7.3 The Jacobi and Gauss-Seidel Iterative Methods

The Jacobi Method

***Two assumptions made on Jacobi Method*:**

1. The system given by

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n = b_n \end{gathered}$$

Has a unique solution.

2. The coefficient matrix A has no zeros on its main diagonal, namely, $a_{11}, a_{22}, \ldots, a_{nn}$ are nonzeros.

Main idea of Jacobi

To begin, solve the 1st equation for x_1, the 2nd equation for x_2 and so on to obtain the rewritten equations:

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots a_{1n}x_n)$$

$$x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots a_{2n}x_n)$$

$$\vdots$$

$$x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots a_{n,n-1}x_{n-1})$$

Then make an initial guess of the solution $\boldsymbol{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \dots x_n^{(0)})$. Substitute these values into the right hand side the of the rewritten equations to obtain the *first approximation*, $\left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \dots x_n^{(1)}\right)$.

This accomplishes one **iteration**.

In the same way, the *second approximation* $\left(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \ldots x_n^{(2)}\right)$ is computed by substituting the first approximation's value $\left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \ldots x_n^{(1)}\right)$ into the right hand side of the rewritten equations.

By repeated iterations, we form a sequence of approximations $\boldsymbol{x}^{(k)} = \left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \ldots x_n^{(k)}\right)^t, \quad k = 1,2,3, \ldots$

<u>The Jacobi Method.</u> For each $k \geq 1$, generate the components $x_i^{(k)}$ of $\boldsymbol{x}^{(\boldsymbol{k})}$ from $\boldsymbol{x}^{(\boldsymbol{k-1})}$ by

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[\sum_{\substack{j=1,\\ j\neq i}}^{n}\left(-a_{ij}x_j^{(k-1)}\right) + b_i\right], \qquad \text{for } i = 1,2,, \ldots n$$

Example. Apply the Jacobi method to solve

$$5x_1 - 2x_2 + 3x_3 = -1$$
$$-3x_1 + 9x_2 + x_3 = 2$$
$$2x_1 - x_2 - 7x_3 = 3$$

Continue iterations until two successive approximations are identical when rounded to three significant digits.

Solution

n	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$x_1^{(k)}$	0.000	-0.200	0.146	0.192			
$x_2^{(k)}$	0.000	0.222	0.203	0.328			
$x_2^{(k)}$	0.000	-0.429	-0.517	-0.416			

When to stop: 1. $\frac{||\boldsymbol{x}^{(k)}-\boldsymbol{x}^{(k-1)}||}{||\boldsymbol{x}^{(k)}||} < \varepsilon$; or 2 $\left\|\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}\right\| < \varepsilon$. Here ε is a given small number .

Definition 7.1 A **vector norm** on R^n is a function, $||\cdot||$, from R^n to R with the properties:

(i) $||\boldsymbol{x}|| \geq 0$ for all $\boldsymbol{x} \in R^n$

(ii) $||\boldsymbol{x}|| = 0$ if and only if $\boldsymbol{x} = \boldsymbol{0}$

(iii) $||\alpha\boldsymbol{x}|| = |\alpha|\,||\boldsymbol{x}||$ for all $\alpha \in R$ and $\boldsymbol{x} \in R^n$

(iv) $||\boldsymbol{x} + \boldsymbol{y}|| \leq ||\boldsymbol{x}|| + ||\boldsymbol{y}||$ for all $\boldsymbol{x}, \boldsymbol{y} \in R^n$

Definition 7.2 The **Euclidean norm** l_2 and the **infinity norm** l_∞ for the vector $\boldsymbol{x} = [x_1, x_2, \dots, x_n]^t$ are defined by

$$||\boldsymbol{x}||_2 = \left\{\sum_{i=1}^{n} {x_i}^2\right\}^{\frac{1}{2}}$$

and

$$||\boldsymbol{x}||_\infty = \max_{1 \leq i \leq n} |x_i|$$

The Jacobi Method in Matrix Form

Consider to solve an $n \times n$ size system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ with

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \text{ and } \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \text{ for } \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

