4.4 Composite Numerical Integration
Motivation: 1) on large interval, use low order Newton-Cotes formulas are not accurate.

2) on large interval, interpolation using high degree polynomial is unsuitable because of oscillatory nature of high degree polynomials.

Main idea: divide integration interval $[a, b]$ into subintervals and use simple integration rule for each subinterval.
Example 1.  a) Use Simpson’s rule to approximate \( \int_0^4 e^x \, dx \). The exact value is 53.59819.  

b) Divide \([0,4]\) into \([0,1] + [1,2] + [2,3] + [3,4]\). Use Simpson’s rule to approximate \( \int_0^1 e^x \, dx \), \( \int_1^2 e^x \, dx \), \( \int_2^3 e^x \, dx \) and \( \int_3^4 e^x \, dx \). Then approximate \( \int_0^4 e^x \, dx \) by adding approximations for \( \int_0^1 e^x \, dx \), \( \int_1^2 e^x \, dx \), \( \int_2^3 e^x \, dx \) and \( \int_3^4 e^x \, dx \). Compare with accurate value.

Solution:

a) \( h = \frac{4-0}{2} \). \( \int_0^4 e^x \, dx \approx \frac{2}{3} (e^0 + 4e^2 + e^4) = 56.76958 \).

Error = \( |53.59819 - 56.76958| = 3.17143 \)

b) \( \int_0^4 e^x \, dx = \int_0^1 e^x \, dx + \int_1^2 e^x \, dx + \int_2^3 e^x \, dx + \int_3^4 e^x \, dx \approx \frac{0.5}{3} (e^0 + 4e^{0.5} + e^1) + \frac{0.5}{3} (e^1 + 4e^{1.5} + e^2) + \frac{0.5}{3} (e^2 + 4e^{2.5} + e^3) + \frac{0.5}{3} (e^3 + 4e^{3.5} + e^4) = 53.61622 \)

Error = \( |53.59819 - 53.61622| = 0.01807 \)

b) is much more accurate than a).
Let $f \in C^2[a, b]$, $h = \frac{b-a}{n}$, and $x_j = a + jh$ for $j = 0, \cdots, n$.

On each subinterval $[x_{j-1}, x_j]$, for $j = 1, \cdots, n$, apply Trapezoidal rule:
\[ \int_{a}^{b} f(x) \, dx \]

\[ = \left[ \frac{h}{2} \left( f(x_0) + f(x_1) \right) - \frac{h^3}{12} f''(\xi_1) \right] \]

\[ + \left[ \frac{h}{2} \left( f(x_1) + f(x_2) \right) - \frac{h^3}{12} f''(\xi_2) \right] + \ldots \]

\[ + \left[ \frac{h}{2} \left( f(x_{n-1}) + f(x_n) \right) - \frac{h^3}{12} f''(\xi_n) \right] \]

\[ = \frac{h}{2} \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right] - \frac{h^3}{12} \sum_{j=1}^{n} f''(\xi_j) \]

\[ = \frac{h}{2} \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right] - \frac{b - a}{12} h^2 f''(\mu) \]

Error, which can be simplified.
Theorem 4.5 Let $f \in C^2[a, b]$, $h = \frac{b-a}{n}$, and $x_j = a + jh$ for each $j = 0, \ldots, n$. There exists a $\mu \in (a, b)$ for which Composite Trapezoidal rule with its error term is

$$
\int_a^b f(x) \, dx = \frac{h}{2} \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right] - \frac{b-a}{12} h^2 f''(\mu)
$$

Error term
Let \( f \in C^2[a, b] \), \( n \) be an even integer, \( h = \frac{b-a}{n} \), and \( x_j = a + jh \) for \( j = 0, \ldots, n \).

On each consecutive pair of subintervals (for example \([x_0, x_2], [x_2, x_4], \) and \([x_{2j-2}, x_{2j}]\)) for each \( j = 1, \ldots, n/2 \), apply a Simpson’s rule:
\[
\int_a^b f(x) \, dx = \sum_{j=1}^{n/2} \int_{x_{2j-2}}^{x_{2j}} f(x) \, dx
\]

\[
= \sum_{j=1}^{n/2} \frac{h}{3} \left( f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j}) - \frac{h^5}{90} f^{(4)}(\xi_j) \right)
\]

\[
= \frac{h}{3} \left( f(x_0) + 2 \sum_{j=1}^{(n/2)-1} f(x_{2j}) + 4 \sum_{j=1}^{n/2} f(x_{2j-1}) + f(x_n) \right)
\]

\[
- \frac{h^5}{90} \sum_{j=1}^{n/2} f^{(4)}(\xi_j)
\]

Error, which can be simplified
Theorem 4.4 Let \( f \in C^4[a, b], n \) be even integer, \( h = \frac{b-a}{n} \), and \( x_j = a + jh \) for each \( j = 0, \cdots, n \). There exists a \( \mu \in (a, b) \) for which Composite Simpson’s rule with its error term is

\[
\int_{a}^{b} f(x)dx = \frac{h}{3} \left[ f(a) + 2 \sum_{j=1}^{\left(\frac{n}{2}\right)-1} f(x_{2j}) + 4 \sum_{j=1}^{\frac{n}{2}} f(x_{2j-1}) + f(b) \right] - \frac{b-a}{180} h^4 f^{(4)}(\mu)
\]

Error Term
Theorem 4.6  Let \( f \in C^2[a, b] \), \( n \) be even, \( h = \frac{b-a}{n+2} \), and \( x_j = a + (j + 1)h \) for each \( j = -1, 0, \ldots, n, n + 1 \). There exists a \( \mu \in (a, b) \) for which the Composite Midpoint rule with its error term is

\[
\int_a^b f(x) \, dx = 2h \sum_{j=0}^{\frac{n}{2}} f(x_{2j}) + \frac{b-a}{6} h^2 f''(\mu)
\]
Exercise 13. Determine the values of $n$ and $h$ required to approximate $\int_{0}^{2} \frac{1}{x+4} \, dx$ to within $10^{-5}$ and compute the approximation. Use

a. Composite Trapezoidal rule.

b. Composite Simpson’s rule.

c. Composite Midpoint rule.