5.3 High-Order Taylor Methods
Consider the IVP
\[
\begin{aligned}
\{ y' & = f(t, y), \quad a \leq t \leq b \\
y(a) & = \beta
\end{aligned}
\]

**Definition 5.11:** The difference method
\[
\begin{aligned}
w_0 & = \beta \\
w_{i+1} & = w_i + hf(t_i, w_i) \quad \text{for each } i = 0, 1, 2, \ldots, N-1
\end{aligned}
\]

with step size \( h = \frac{b-a}{N} \) has **Local Truncation Error**
\[
\tau_{i+1}(h) = \frac{y_{i+1} - (y_i + hf(t_i, y_i))}{h} = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i) \quad \text{for each } i = 0, 1, 2, \ldots, N-1.
\]

Note: \( y_i := y(t_i) \) and \( y_{i+1} := y(t_{i+1}) \).
Geometric view of local truncation error

\[ t_i + 1 \]

\[ y_{i+1} - w_{i+1} \]

\[ |h \sigma_{i+1}(h)| \]
Example. Analyze the local truncation error of Euler’s method for solving
\[ y' = f(t,y), \quad a \leq t \leq b, \quad y(a) = \beta. \]
Assume \(|y''(t)| < M\) with \(M > 0\) constant.

Consider the IVP
\[ y' = f(t,y), \quad a \leq t \leq b, \quad y(a) = \beta. \]
Compute \(y'', y^{(3)} \ldots\) using \(f(t,y)\) and its derivatives.
**Derivation of higher-order Taylor methods**

Consider the IVP

\[ y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \beta, \quad \text{with step size} \]

\[ h = \frac{b - a}{N}, \quad t_{i+1} = a + ih. \]

Expand \( y(t) \) in the \( n \)th Taylor polynomial about \( t_i \), evaluate at \( t_{i+1} \)

\[
y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(t_i) + \cdots + \frac{h^n}{n!} y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi_i)
\]

\[
y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} f'(t_i, y(t_i)) + \cdots + \frac{h^n}{n!} f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))
\]
for some $\xi_i \in (t_i, t_{i+1})$. Delete remainder term to obtain the $n$th Taylor method of order $n$.

Denote

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i)$$

**Taylor method of order $n$**

$$w_0 = \beta$$

$$w_{i+1} = w_i + hT^{(n)}(t_i, w_i) \quad \text{for each } i = 0, 1, 2, \ldots, N - 1.$$  

Remark: Euler’s method is the Taylor method of order one.
Example 1. Use Taylor method of orders (a) two and (b) four with \( N = 10 \) to the IVP

\[
y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5.
\]

Solution (b):

\[
h = \frac{2-0}{N} = \frac{2-0}{10} = 0.2
\]

So \( t_i = 0 + 0.2i = 0.2i \) for each \( i = 0, 1, 2, \ldots, 10 \).

\[
f'(t, y(t)) = \frac{d}{dt}(y - t^2 + 1) = y' - 2t = y - t^2 + 1 - 2t
\]

\[
f''(t, y(t)) = \frac{d}{dt}(f'(t, y(t))) = (y - t^2 + 1 - 2t)' = y' - 2t - 2 = y - t^2 + 1 - 2t - 2 = y - t^2 - 2t - 1
\]

\[
f^{(3)}(t, y(t)) = \frac{d}{dt}(f''(t, y(t))) = (y - t^2 - 2t - 1)' = y' - 2t - 2 = y - t^2 + 1 - 2t - 2 = y - t^2 - 2t - 1
\]
\[ T^{(4)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \frac{h^2}{3!} f''(t_i, w_i) + \frac{h^3}{4!} f^{(3)}(t_i, w_i) \]

\[ = (w_i - t_i^2 + 1) + \frac{h}{2} (w_i - t_i^2 + 1 - 2t_i) + \]

\[ \frac{h^2}{6} (w_i - t_i^2 - 2t_i - 1) + \frac{h^3}{24} (w_i - t_i^2 - 2t_i - 1) \]

\[ = \left( 1 + \frac{h}{2} + \frac{h^2}{6} + \frac{h^3}{24} \right) (w_i - t_i^2) - \]