We split A into

$$A = \begin{bmatrix} a_{11} & 0 & \ldots & 0 \\ 0 & a_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{nn} \end{bmatrix} - \begin{bmatrix} 0 & \ldots & 0 & 0 \\ -a_{21} & \ldots & 0 & 0 \\ \vdots & & \ddots & \vdots \\ -a_{n1} & \ldots & -a_{n,n-1} & 0 \end{bmatrix}$$

$$- \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ 0 & 0 & & \vdots \\ \vdots & \vdots & \ddots & -a_{n-1,n} \\ 0 & 0 & \ldots & 0 \end{bmatrix} = D - L - U$$

$A\boldsymbol{x} = \boldsymbol{b}$ is transformed into $(D - L - U)\boldsymbol{x} = \boldsymbol{b}$

$$D\boldsymbol{x} = (L + U)\boldsymbol{x} + \boldsymbol{b}$$

Assume D^{-1} exists and $D^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & \ldots & 0 \\ 0 & \frac{1}{a_{22}} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \frac{1}{a_{nn}} \end{bmatrix}$

Then

$$\boldsymbol{x} = D^{-1}(L + U)\boldsymbol{x} + D^{-1}\boldsymbol{b}$$

The matrix form of Jacobi iterative method is

$$\boldsymbol{x}^{(k)} = D^{-1}(L + U)\boldsymbol{x}^{(k-1)} + D^{-1}\boldsymbol{b} \qquad\qquad k = 1,2,3, \ldots$$

Define $T_j = D^{-1}(L+U)$ and $\boldsymbol{c} = D^{-1}\boldsymbol{b}$, Jacobi iteration method can also be written as

$$\boldsymbol{x}^{(k)} = T_j\boldsymbol{x}^{(k-1)} + \boldsymbol{c} \qquad\qquad k = 1,2,3,\ldots$$

Numerical Algorithm of Jacobi Method

Input: $A = [a_{ij}]$, $\boldsymbol{b}, \boldsymbol{XO} = \boldsymbol{x}^{(0)}$, tolerance TOL, maximum number of iterations N.

Step 1 Set $k = 1$

Step 2 while $(k \leq N)$ do Steps 3-6

Step 3 For for $i = 1,2,,\ldots n$

$$x_i = \frac{1}{a_{ii}}\left[\sum_{\substack{j=1,\\ j\neq i}}^{n}(-a_{ij}\boldsymbol{XO}_j) + b_i\right],$$

Step 4 If $||\boldsymbol{x} - \boldsymbol{XO}|| < TOL$, then OUTPUT $(x_1, x_2, x_3, \ldots x_n)$; STOP.

Step 5 Set $k = k + 1$.

Step 6 For for $i = 1,2,,\ldots n$

Set $\boldsymbol{XO}_i = x_i$.

Step 7 OUTPUT $\left(x_1\ ,x_2\ ,x_3\ ,\ldots x_n\ \right)$;

STOP.

Another stopping criterion in Step 4: $\frac{||\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}||}{||\boldsymbol{x}^{(k)}||}$

The Gauss-Seidel Method

Main idea of Gauss-Seidel

With the Jacobi method, only the values of $x_i^{(k)}$ obtained in the kth iteration are used to compute $x_i^{(k+1)}$. With the Gauss-Seidel method, we use the new values $x_i^{(k+1)}$ as soon as they are known. For example, once we have computed $x_1^{(k+1)}$ from the first equation, its value is then used in the second equation to obtain the new $x_2^{(k+1)}$, and so on.

Example. Use the Gauss-Seidel method to solve

$$5x_1 - 2x_2 + 3x_3 = -1$$
$$-3x_1 + 9x_2 + x_3 = 2$$
$$2x_1 - x_2 - 7x_3 = 3$$

Choose the initial guess $x_1 = 0, x_2 = 0, x_3 = 0$

n	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
$x_1^{(k)}$	0.000	-0.200	0.167				
$x_2^{(k)}$	0.000	0.156	0.334				
$x_2^{(k)}$	0.000	-0.508	-0.429				

The Gauss-Seidel Method. For each $k \geq 1$, generate the components $x_i^{(k)}$ of $\boldsymbol{x}^{(\boldsymbol{k})}$ from $\boldsymbol{x}^{(\boldsymbol{k-1})}$ by

