarrow (E_3)}{\rightarrow}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & -\frac{5}{2} & -3 \end{array}\right]$$

Now use backward substation to solve for values of $x_1, x_2, x_3$ (see **Example 1**).

**Remark**: Gaussian elimination is computationally expensive. The total number of multiplication and divisions is about $n^3/3$, where $n$ is the number of unknowns.

# 6.2 Pivoting Strategies

**Example 1**. Apply Gaussian elimination to solve

$E1$: $\quad 0.003000x_1 + 59.14x_2 = 59.17$

$E2$: $\quad 5.291x_1 - 6.130x_2 = 46.78$

using 4-digit arithmetic with rounding (The exact solution is $x_1 = 10.00,\ \ x_2 = 1.000$).

**Remark.** In Gaussian elimination, if a pivot element $a_{kk}^{(k)}$ is small compared to an element $a_{jk}^{(k)}$ below, the multiplier $m_{jk} = \frac{a_{jk}^{(k)}}{a_{kk}^{(k)}}$ will be large, leading to large round-off errors.

To solve a system of linear equations

$$E_1: \ a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1$$
$$E_2: \ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2$$
$$\vdots$$
$$E_n: \ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n = b_n$$

for $x_1, x_2, \ldots, x_n$ by **Gaussian elimination** where $a_{kk}^{(k)}$ are numbers with small magnitude.

**Ideas of Partial Pivoting.**

Partial pivoting finds the smallest $p \geq k$ such that

$$\left|a_{pk}^{(k)}\right| = \max_{k \leq i \leq n} |a_{ik}^{(k)}|$$

and interchanges the rows $(E_k) \leftrightarrow (E_p)$. $a_{pk}^{(k)}$ is used as the pivot element.

**Example 2.** Apply Gaussian elimination with partial pivoting to solve

$E1$: $0.003000x_1 + 59.14x_2 = 59.17$

$E2$: $5.291x_1 - 6.130x_2 = 46.78$

using 4-digit arithmetic with rounding.

**Solution**:

Step 1 of partial pivoting

$$\max\left\{|a_{11}^{(1)}|, |a_{21}^{(1)}|\right\} = \{|0.003000|, |5.291|\} = 5.291 = |a_{21}^{(1)}|.$$

So perform $(E_1) \leftrightarrow (E_2)$ to make 5.291 the pivot element.

$$E1: \quad 5.291x_1 - 6.130x_2 = 46.78$$

$$E2: \quad 0.003000x_1 + 59.14x_2 = 59.17$$

$$m_{21} = \frac{a_{21}^{(1)}}{a_{11}^{(1)}} = \frac{0.003000}{5.29} = 0.0005670$$

Perform $\left(E_2 - m_{21}E_1\right) \rightarrow (E_2)$

$$5.291x_1 - 6.130x_2 = 46.78$$

$$59.14x_2 \approx 59.14$$

Backward substitution with 4-digit rounding: $x_1 = 10.00$; $x_2 = 1.000$.

**Gaussian Elimination with Partial Pivoting (Algorithm 6.2)**

$$
\begin{array}{ll}
E_1 & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = a_{1,n+1} \\
E_2 & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = a_{2,n+1} \\
 & \vdots \\
E_n & a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = a_{n,n+1}
\end{array}
$$

**INPUT:** number of equations $n$; augmented matrix $A = [a_{ij}]$. Here $1 \le i \le N$; $1 \le j \le N + 1$

**OUTPUT:** solution $x_1$, $x_2$, ... , $x_n$

**STEP 1** For $i = 1, \dots, n$ set *NROW(i)=i.*

**STEP 2** For $i = 1, \dots, n - 1$ do **STEP**s 3 – 6

**STEP 3** Let $p$ be the smallest integer with $i \le p \le n$ and

$$\left|a_{NROW(p),i}\right| = \max_{i \le j \le n} \left|a_{NROW(j),i}\right|.$$

**STEP 4** If $a_{NROW(p),i} = 0$ then OUTPUT('no unique solution exists'); STOP.

**STEP 5** If $NROW(i) \neq NROW(p)$ then set $NCOPY = NROW(i)$;

$NROW(i) = NROW(p)$;

$NROW(p) = NCOPY$.

