4.4 Composite Numerical Integration

Motivation: 1) on large interval, use low order Newton-Cotes formulas are not accurate.

2) on large interval, interpolation using high degree polynomial is unsuitable because of oscillatory nature of high degree polynomials.

Main idea: divide integration interval $[a, b]$ into subintervals and use simple integration rule for each subinterval.

Example 1. a) Use Simpson's rule to approximate $\int_0^4 e^x dx$. The exact value is 53.59819. **b)** Divide [0,4] into [0,1] + [1,2] + [2,3] + [3,4]. Use Simpson's rule to approximate $\int_0^1 e^x dx$, $\int_1^2 e^x dx$, $\int_2^3 e^x dx$ and $\int_3^4 e^x dx$. Then approximate $\int_0^4 e^x dx$ by adding approximations for $\int_0^1 e^x dx$, $\int_1^2 e^x dx$, $\int_2^3 e^x dx$ and $\int_3^4 e^x dx$. Compare with accurate value.

Solution:

a) $h = \frac{4-0}{2}$. $\int_0^4 e^x dx \approx \frac{2}{3}(e^0 + 4e^2 + e^4) = 56.76958.$

Error= |53.59819 -56.76958| = 3.17143

b) $\int_0^4 e^x dx = \int_0^1 e^x dx + \int_1^2 e^x dx + \int_2^3 e^x dx + \int_3^4 e^x dx \approx \frac{0.5}{3}(e^0 + 4e^{0.5} + e^1) + \frac{0.5}{3}(e^1 + 4e^{1.5} + e^2) + \frac{0.5}{3}(e^2 + 4e^{2.5} + e^3) + \frac{0.5}{3}(e^3 + 4e^{3.5} + e^4) = 53.61622$

Error=|53.59819 -53.61622| = 0.01807

b) is much more accurate than **a)**.

Composite Trapezoidal rule

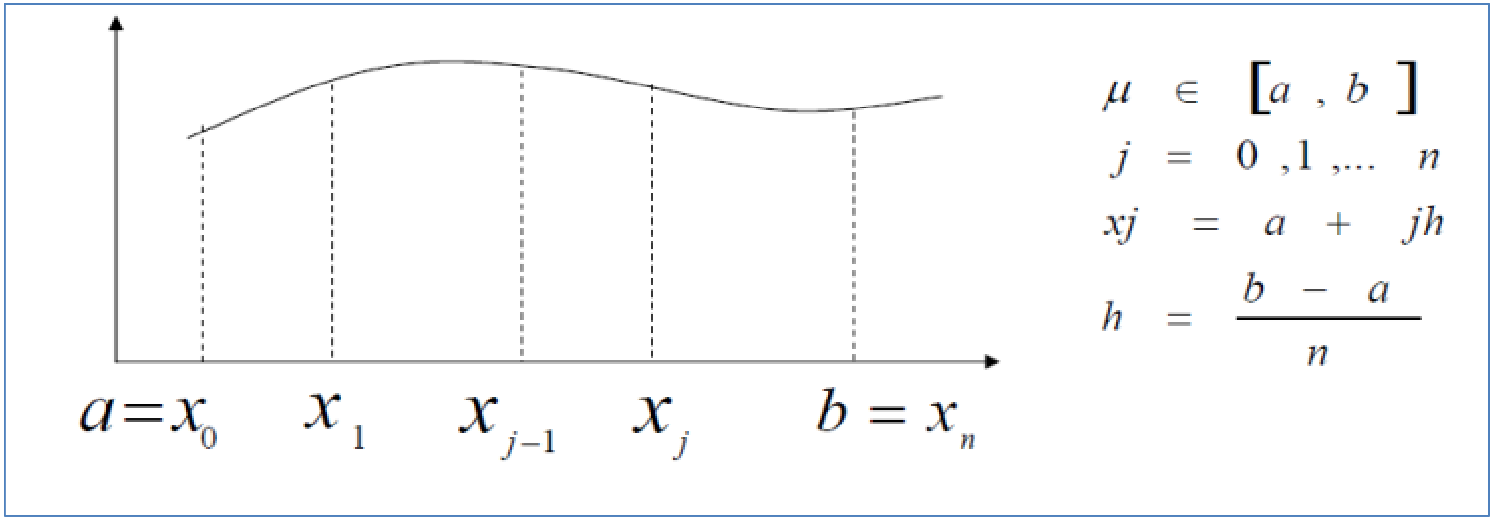

Figure 1 Composite Trapezoidal Rule

Let $f \in C^2[a, b], h = \frac{b-a}{n}$, and $x_j = a + jh$ for $j = 0, \cdots, n$.