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i-1}(a_{ij}x_j^{(k)}) - \sum_{j=i+1}^{n}(a_{ij}x_j^{(k-1)}) + b_i\right], \qquad \text{for } i = 1,2,,\ldots n$$

Namely,

$$a_{11}x_1^{(k)} = -a_{12}x_2^{(k-1)} - \cdots - a_{1n}x_n^{(k-1)} + b_1$$
$$a_{22}x_2^{(k)} = -a_{21}x_1^{(k)} - a_{23}x_3^{(k-1)} - \cdots - a_{2n}x_n^{(k-1)} + b_2$$
$$a_{33}x_3^{(k)} = -a_{31}x_1^{(k)} - a_{32}x_2^{(k)} - a_{34}x_4^{(k-1)} - \cdots - a_{3n}x_n^{(k-1)} + b_3$$
$$\vdots$$
$$a_{nn}x_n^{(k)} = -a_{n1}x_1^{(k)} - a_{n2}x_2^{(k)} - \cdots - a_{n,n-1}x_{n-1}^{(k)} + b_n$$

Matrix form of Gauss-Seidel method.

$$(D - L)\boldsymbol{x}^{(\boldsymbol{k})} = U\boldsymbol{x}^{(k-1)} + \boldsymbol{b}$$
$$\boldsymbol{x}^{(\boldsymbol{k})} = (D - L)^{-1}U\boldsymbol{x}^{(k-1)} + (D - L)^{-1}\boldsymbol{b}$$

Define $T_g = (D - L)^{-1}U$ and $\boldsymbol{c}_g = (D - L)^{-1}\boldsymbol{b}$**,** Gauss-Seidel method can be written as

$$\boldsymbol{x}^{(k)} = T_g\boldsymbol{x}^{(k-1)} + \boldsymbol{c}_g \qquad\qquad k = 1,2,3, \ldots$$

Numerical Algorithm of Gauss-Seidel Method

Input: $A = [a_{ij}]$, $\boldsymbol{b}, \boldsymbol{XO} = \boldsymbol{x}^{(0)}$, tolerance *TOL*, maximum number of iterations N.

Step 1 Set $k = 1$

Step 2 while ($k \leq N$) do Steps 3-6

Step 3 For for $i = 1,2,, \dots n$

$$x_i = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i-1}(a_{ij}x_j) - \sum_{j=i+1}^{n}(a_{ij}\boldsymbol{XO}_j) + b_i\right],$$

Step 4 If $||\boldsymbol{x} - \boldsymbol{XO}|| < TOL$, then OUTPUT $(x_1, x_2, x_3, \ldots x_n)$;
STOP.

Step 5 Set $k = k + 1$.

Step 6 For for $i = 1,2,, \ldots n$

Set $\boldsymbol{XO}_i = x_i$.

Step 7 OUTPUT $(x_1, x_2, x_3, \ldots x_n)$;

STOP.

<u>Convergence theorems of the iteration methods</u>

Let the iteration method be written as

$$\boldsymbol{x}^{(k)} = T\boldsymbol{x}^{(k-1)} + \boldsymbol{c} \qquad \text{for each } k = 1,2,3, \ldots$$

Definition 7.14 The **spectral radius** $\rho(A)$ of a matrix A is defined by

$$\rho(A) = \max|\lambda|, \qquad \text{where } \lambda \text{ is an eigenvalue of } A.$$

Remark: For complex $\lambda = a + bj$, we define $|\lambda| = \sqrt{a^2 + b^2}$.

Lemma 7.18 If the spectral radius satisfies $\rho(T) < 1$, then $(I - T)^{-1}$ exists, and

$$(I - T)^{-1} = I + T + T^2 + \cdots = \sum_{j=0}^{\infty} T^j$$

Theorem 7.19 For any $\boldsymbol{x}^{(0)} \in R^n$, the sequence $\{\boldsymbol{x}^{(k)}\}_{k=0}^{\infty}$ defined by

$$\boldsymbol{x}^{(k)} = T\boldsymbol{x}^{(k-1)} + \boldsymbol{c} \qquad \text{for each } k \geq 1$$

converges to the unique solution of $\boldsymbol{x} = T\boldsymbol{x} + \boldsymbol{c}$ if and only if $\rho(T) < 1$.