**STEP 6** For $j = i + 1, \ldots, n$ do **STEP**s 7 and 8.

**STEP 7** Set $m_{NROW(j),i} = \dfrac{a_{NROW(j),i}}{a_{NROW(j),i}}$

**STEP 8** Perform $\left(E_{NROW(j)} - m_{NROW(j),i}E_{NROW(i)}\right) \to (E_{NROW(j)})$

**STEP 9** If $a_{NROW(n),n} = 0$ then OUTPUT('no unique solution exists');

STOP.

**STEP 10** Set $x_n = a_{NROW(n),n+1}/a_{NROW(n)n}$ *// Start backward substitution*

**STEP 11** For $i = n - 1, \ldots, 1$

set $x_i = (a_{NROW(i),n+1} - \sum_{j=i+1}^{n} a_{NROW(i),j}x_j)/a_{NROW(i),i}$

**STEP 12** OUTPUT($x_1, x_2, \ldots, x_n$);

STOP.

**Example 3**. Apply Gaussian elimination with partial pivoting to solve

$E1$: $30.00x_1 + 591400x_2 = 591700$

$E2$: $5.291x_1 - 6.130x_2 = 46.78$

using 4-digit arithmetic with rounding.

**Solution**:

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{5.291}{30.00} = 0.1764$$

$$30.00x_1 + 591400x_2 = 591700$$

$$-104300x_2 \approx -104400$$

Using backward substitution with 4-digit arithmetic:

$$x_1 = -10.00, \qquad x_2 = 1.001.$$

## Scaled Partial Pivoting

- If there are large variations in magnitude of the elements within a row, scaled partial pivoting should be used.
- Define a scale factor $s_i$ for each row

$$s_i = \max_{1 \le j \le n} |a_{ij}|$$

- At step $i$, find $p$ (the element which will be used as pivot) such that

$$\frac{a_{pi}}{s_p} = \max_{i \le k \le n} \frac{|a_{ki}|}{s_k} \quad \text{and interchange the rows } (E_i) \leftrightarrow (E_p)$$

*NOTE:* The effect of scaling is to ensure that the largest element in each row has a relative magnitude of 1 before the comparison for row interchange is performed.

**Example 3**. Apply Gaussian elimination with scaled partial pivoting to solve

$E1$: $30.00x_1 + 591400x_2 = 591700$

$E2$: $5.291x_1 - 6.130x_2 = 46.78$

using 4-digit arithmetic with rounding.

**Solution**:

$$s_1 = 591400$$

$$s_2 = 6.130$$

Consequently

$$\frac{|a_{11}|}{s_1} = \frac{30.00}{591400} = 0.5073 \times 10^{-4}$$

$$\frac{|a_{21}|}{s_2} = \frac{5.291}{6.130} = 0.8631$$

5.291 should be used as pivot element. So $(E_1) \leftrightarrow (E_2)$

Solve
$$5.291x_1 - 6.130x_2 = 46.78$$
$$30.00x_1 + 591400x_2 = 591700$$
$x_1 = 10.00,\ x_2 = 1.000.$

**Gaussian Elimination with Scaled Partial Pivoting (Algorithm 6.3)**

The only steps in Alg. 6.3 that differ from those of Alg. 6.2 are:

**STEP 1** For $i = 1, \ldots, n$ set $s_i = \max_{1 \le j \le n} |a_{ij}|$;

If $s_i = 0$ then OUTPUT('no unique solution exists');

STOP.

set *NROW(i)=i.*

**STEP 2** For $i = 1, \ldots, n-1$ do **STEP**s 3 – 6

**STEP 3** Let $p$ be the smallest integer with $i \le p \le n$ and

$$\frac{\left|a_{NROW(p),i}\right|}{s_{NROW(p)}} = \max_{i \le j \le n} \frac{|a_{NROW(j),i}|}{s_{NROW(j)}}.$$

**Example 3.** Use scaled partial pivoting with three-digit rounding to solve

$$2.11x_1 - 4.21x_2 + 0.921x_3 = 2.01$$
$$4.01x_1 + 10.2x_2 - 1.12x_3 = -3.09$$
$$1.09x_1 + 0.987x_2 + 0.832x_3 = 4.21$$