5.1 Elementary Theory of Initial-Value Problems

Definition 5.1 A function $f(t,y)$ is said to satisfy a **Lipschitz condition** in the variable y on a set $D \subset R^2$ if a constant $L > 0$ exists with

$$|f(t,y_1) - f(t,y_2)| \le L|y_1 - y_2|$$

whenever (t,y_1) and (t,y_2) are in D. The constant L is called a **Lipschitz constant** for f.

Exercise 5.1 Show that $f(t,y) = \frac{2}{t}y + t^2e^t$ satisfies a Lipschitz condition on the interval $D = \{(t,y)|1 \le t \le 2 \; and -2 \le y \le 5\}$.

Definition 5.2 A set $D \subset R^2$ is said to be **convex** if whenever (t_1,y_1) and (t_2,y_2) belongs to D and $\lambda \in [0,1]$, the point $((1-\lambda)t_1 + \lambda t_2,\ (1-\lambda)y_1 + \lambda y_2)$ also belongs to D.

***Remark*:**

1. Convex means that line segment connecting (t_1,y_1) and (t_2,y_2) is in D whenever (t_1,y_1) and (t_2,y_2) belongs to D.

2. The set $D = \{(x, y)\ \ a\ \le t \le b\ and - \infty \le y \le \infty\}$ is convex.

Theorem 5.3 Suppose $f(t, y)$ is defined on a convex set $D\ \subset R^2$. If a constant $L > 0$ exists with

$$|\frac{\partial f}{\partial y}(t, y)| \le L$$

for all $(t, y) \in D$, then f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L.

Theorem 5.4 (existence & uniqueness) Suppose that $D = \{(x, y)\ \ a\ \le t \le b\ and - \infty < y < \infty\}$ and that $f(t, y)$ is continuous on D. If f satisfies a Lipschitz condition on D in the variable y, then the initial-value problem (IVP)

$$y' = f(t, y), \quad a\ \le t\ \le b,\ \ y(a) =\ \beta,$$

has a unique solution $y(t)$ for $a\ \le t\ \le b$.

Example 2. Show that there is a unique solution to the IVP

$$y' = 1 + tsin(ty), \quad 0\ \le t \le 2, \quad y(0) = 0.$$

Definition 5.5 The IVP $\frac{dy}{dt} = f(t,y), \quad a \le t \le b, \quad y(a) = \beta$ is said to be **a well-posed problem** if:

1. A unique solution $y(t)$, to the problem exists, and
2. There exist constant $\varepsilon_0 > 0$ and $k > 0$ such that for any ε with $\varepsilon_0 > \varepsilon > 0$, whenever $\delta(t)$ is continuous with $|\delta(t)| < \varepsilon$ for all t in $[a,b]$, and when $|\delta_0| < \varepsilon$, the IVP (*a perturbed problem associated with original* $\frac{dy}{dt} = f(t,y)$)

$$\frac{dz}{dt} = f(t,z) + \delta(t), \quad a \le t \le b, \quad z(a) = \beta + \delta_0$$

has a unique solution $z(t)$ that satisfies

$$|z(t) - y(t)| < k\varepsilon \qquad \text{for all } t \text{ in } [a,b].$$

Why well-posedness? Numerical methods always solve perturbed problem because of round-off errors.

Example 3. Show the IVP $y' = y - t^2 + 1, \quad 0 \le t \le 2, \; y(0) = 0.5$ is well-posed on $D = \{(t, y) | 0 \le t \le 2 \; and - \infty \le y \le \infty\}$

Theorem 5.6 Suppose $D = \{(x, y) \;\; a \le t \le b \; and - \infty < y < \infty\}$ and that $f(t, y)$ is continuous on D and satisfies a Lipschitz condition on D in the variable y, then IVP

$$y' = f(t, y), \quad a \le t \le b, \; y(a) = \beta,$$

is well-posed.

Example. Show the IVP $y' = y - t^2 + 1, \quad 0 \le t \le 2, \; y(0) = 0.5$ is well-posed on $D = \{(x, y) \;\; 0 \le t \le 2 \; and - \infty < y < \infty\}$

Solution: $\left|\frac{\partial}{\partial y}(y - t^2 + 1)\right| = |1| = 1.$

Function $(y - t^2 + 1)$ satisfies Lipschitz condition with L =1.