Proof (only show $\rho(T) < 1$ is sufficient condition)

$\boldsymbol{x}^{(k)} = T\boldsymbol{x}^{(k-1)} + \boldsymbol{c} = T\left(T\boldsymbol{x}^{(k-2)} + \boldsymbol{c}\right) + \boldsymbol{c} = \cdots = T^k\boldsymbol{x}^{(0)} + (T^{k-1} + \cdots + T + I)\boldsymbol{c}$

Since $\rho(T) < 1$, $\lim_{k\to\infty} T^k \boldsymbol{x}^{(0)} = \boldsymbol{0}$

$$\lim_{k\to\infty} \boldsymbol{x}^{(k)} = \boldsymbol{0} + \lim_{k\to\infty} \left(\sum_{j=0}^{k-1} T^j\right) \boldsymbol{c} = (I - T)^{-1}\boldsymbol{c}$$

Definition 7.8 A **matrix norm** $||\cdot||$ on $n \times n$ matrices is a real-valued function satisfying

(i) $||A|| \geq 0$

(ii) $||A|| = 0$ if and only if $A = 0$

(iii) $||\alpha A|| = |\alpha| ||A||$

(iv) $||A + B|| \leq ||A|| + ||B||$

(v) $||AB|| \leq ||A||\,||B||$

Theorem 7.9. If $||\cdot||$ is a vector norm, the **induced** (or **natural**) **matrix norm** is given by

$$||A|| = \max_{||\boldsymbol{x}||=1} ||A\boldsymbol{x}||$$

Example. $||A||_\infty = \max_{||\boldsymbol{x}||_\infty=1} ||A\boldsymbol{x}||_\infty$, the l_∞ induced norm.

$||A||_2 = \max_{||\boldsymbol{x}||_2=1} ||A\boldsymbol{x}||_2$, the l_2 induced norm.

Theorem 7.11. If $A = [a_{ij}]$ is an $n \times n$ matrix, then

$$||A||_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|$$

Example. Determine $||A||_\infty$ for the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 5 & -1 & 1 \end{bmatrix}$

Corollary 7.20 If $||T|| < 1$ for any natural matrix norm and $\boldsymbol{c}$ is a given vector, then the sequence $\{\boldsymbol{x}^{(k)}\}_{k=0}^{\infty}$ defined by
$\boldsymbol{x}^{(k)} = T\boldsymbol{x}^{(k-1)} + \boldsymbol{c}$ converges, for any $\boldsymbol{x}^{(0)} \in R^n$, to a vector $\boldsymbol{x} \in R^n$, with $\boldsymbol{x} = T\boldsymbol{x} + \boldsymbol{c}$ **,** and the following error bound hold:

(i) $||\boldsymbol{x} - \boldsymbol{x}^{(k)}|| \le ||T||^k ||\boldsymbol{x}^{(0)} - \boldsymbol{x}||$

(ii) $||\boldsymbol{x} - \boldsymbol{x}^{(k)}|| \le \frac{||T||^k}{1-||T||} ||\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(0)}||$

Theorem 7.21 If A is strictly diagonally dominant, then for any choice of $\boldsymbol{x}^{(0)}$, both the Jacobi and Gauss-Seidel methods give sequences $\{\boldsymbol{x}^{(k)}\}_{k=0}^{\infty}$ that converges to the unique solution of $A\boldsymbol{x} = \boldsymbol{b}$**.**

Rate of Convergence

Corollary 7.20 (i) implies $||\boldsymbol{x} - \boldsymbol{x}^{(k)}|| \approx \rho(T)^k ||\boldsymbol{x}^{(0)} - \boldsymbol{x}||$

Theorem 7.22 (Stein-Rosenberg) If $a_{ij} \leq 0$, for each $i \neq j$ and $a_{ii} \geq 0$, for each $i = 1,2, \ldots, n$, then one and only one of following statements holds:

(i) $0 \leq \rho(T_g) < \rho(T_j) < 1$;
(ii) $1 < \rho(T_j) < \rho(T_g)$;
(iii) $\rho(T_j) = \rho(T_g) = 0$;
(iv) $\rho(T_j) = \rho(T_g) = 1$.