# 1.1 Review of Calculus

# Limits and Continuity

## Definition

A function $f$ defined on a set $X$ of real numbers has the *limit* $L$ at $x_0$, written $\lim_{x \to x_0} f(x) = L$, if, given any real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon, \quad \text{whenever} \quad x \in X \text{ and } 0 < |x - x_0| < \delta.$$

## Definition

Let $f$ be a function defined on a set $X$ of real numbers and $x_0 \in X$. Then $f$ is *continuous* at $x_0$ if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

The function $f$ is continuous on the set $X$ if it is continuous at each number in $X$.

# Limits of Sequence

## Definition

Let $\{x_n\}_{n=1}^{\infty}$ be an infinite sequence of real of complex numbers. The sequence $\{x_n\}_{n=1}^{\infty}$ has the *limit* $x$ is, for any $\varepsilon > 0$, there exists a positive integer $N(\varepsilon)$ such that $|x_n - x| < \varepsilon$, whenever $n > N(\varepsilon)$. The notation

$$\lim_{n\to\infty} x_n = x, \text{ or } x_n \to x \text{ as } n \to \infty,$$

means that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to $x$.

## Theorem

*If $f$ is a function defined on a set $X$ of real numbers and $x_0 \in X$, then the following statements are equivalent:*

1. *$f$ is continuous at $x_0$;*
2. *If the sequence $\{x_n\}_{n=1}^{\infty}$ in $X$ converges to $x_0$, then $\lim_{n\to\infty} f(x_n) = f(x_0)$.*

## Notations

$C(X)$ : the set of all functions that are continuous on the set $X$.

$C[a, b]$ : the set of all functions that are continuous on the closed interval $[a, b]$.

$C(-\infty, \infty)$ : the set of all functions that are continuous on $(-\infty, \infty)$.

# Differentiability

**Definition**

Let $f$ be a function defined in an open interval containing $x_0$. The function $f$ is **differentiable** at $x_0$ if

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. The number $f'(x_0)$ is called the derivative of $f$ at $x_0$. A function that has a derivative at each number in a set $X$ is **differentiable** on $X$.

- **Theorem**

If the function $f$ is differentiable at $x_0$, then $f$ is continuous at $x_0$.

## Notations

$C^n(X)$ : the set of all functions that have $n$ continuous derivatives on the set $X$.
$C^n[a,b]$ : the set of all functions that have $n$ continuous derivatives on the closed interval $[a,b]$.
$C^\infty(X)$ : the set of all functions that have derivatives of all orders on $X$.

- **Rolle's Theorem**

Suppose $f \in C[a,b]$ and $f$ is differentiable on $(a,b)$. If $f(a) = f(b)$, then a number $c$ in $(a,b)$ exists with $f'(c) = 0$.

- **Mean Value Theorem**

If $f \in C[a,b]$ and $f$ is differentiable on $(a,b)$, then a number $c$ in $(a,b)$ exists with $f'(c) = \frac{f(b)-f(a)}{b-a}$.

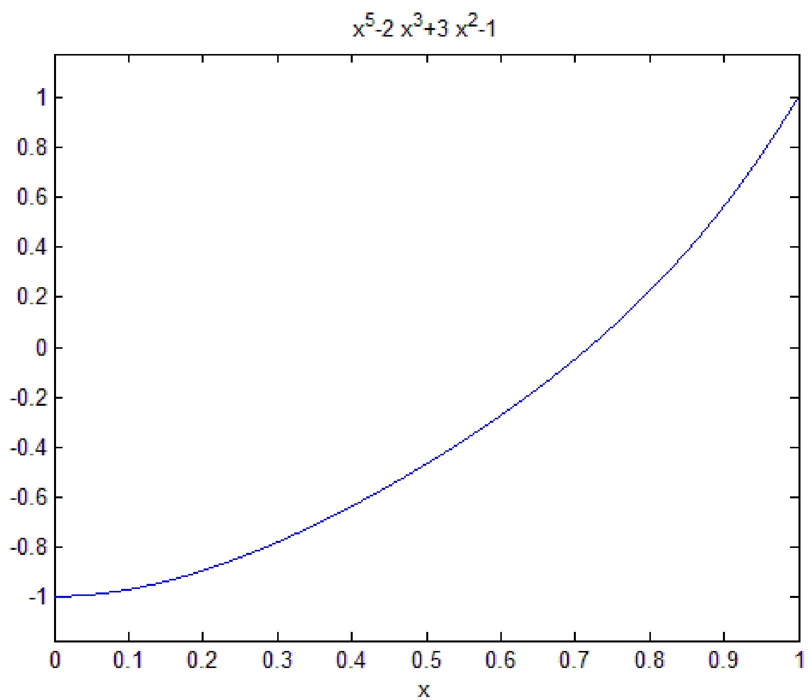
- **Extreme Value Theorem**

If $f \in C[a,b]$, then $c_1, c_2 \in [a,b]$ exist with $f(c_1) \leq f(x) \leq f(c_2)$ for all $x \in [a,b]$. Additionally, if $f$ is differentiable on $(a,b)$, then the numbers $c_1\ and\ c_2$ occur either at the endpoints of $[a,b]$ or where $f'$ is zero.

- **Intermediate Value Theorem**

If $f \in C[a,b]$, and $K$ is any number between $f(a)$ and $f(b)$, then there exists a number $c$ in $(a,b)$ for which $f(c) = K$.

**Example 2.** Show that $x^5 - 2x^3 + 3x^2 - 1 = 0$ has a solution in the interval $[0,1]$.



# Taylor’s Theorem

Suppose $f \in C^n[a,b]$, that $f^{(n+1)}$ exists on $[a,b]$ and $x_0 \in [a,b]$. For every $x \in [a,b]$, there exists a number $\xi(x)$ between $x_0$ and $x$ with

$$f(x) = P_n(x) + R_n(x),$$

Where

$$P_n(x)$$
$$= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$
$$= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

and

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)^{n+1}$$

- $P_n(x)$ – nth Taylor polynomial
- $R_n(x)$ – remainder term (truncation error)
- When $x_0 = 0$, $P_n(x)$ is also called Maclaurin polynomial

**Example 1.1.3.** Let $f(x) = \cos(x)$ and $x_0 = 0$. Determine

(a) the 2nd Taylor polynomial $P_2(x)$ for $f$ about $x_0$, use $P_2(x)$ to approximate $\cos(0.01)$ and find a bound for the accuracy of the approximation;

(b) the 3rd Taylor polynomial $P_3(x)$ for $f$ about $x_0$, use $P_3(x)$ to approximate $\cos(0.01)$ and find a bound for the accuracy of the approximation;

(c) use the third Taylor polynomial and its remainder term found in (b) to approximate $\int_0^{0.1} \cos(x)\,dx$, find a bound for the accuracy of the approximation.

**Exercise 1.1.18**. Let $f(x) = (1 - x)^{-1}$ and $x_0 = 0$. Find the nth Taylor polynomial $P_n(x)$ for $f(x)$ about $x_0$. Find a necessary $n$ for $P_n(x)$ to approximate $f(x)$ to within $10^{-6}$ on [0, 0.5].