On each subinterval $[x_{j-1}, x_j]$, for for $j = 1, \cdots, n$, apply Trapezoidal rule:

$$\int_a^b f(x)dx$$

$$= \left[\frac{h}{2}\left(f(x_0) + f(x_1)\right) - \frac{h^3}{12} f''(\xi_1)\right]$$

$$+ \left[\frac{h}{2}\left(f(x_1) + f(x_2)\right) - \frac{h^3}{12} f''(\xi_2)\right] + \cdots$$

$$+ \left[\frac{h}{2}\left(f(x_{n-1}) + f(x_n)\right) - \frac{h^3}{12} f''(\xi_n)\right]$$

$$= \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f\left(x_j\right) + f(b)\right] - \frac{h^3}{12}\sum_{j=1}^{n} f''(\xi_j)$$

Error, which can be simplified

$$= \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f\left(x_j\right) + f(b)\right] - \frac{b-a}{12} h^2 f''(\mu)$$

Theorem 4.5 Let $f \in C^2[a,b]$, $h = \frac{b-a}{n}$, and $x_j = a + jh$ for each $j = 0, \cdots, n$. There exists a $\mu \in (a,b)$ for which **Composite Trapezoidal rule** with its error term is

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \boxed{\frac{b-a}{12}h^2 f''(\mu)}$$

Error term

Composite Simpson's rule

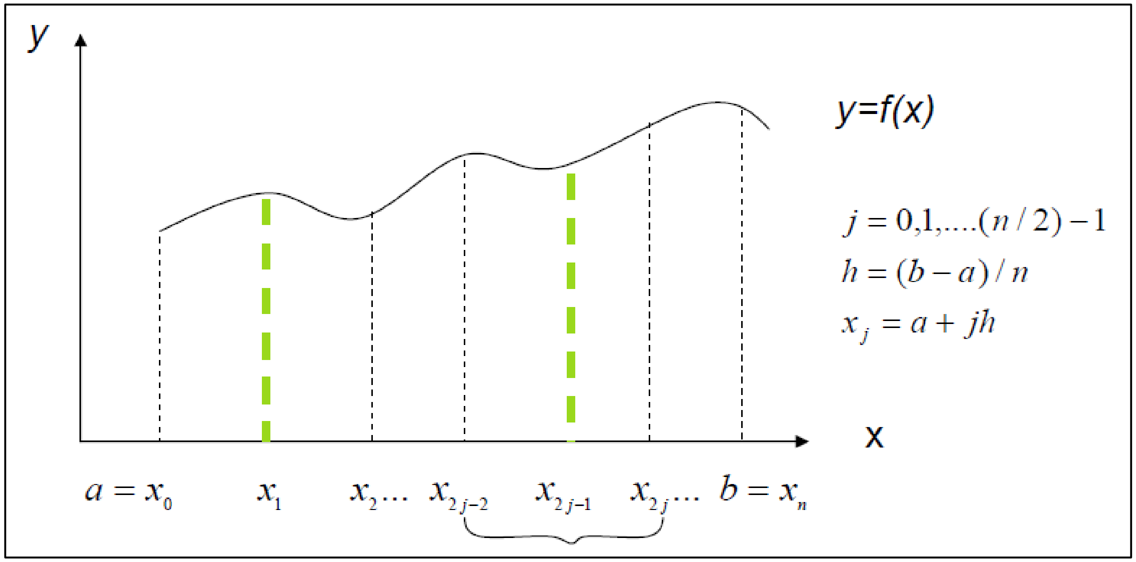

Figure 2 Composite Simpson's rule

Let $f \in C^2[a,b]$, n ***be an even integer***, $h = \frac{b-a}{n}$, and $x_j = a + jh$ for $j = 0, \cdots, n$.