So **Theorem 5.6** implies the IVP is well posed.

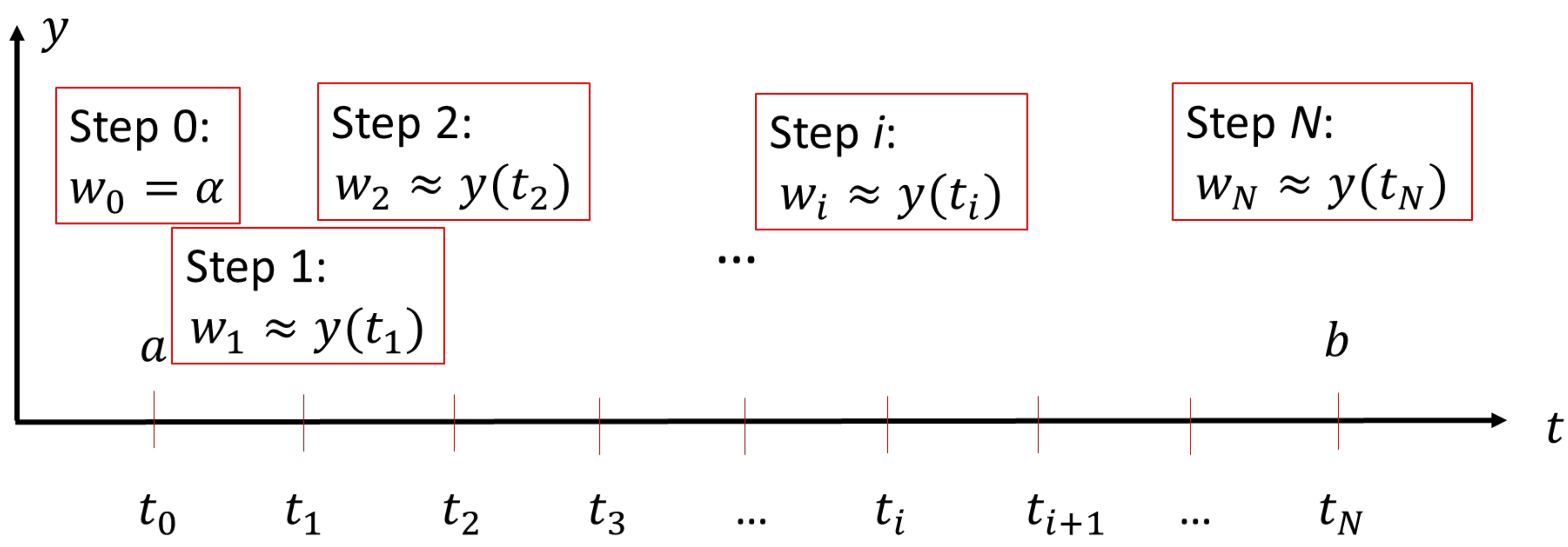
5.2 Euler's Method

Consider to solve $\begin{cases} \frac{dy}{dt} = f(t, y) \\ y(t = a) = \alpha \end{cases}$ $\quad a \le t \le b$

Choose integer N. Let $h = \frac{b-a}{N}$, and $t_i = a + ih$ with $i = 0,1, \dots, N$. t_i are called mesh points.

We want to compute approximate solutions

$w_0, w_1, w_2, \dots, w_i, w_{i+1}, \dots w_N$ step by step with



Let $w_0 = \alpha$, $w_i \approx y(t_i)$, for each $i = 0,1, \dots, N$.

Euler's method:

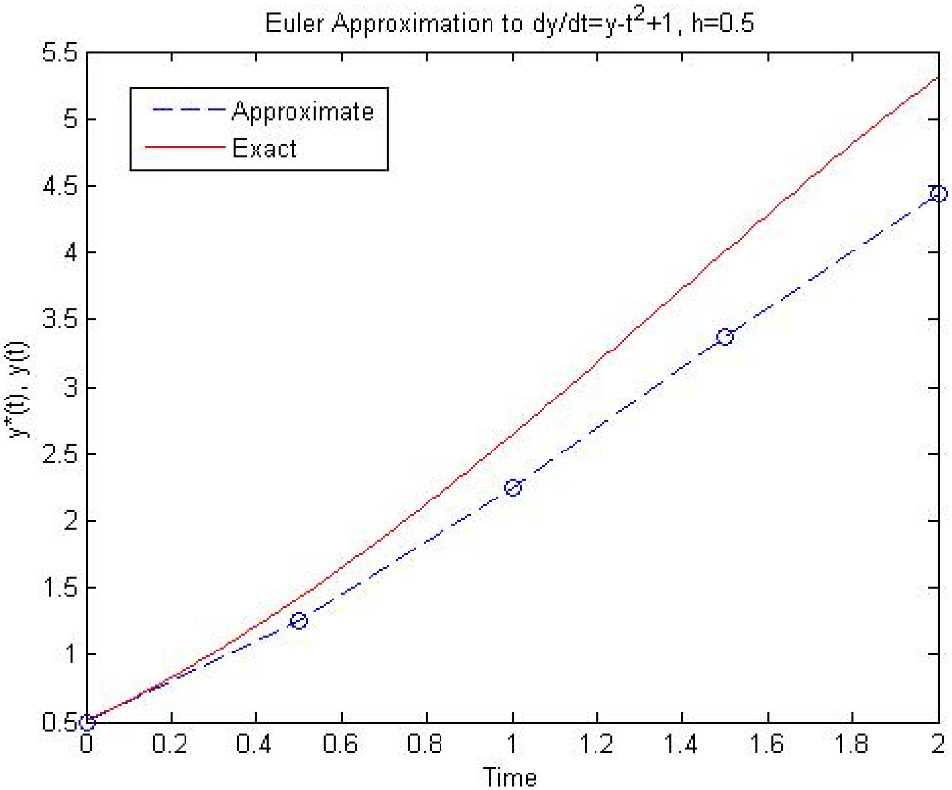
$w_0 = \alpha$

$w_{i+1} = w_i + hf(t_i, w_i)$, for each $i = 0,1, \dots, N - 1$.

Example 1. Solve $y' = y - t^2 + 1, \quad 0 \le t \le 2, \; y(0) = 0.5$ numerically with time step size $h = 0.2$.

Geometric interpretation of Euler's Method

$f(t_i, w_i) \approx y'(t_i) = f(t_i, y(t_i))$ implies $f(t_i, w_i)$ is an approximation to slope of $y(t)$ at t_i



Error bound

Theorem 5.9 Suppose $D = \{(x, y)\ \ a \leq t \leq b \ and - \infty < y < \infty\}$ and that $f(t, y)$ is continuous on D and satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L and that a constant M exists with

$$|y''(t)| \leq M, \quad \text{for all } t \in [a, b].$$

Let $y(t)$ denote the unique solution to the IVP

$$y' = f(t, y), \quad a \leq t \leq b, \ \ y(a) = \beta,$$

and $w_0, w_1, \cdots, w_n$ as in Euler's method. Then

$$|y(t_i) - w_i| \leq \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right].$$

Example 2 The solution to the IVP $y' = y - t^2 + 1, 0 \leq t \leq 2$, $y(0) = 0.5$ was approximated by Euler's method with $h = 0.2$. Find the bound for approximation.

Effects of round-off error

Theorem 5.10 Let $y(t)$ be the unique solution to the IVP

$$y' = f(t, y), \quad a \le t \le b, \; y(a) = \alpha,$$

and $u_0, u_1, \dots, u_N$ be approximate solution obtained using

$$\begin{cases} u_0 = \alpha + \delta_0 \\ u_{i+1} = u_i + hf(t_i, u_i) + \delta_{i+1}, \quad \text{for each } \; i = 0,1,\dots,N-1 \end{cases}$$

If $|\delta_i| < \delta$ for each $i = 0,1,\dots,N$ and the hypotheses of Theorem 5.9 holds for $y' = f(t, y)$, then

$$|y(t_i) - u_i| \le \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)\left[e^{L(t_i - a)} - 1\right] + |\delta_0| e^{L(t_i - a)}$$

for each $i = 0,1,\dots,N$.

Let $E(h) = \frac{hM}{2} + \frac{\delta}{h}$. Minimal value of $E(h)$ occurs when $h = \sqrt{\frac{2\delta}{M}}$.