5.4 Runge-Kutta Methods

Motivation: Obtain high-order accuracy of Taylor method without knowledge of derivatives of $f(t, y)$.

Theorem 5.13(Taylor's Theorem in Two Variables) Suppose $f(t, y)$ and partial derivative up to order $n + 1$ continuous on $D = \{(t, y) | a \leq t \leq b, c \leq y \leq d\}$, let $(t_0, y_0) \in D$. For $(t, y) \in D$, there is $\xi \in [t, t_0]$ and $\mu \in [y, y_0]$ with

$$f(t, y) = P_n(t, y) + R_n(t, y).$$

Here

$$P_n(t, y) = f(t_0, y_0) + \left[(t - t_0)\frac{\partial f}{\partial t}(t_0, y_0) + (y - y_0)\frac{\partial f}{\partial y}(t_0, y_0)\right]$$

$$+\left[\frac{(t-t_0)^2}{2}\frac{\partial^2 f}{\partial t^2}(t_0, y_0) + (t - t_0)(y - y_0)\frac{\partial^2 f}{\partial t \partial y}(t_0, y_0)\right.$$

$$\left. + \frac{(y-y_0)^2}{2}\frac{\partial^2 f}{\partial y^2}(t_0, y_0)\right] + \cdots +$$

$$\left[\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}(t - t_0)^{n-j}(y - y_0)^j \frac{\partial^n f}{\partial t^{n-j}\partial y^j}(t_0, y_0)\right]$$

$$R_n(t,y) = \frac{1}{(n+1)!} \sum_{j=0}^{n+1} \binom{n+1}{j} (t-t_0)^{n+1-j} (y-y_0)^j \frac{\partial^{n+1} f}{\partial t^{n+1-j} \partial y^j}(\xi, \mu).$$

$P_n(t,y)$ is the nth Taylor polynomial in two variables. $R_n(t,y)$ is the remainder term associated with $P_n(t,y)$.

$\xi \in (t, t_0), \quad \mu \in (y, y_0)$

Derivation of Runge-Kutta method of order two

Determine a_1, α_1, β_1 such that

$$a_1 f(t+\alpha_1, y+\beta_1) \approx f(t,y) + \frac{h}{2} f'(t,y) \text{ with } O(h^2) \text{ error.}$$

Notice $f'(t,y) = \frac{df(t,y(t))}{dt} = \frac{\partial f}{\partial t}\big(t, y(t)\big) + \frac{\partial f}{\partial y}\big(t, y(t)\big) \cdot y'(t) = \frac{\partial f}{\partial t}\big(t, y(t)\big) + \frac{\partial f}{\partial y}\big(t, y(t)\big) \cdot f\big(t, y(t)\big).$

Define:

$$T^{(2)}(t,y) = f(t,y) + \frac{h}{2}\frac{\partial f}{\partial t}\big(t, y(t)\big) + \frac{h}{2}\frac{\partial f}{\partial y}\big(t, y(t)\big) \cdot f\big(t, y(t)\big) \qquad (1)$$

1.Expand $a_1 f(t+\alpha_1, y+\beta_1)$ in 1st degree Taylor polynomial:

$$a_1 f(t+\alpha_1, y+\beta_1)$$
$$= a_1 f(t,y) + a_1\alpha_1 \frac{\partial f}{\partial t}(t,y) + a_1\beta_1 \frac{\partial f}{\partial y}(t,y)$$
$$+ a_1 R_1(t+\alpha_1, y+\beta_1) \qquad (2)$$

2.Match coefficients of equation (1) and (2) gives

$$a_1 = 1, \qquad a_1\alpha_1 = \frac{h}{2}, \qquad a_1\beta_1 = \frac{h}{2} f\big(t, y(t)\big)$$

with unique solution

$$a_1 = 1, \qquad \alpha_1 = \frac{h}{2}, \qquad \beta_1 = \frac{h}{2} f\big(t, y(t)\big)$$

