3.1 Interpolation and Lagrange Polynomial
Example. Daily Treasury Yield Curve Rates

<table>
<thead>
<tr>
<th>Date</th>
<th>1 Mo</th>
<th>3 Mo</th>
<th>6 Mo</th>
<th>1 Yr</th>
<th>2 Yr</th>
<th>3 Yr</th>
<th>5 Yr</th>
<th>7 Yr</th>
<th>10 Yr</th>
<th>20 Yr</th>
<th>30 Yr</th>
</tr>
</thead>
<tbody>
<tr>
<td>09/01/15</td>
<td>0.01</td>
<td>0.03</td>
<td>0.26</td>
<td>0.39</td>
<td>0.70</td>
<td>1.03</td>
<td>1.49</td>
<td>1.89</td>
<td>2.17</td>
<td>2.62</td>
<td>2.93</td>
</tr>
</tbody>
</table>

Suppose we want yield rate for a four-years maturity bond, what shall we do?

**Solution:** Draw a smooth curve passing through these data points (interpolation).

• **Interpolation problem**: Find a smooth function $P(x)$ which interpolates (passes) the data $\{(x_i, y_i)\}_{i=0}^{N}$.

• **Remark**: In this class, we always assume that the data $\{y_i\}_{i=0}^{N}$ represent measured or computed values of a underlying function $f(x)$, i.e., $y_i = f(x_i)$ Thus $P(x)$ can be considered as an approximation to $f$. 
Polynomial Interpolation

Polynomials $P_n(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$ are commonly used for interpolation.

- Advantages for using polynomial: efficient, simple mathematical operation such as differentiation and integration.

Theorem 3.1 Weierstrass Approximation theorem

Suppose $f \in C[a, b]$. Then $\forall \epsilon > 0$, $\exists$ a polynomial $P(x)$: $|f(x) - P(x)| < \epsilon$, $\forall x \in [a, b]$.

Remark:
1. The bound is uniform, i.e. valid for all $x$ in $[a, b]$. This means polynomials are good at approximating general functions.
2. The way to find $P(x)$ is unknown.
• **Question:** Can Taylor polynomial be used here?
• Taylor expansion is accurate in the neighborhood of one point. We need to the (interpolating) polynomial to pass many points.

• **Example.** Taylor polynomial approximation of $e^x$ for $x \in [0,3]$
Example. Taylor polynomial approximation of \( \frac{1}{x} \) for \( x \in [0.5, 5] \). Taylor polynomials of different degrees are expanded at \( x_0 = 1 \).
2nd-degree Lagrange Interpolating Polynomial

**Goal:** construct a polynomial of degree 2 passing 3 data points \((x_0, y_0), (x_1, y_1), (x_2, y_2)\).

**Step 1:** construct a set of *basis polynomials* \(L_{2,k}(x)\), \(k = 0,1,2\) satisfying

\[
L_{2,k}(x_j) = \begin{cases} 
1, & \text{when } j = k \\
0, & \text{when } j \neq k 
\end{cases}
\]

These polynomials are:

\[
L_{2,0}(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)},
\]

\[
L_{2,1}(x) = \frac{(x_1 - x_0)(x_1 - x_2)}{(x - x_0)(x - x_2)},
\]

\[
L_{2,2}(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}.
\]
Step 2: form the 2\textsuperscript{nd}-degree Lagrange interpolating polynomial $P(x)$:

$$P(x) = y_0L_{2,0}(x) + y_1L_{2,1}(x) + y_2L_{2,2}(x)$$
Exercise 3.1.2(a) Use nodes $x_0 = 1, x_1 = 1.25, x_2 = 1.6$ to find 2\textsuperscript{nd} Lagrange interpolating polynomial $P(x)$ for $f(x) = \sin(\pi x)$. And use $P(x)$ to approximate $f(1.4)$. 
Constructing a Lagrange interpolating polynomial $P(x)$ passing through the points $(x_0, f(x_0))$, $(x_1, f(x_1))$, $(x_2, f(x_2))$, ..., $(x_n, f(x_n))$.

1. Define Lagrange basis functions $L_{n,k}(x) = \prod_{i=0, i \neq k}^{n} \frac{x-x_i}{x_k-x_i} = \frac{x-x_0}{x_k-x_0} \cdots \frac{x-x_{k-1}}{x_k-x_{k-1}} \frac{x-x_k+1}{x_k-x_k+1} \cdots \frac{x-x_n}{x_k-x_n}$ for $k = 0, 1 \ldots n$.

Remark: $L_{n,k}(x_k) = 1; L_{n,k}(x_i) = 0, \forall i \neq k$

2. $P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x)$. 
• $L_{6,3}(x)$ for points $x_i = i, \ i = 0, \ldots, 6$. 
• **Theorem 3.2** If \( x_0, \ldots, x_n \) are \( n + 1 \) distinct numbers (called nodes) and \( f \) is a function whose values are given at these numbers, then a unique polynomial \( P(x) \) of degree at most \( n \) exists with \( P(x_k) = f(x_k) \), for each \( k = 0, 1, \ldots, n \).

\[
P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x).
\]

Where \( L_{n,k}(x) = \prod_{i=0,i\neq k}^{n} \frac{x-x_i}{x_k-x_i} \).
Error Bound for the Lagrange Interpolating Polynomial

**Theorem 3.3** Suppose $x_0, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then for each $x$ in $[a, b]$, a number $\xi(x)$ (generally unknown) between $x_0, \ldots, x_n$, and hence in $(a, b)$, exists with $f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n + 1)!} (x - x_0)(x - x_1) \ldots (x - x_n)$.

Where $P(x)$ is the Lagrange interpolating polynomial.
Remark:
1. Applying the error term may be difficult. 
   \( \xi(x) \) is generally unknown.
2. \((x - x_0)(x - x_1) \ldots (x - x_n)\) is oscillatory.

Graph of \((x - 0)(x - 1)(x - 2)(x - 3)(x - 4)\)

Remark: In general, \(|f(x) - p(x)|\) is small when \(x\) is close to the center of \([x_0, x_n]\).

3. The error formula is important as they are used for numerical differentiation and integration.
Example 3. 2\textsuperscript{nd} Lagrange polynomial for $f(x) = \frac{1}{x}$ on [2, 4] using nodes $x_0 = 2, x_1 = 2.75, x_2 = 4$ is $P(x) = \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}$. Determine the error form for $P(x)$, and maximum error when polynomial is used to approximate $f(x)$ for $x \in [2,4]$.
Exercise 3.1.6(a). Use appropriate Lagrange polynomials of degree 2 to approximate $f(0.43)$ if $f(0) = 1$, $f(0.25) = 1.64872$, $f(0.5) = 2.71828$, $f(0.75) = 4.48169$. 
Example 4 Suppose a table is to be prepared for $f(x) = e^x$, $x \in [0,1]$. Assume the number of decimal places to be given per entry is $d \geq 8$ and that the difference between adjacent $x$-values, the step size is $h$. What step size $h$ will ensure that linear interpolation gives an absolute error of at most $10^{-6}$ for all $x$ in $[0,1]$. 