

2.4 Error Analysis for Iterative Methods

Definition 2.7. Order of Convergence

Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p with $p_n \neq p$ for all n . If **positive constants** λ and α exist with

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda$$

then $\{p_n\}_{n=0}^{\infty}$ is said to **converges to p of order α with asymptotic error constant λ** .

An iterative technique $p_n = g(p_{n-1})$ is said to be of order α if the sequence $\{p_n\}_{n=0}^{\infty}$ converges to the solution $p = g(p)$ of order α .

- Special cases
 1. If $\alpha = 1$ (and $\lambda < 1$), the sequence is **linearly convergent**
 2. If $\alpha = 2$, the sequence is **quadratically convergent**
 3. If $\alpha < 1$, the sequence is **sub-linearly convergent** (undesirable, very slow)
 4. If $\alpha = 1$ and $\lambda = 0$ or $1 < \alpha < 2$, the sequence is **super-linearly convergent**

- **Remark:**

High order (α) \Rightarrow faster convergence (more desirable)

λ is less important than the order α .

Linear vs. Quadratic Convergence

Suppose we have two sequences converging to 0 with:

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1}|}{|p_n|} = 0.9, \quad \lim_{n \rightarrow \infty} \frac{|q_{n+1}|}{|q_n|^2} = 0.9$$

Roughly we have:

$$|p_n| \approx 0.9|p_{n-1}| \approx \dots \approx 0.9^n |p_0|,$$
$$|q_n| \approx 0.9|q_{n-1}|^2 \approx \dots \approx 0.9^{2^n - 1} |q_0|,$$

Assume $p_0 = q_0 = 1$

n	p_n	q_n
0	1	1
1	0.9	0.9
2	0.81	0.729
3	0.729	0.4782969
4	0.6561	0.205891132094649
5	0.59049	0.0381520424476946
6	0.531441	0.00131002050863762
7	0.4782969	0.00000154453835975
8	0.43046721	0.0000000000000021470

Convergence of Fixed-Point Iteration

• Theorem 2.8

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$ for all $x \in [a, b]$. Suppose g' is continuous on (a, b) and that $0 < k < 1$ exists with $|g'(x)| \leq k$ for all $x \in (a, b)$.

If $g'(p) \neq 0$, then for all number p_0 in $[a, b]$, the sequence $p_n = g(p_{n-1})$ converges only **linearly** to the **unique fixed point** p in $[a, b]$.

Proof:

$$p_{n+1} - p = g(p_n) - g(p) = g'(\xi_n)(p_n - p), \xi_n \in (p_n, p)$$

Since $\{p_n\}_{n=0}^{\infty}$ converges to p , $\{\xi_n\}_{n=0}^{\infty}$ converges to p .

Since g' is continuous, $\lim_{n \rightarrow \infty} g'(\xi_n) = g'(p)$

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n \rightarrow \infty} |g'(\xi_n)| = |g'(p)| \Rightarrow \text{linear convergence}$$

Comparison of fixed-point iteration and Newton's method

Revisit Example 2.3.1

Consider the function $f(x) = \cos(x) - x$. Solve $f(x) = 0$ using (a) fixed-point method, and (b) Newton's method.

Solution (a): Define $g(x) = \cos(x)$.

Then the fixed-point iteration alg. defined by $p_n = \cos(p_{n-1})$ with $p_0 = 0.3$ computes the fixed-point $p \approx 0.7390851$.

$$g'(x) = -\sin(x).$$

Apparently, $g'(0.7390851) = -\sin(0.7390851) \neq 0$.

Speed up Convergence of Fixed Point Iteration

- If we look for faster convergence methods, we must have $g'(p) = 0$ where p is the fixed-point.

- **Theorem 2.9**

Let p be a solution of $x = g(x)$. Suppose $g'(p) = 0$ and g'' is continuous with $|g''(x)| < M$ on an open interval I containing p . Then there exists a $\delta > 0$ such that for $p_0 \in [p - \delta, p + \delta]$, the sequence defined by $p_{n+1} = g(p_n)$, when $n \geq 0$, converges **at least quadratically** to p . For sufficiently large n

$$|p_{n+1} - p| < \frac{M}{2} |p_n - p|^2$$

Remark:

Look for quadratically convergent fixed-point methods which $g(p) = p$ and $g'(p) = 0$.

Newton's Method as Fixed-Point Problem

Consider to solve $f(x) = 0$ by Newton's method:

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}.$$

Let's define function $g(x)$ by $g(x) = x - \frac{f(x)}{f'(x)}$. The zero p of $f(x) = 0$ is also the fixed-point of $g(x)$ (assuming $f'(p) \neq 0$).

Compute $g'(x)$ to see: $g'(x) = 1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2}$

Thus $g'(p) = 1 - \frac{(f'(p))^2 - (0)f''(p)}{(f'(p))^2} = 0$.

Note: Newton's method will converge at least quadratically if $f(p) = 0$ and $f'(p) \neq 0$.

Revisit Example 2.3.1

Fixed-point method and Newton's method are used to solve $\cos(x) - x = 0$ for $x \in [0,1]$, respectively. Compare the order of convergence of these two methods.

Multiple Roots

- Newton's method and Secant method have **difficulty** to solve $f(x) = 0$ when $f(p) = 0$ and $f'(p) = 0$.
- How to modify Newton's method when $f'(p) = 0$? Here p is the root of $f(x) = 0$.

- **Definition 2.10. Multiplicity of a Root**

A solution p of $f(x) = 0$ is a zero of multiplicity m of f if for $x \neq p$, we can write $f(x) = (x - p)^m q(x)$, where $\lim_{x \rightarrow p} q(x) \neq 0$.

- **Theorem 2.11**

$f \in C^1[a, b]$ has a **simple zero** at p in (a, b) if and only if $f(p) = 0$, but $f'(p) \neq 0$.

- **Theorem 2.12**

The function $f \in C^m[a, b]$ has a zero of multiplicity m at point p in (a, b) if and only if

$$0 = f(p) = f'(p) = f''(p) = \dots = f^{(m-1)}(p), \text{ but } f^{(m)}(p) \neq 0$$

Example 1.

Let $f(x) = e^x - x - 1$. Show that f has a zero of multiplicity 2 at $x = 0$.

Modified Newton's Method for Zeroes of Higher Multiplicity ($m > 1$)

Define the new function $\mu(x) = \frac{f(x)}{f'(x)}$.

Write $f(x) = (x - p)^m q(x)$, hence

$$\mu(x) = \frac{f(x)}{f'(x)} = (x - p) \frac{q(x)}{mq(x) + (x - p)q'(x)}$$

Note that $f(p) = 0$ and p is a simple zero of $\mu(x)$.

- Apply Newton's method to solve $\mu(x) = 0$ to give:

$$\begin{aligned} x = g(x) &\equiv x - \frac{\mu(x)}{\mu'(x)} \\ &= x - \frac{f(x)f'(x)}{[f'(x)]^2 - f(x)f''(x)} \end{aligned}$$

- Quadratic convergence of the modified Newton's method:

$$p_n = p_{n-1} - \frac{f(p_{n-1})f'(p_{n-1})}{[f'(p_{n-1})]^2 - f(p_{n-1})f''(p_{n-1})}$$

Drawbacks of modified Newton's method:

- Compute $f''(x)$ is expensive
- Iteration formula is more complicated – more expensive to compute
- Roundoff errors in denominator – both $f'(x)$ and $f(x)$ approach zero.

Example 2.

Let $f(x) = e^x - x - 1$. Use Newton's method and modified Newton's method to solve $f(x) = 0$ with $p_0 = 1$.