\[ \left( 1 + \frac{h}{3} + \frac{h^2}{12} \right) (ht_i) + 1 + \frac{h}{2} - \frac{h^2}{6} - \frac{h^3}{24} \]
The 4\textsuperscript{th} order Taylor method is

\[ w_0 = 0.5 \]

\[ w_{i+1} = w_i \]

\[ + h \left[ \left( 1 + \frac{h}{2} + \frac{h^2}{6} + \frac{h^3}{24} \right) (w_i - t_i^2) - \left( 1 + \frac{h}{3} + \frac{h^2}{12} \right) (ht_i) \right. \]

\[ + \left. 1 + \frac{h}{2} - \frac{h^2}{6} - \frac{h^3}{24} \right] \]

for each \( i = 0, 1, 2, \ldots, 9 \).

Now compute approximate solutions at each time step:

\[ w_1 = 0.5 \]

\[ + 0.2 \left( \left( 1 + \frac{0.2}{2} + \frac{0.2^2}{6} + \frac{0.2^3}{24} \right) (0.5 - 0) \right. \]

\[ - \left(1 + \frac{0.2}{3} + \frac{0.2^2}{12}\right) (0) + 1 + \frac{0.2}{2} - \frac{0.2^2}{6} - \frac{0.2^3}{24} \right) = 0.8293 \]
abs. error of 4th order Taylor at $t_1$:  $|w_1 - y_1| = 0.000001$

$w_2 = 0.8293$

$$+ 0.2 \left( \left( 1 + \frac{0.2}{2} + \frac{0.2^2}{6} + \frac{0.2^3}{24} \right) (0.8293 - 0.2^2) 
- \left( 1 + \frac{0.2}{3} + \frac{0.2^2}{12} \right) (0.2(0.2)) + 1 + \frac{0.2}{2} - \frac{0.2^2}{6} - \frac{0.2^3}{24} \right)$$

$= 1.214091$

abs. error 4th order Taylor at $t_2$:  $|w_2 - y_2| = 0.000003$
Finding approximations at time other than $t_i$

**Example.** (Table 5.4 on Page 259). Assume the IVP $y' = y - t^2 + 1$, $0 \leq t \leq 2$, $y(0) = 0.5$ is solved by the 4th order Taylors method with time step size $h = 0$. $w_6 = 3.1799640$ ($t_6 = 1.2$), $w_7 = 3.7324321$ ($t_7 = 1.4$). Find $y(1.25)$.

**Solution:**

**Method 1:** use linear Lagrange interpolation.

$$y(1.25) \approx \frac{1.25-1.4}{1.2-1.4}w_6 + \frac{1.25-1.2}{1.4-1.2}w_7 = 3.3180810$$

**Method 2:** use Hermite polynomial interpolation (more accurate than the result by linear Lagrange interpolation).

First use $y' = y - t^2 + 1$ to approximate $y'(1.2)$ and $y'(1.4)$.

$$y'(1.2) = y(1.2) - (1.2)^2 + 1 \approx 3.1799640 - (1.2)^2 + 1 = 2.7399640$$

$$y'(1.4) = y(1.4) - (1.4)^2 + 1 \approx 3.7324321 - (1.4)^2 + 1 = 2.7724321$$

Then use **Theorem 3.9** to construct Hermite polynomial $H_3(x)$. 


\[ y(1.25) \approx H_3(1.25). \]

**Error analysis**

**Theorem 5.12** If Taylor method of order \( n \) is used to approximate the solution to the IVP

\[ y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \beta \]

with step size \( h \) and if \( y \in C^{n+1}[a, b] \), then the **local truncation error** is \( O(h^n) \).

**Remark:**

\[ y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} f'(t_i, y(t_i)) + \cdots + \frac{h^n}{n!} f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i)) \]

\[ \tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - T^{(n)}(t_i, y_i) = \frac{h^n}{(n + 1)!} f^{(n)}(\xi_i, y(\xi_i)). \]

Assume \( y^{(n+1)}(t) = f^{(n)}(t, y(t)) \) is bounded by \( |y^{(n+1)}(t)| \leq M \).

Thus \( |\tau_{i+1}(h)| \leq \frac{h^n}{(n+1)!} M. \)
So the local truncation error in Euler’s method is $O(h^n)$. 