6.5 Matrix Factorization

Application: Consider to solve $A\boldsymbol{x} = \boldsymbol{b}.$ Here A is $n \times n$ matrix. Suppose $A = LU$, where L is a lower triangular matrix and U is an upper triangular matrix.

First solve $L\boldsymbol{y} = \boldsymbol{b}$ for $\boldsymbol{y}$

Then solve $U\boldsymbol{x} = \boldsymbol{y}$ for $\boldsymbol{x}$

Consider the first step of Gaussian elimination (assume no row interchange)

on $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$

Do $(E_j - m_{j1}E_1) \to (E_j)$ for $j = 2,3, \dots, n$. Here $m_{j1} = \frac{a_{ji}}{a_{11}}$ to obtain

$$A^{(1)} = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \dots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \dots & a_{2n}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}^{(2)} & \dots & a_{nn}^{(2)} \end{bmatrix}$$

Note: $a_{11}^{(1)} = a_{11},\ a_{12}^{(1)} = a_{12}, \dots a_{1n}^{(1)} = a_{1n}$.

This is equivalent to

$$A^{(1)} = M^{(1)}A$$

$$M^{(1)} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ -m_{21} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -m_{n1} & 0 & \dots & 1 \end{bmatrix}$$

$M^{(1)}$ is called the **first Gaussian transformation matrix**.

Similarly, the **kth Gaussian transformation matrix is**

$$M^{(k)} = \begin{bmatrix} 1 & 0 & & \cdots & \cdots & & 0 \\ 0 & \ddots & & & & & 0 \\ \vdots & \ddots & \ddots & & \vdots & & \vdots \\ \vdots & & 0 & 1 & \vdots & & \vdots \\ \vdots & & \vdots & -m_{k+1,k} & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & -m_{n,k} & 0 & \cdots & 1 \end{bmatrix}$$

Gaussian elimination (without row interchange) can be written as $A^{(n)} = M^{(n-1)}M^{(n-2)} \ldots M^{(1)}A$ with

$$A^{(n)} = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \ldots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \ldots & a_{2n}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{nn}^{(n)} \end{bmatrix}$$

LU Factorization $A = LU$

Reversing the elimination steps gives the inverses:

$$L^{(k)} = [M^{(k)}]^{-1} = \begin{bmatrix} 1 & 0 & & \cdots & & \cdots & 0 \\ 0 & \ddots & & & & & 0 \\ \vdots & \ddots & \ddots & & \vdots & & \vdots \\ \vdots & & 0 & 1 & \vdots & & \vdots \\ \vdots & & \vdots & m_{k+1,k} & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & m_{n,k} & 0 & \cdots & 1 \end{bmatrix}$$

We define $A = LU = [M^{(n-1)} M^{(n-2)} \ldots M^{(1)}]^{-1} A^{(n)}$

Here $U = A^{(n)}$ is the **upper triangular** matrix.

$L = [M^{(n-1)} M^{(n-2)} \ldots M^{(1)}]^{-1} = [M^{(1)}]^{-1} [M^{(2)}]^{-1} \ldots [M^{(n-1)}]^{-1}$ is the **lower triangular** matrix.

Theorem 6.19 If Gaussian elimination can be performed on the linear system $A\boldsymbol{x} = \boldsymbol{b}$ *without row interchange*, A can be factored into the product of *lower triangular* matrix L and *upper triangular* matrix U as $A = LU$:

$$U = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \dots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \dots & a_{2n}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn}^{(n)} \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 & \dots & 0 \\ m_{21} & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{n,n-1} & 1 \end{bmatrix}$$