On **each consecutive pair** of subintervals (for example $[x_0, x_2]$, $[x_2, x_4]$, and $\left[x_{2j-2}, x_{2j}\right]$) for each $j = 1, \cdots, n/2$, apply a Simpson's rule:

$$\int_a^b f(x)dx = \sum_{j=1}^{n/2} \int_{x_{2j-2}}^{x_{2j}} f(x)dx$$

$$= \sum_{j=1}^{n/2} \frac{h}{3}\left(f\left(x_{2j-2}\right) + 4f\left(x_{2j-1}\right) + f\left(x_{2j}\right) - \frac{h^5}{90} f^{(4)}\left(\xi_j\right)\right)$$

$$= \frac{h}{3}\left(f(x_0) + 2\sum_{j=1}^{\left(\frac{n}{2}\right)-1} f\left(x_{2j}\right) + 4\sum_{j=1}^{\left(\frac{n}{2}\right)} f\left(x_{2j-1}\right) + f(x_n)\right)$$

$$- \frac{h^5}{90} \sum_{j=1}^{\left(\frac{n}{2}\right)} f^{(4)}\left(\xi_j\right)$$

Error, which can be simplified

Theorem 4.4 Let $f \in C^4[a,b]$, n *be* ***even integer***, $h = \frac{b-a}{n}$, and $x_j = a + jh$ for each $j = 0, \cdots, n$. There exists a $\mu \in (a,b)$ for which **Composite Simpson's rule** with its error term is

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{(\frac{n}{2})-1} f(x_{2j}) + 4\sum_{j=1}^{(\frac{n}{2})} f(x_{2j-1}) + f(b)\right] - \frac{b-a}{180}h^4 f^{(4)}(\mu)$$

Error Term

Composite Midpoint rule

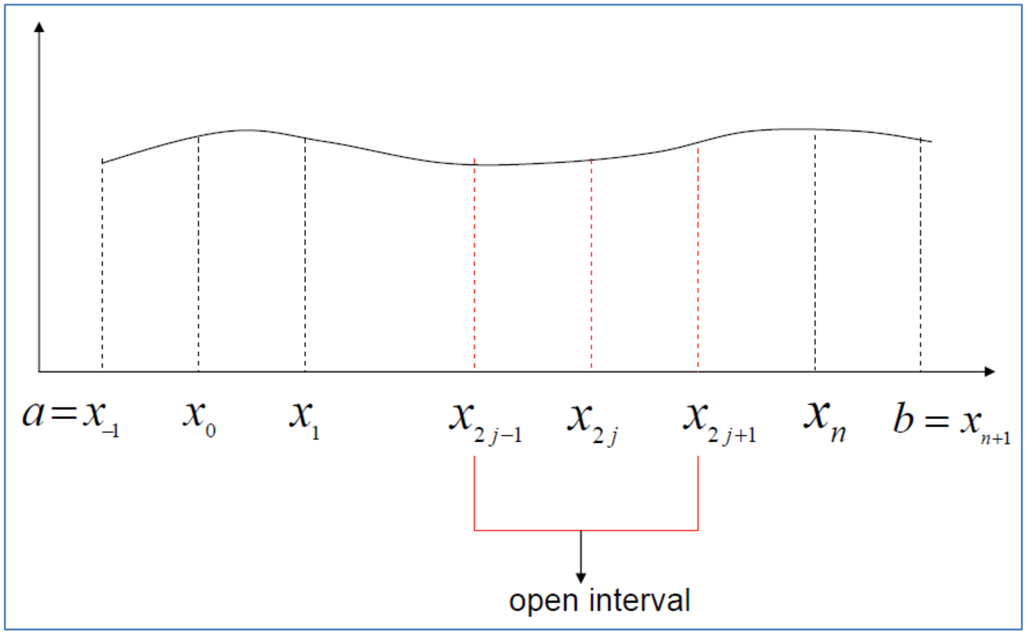

Figure 3 Composite Midpoint rule

Theorem 4.6 Let $f \in C^2[a,b]$, n *be* ***even***, $h = \frac{b-a}{n+2}$, and $x_j = a + (j+1)h$ for each $j = -1, 0, \cdots, n, n+1$. There exists a $\mu \in (a,b)$ for which **Composite Midpoint rule** with its error term is

$$\int_a^b f(x)dx = 2h \sum_{j=0}^{(\frac{n}{2})} f(x_{2j}) \quad + \boxed{\frac{b-a}{6} h^2 f''(\mu)}$$

Exercise 13. Determine the values of n and h required to approximate $\int_0^2 \frac{1}{x+4}\,dx$ to within 10^{-5} and compute the approximation. Use

a. Composite Trapezoidal rule.

b. Composite Simpson's rule.

c. Composite Midpoint rule.