3.This gives:

$$T^{(2)}(t,y) = f\left(t + \frac{h}{2}, y + \frac{h}{2} f\big(t, y(t)\big)\right)$$

$$-R_1\left(t + \frac{h}{2}, y + \frac{h}{2} f\big(t, y(t)\big)\right),$$

with $R_1\left(t + \frac{h}{2}, y + \frac{h}{2} f\big(t, y(t)\big)\right) = O(h^2)$

Since $y(t_{i+1}) = y(t_i) + hT^{(2)}\big(t_i, y(t_i)\big) + \frac{h^3}{3!} f^{(3)}\big(\xi_i, y(\xi_i)\big),$

$$y(t_{i+1})$$

$$= y(t_i) + hf\left(t_i + \frac{h}{2}, y_i + \frac{h}{2} f\big(t_i, y(t_i)\big)\right) + hO(h^2)$$

$$+ \frac{h^3}{3!} f^{(3)}\big(\xi_i, y(\xi_i)\big),$$

$$y(t_{i+1}) = y(t_i) + hf\left(t_i + \frac{h}{2}, y_i + \frac{h}{2} f\big(t_i, y(t_i)\big)\right) + O(h^3).$$

Midpoint Method (one of Runge-Kutta methods of order two)

Consider to solve the IVP

$$y' = f(t, y), \quad a \leq t \leq b, \; y(a) = \beta.$$

with step size $h = \frac{b-a}{N}$.

$$w_0 = \beta$$

$$w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right),$$

$$\text{for each } i = 0, 1, 2, \cdots, N-1.$$

Remark: Local truncation error of Midpoint method is $O(h^2)$.

Two stage formula of Runge-Kutta method of order two:

$$w_0 = \beta$$

$$\begin{cases} k_1 = f(t_i, w_i) \\ k_2 = f\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} k_1\right) \end{cases}$$

$$w_{i+1} = w_i + hk_2$$

for each $i = 0, 1, 2, \cdots, N - 1.$

Example 2. Use the Midpoint method with $N = 10, h = 0.2,$ $t_i = 0.2i$ and $w_0 = 0.5$ to solve the IVP

$$y' = y - t^2 + 1, \qquad 0 \leq t \leq 2, \quad y(0) = 0.5.$$

Modified Euler Method (Another Runge-Kutta method of order two)

Consider to solve the IVP

$y' = f(t, y),\ a \le t \le b,\ y(a) = \beta$. with step size $h = \frac{b-a}{N}$.

$$w_0 = \beta$$

$$w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f\left(t_{i+1}, w_i + hf(t_i, w_i)\right)\right]$$

$$\text{for each } i = 0, 1, 2, \cdots, N-1.$$

Local truncation error is $O(h^2)$.

Two stage formula of the Modified Euler Method:

$$w_0 = \beta$$

$$k_1 = f(t_i, w_i)$$

$$k_2 = f(t_{i+1}, w_i + hk_1)$$

$$w_{i+1} = w_i + \frac{h}{2}[k_1 + k_2]$$

$$\text{for each } i = 0, 1, 2, \cdots, N-1.$$

Example. Use the Modified Euler method with $N = 10, h = 0.2$, $t_i = 0.2i$ and $w_0 = 0.5$ to solve the IVP

$$y' = y - t^2 + 1, \qquad 0 \le t \le 2, \quad y(0) = 0.5.$$

Heun's Method (Runge-Kutta Method of order three)

Idea: Approximate $T^{(3)}(t, y)$ with $O(h^3)$ error by $f(t + \alpha_1, y + \delta_1 f(t + \alpha_2, y + \delta_2 f(t, y)))$

$$w_0 = \beta$$

$$w_{i+1} = w_i + \frac{h}{4}[f(t_i, w_i) + 3f(t_i + \frac{2h}{3}, w_i + \frac{2h}{3} f\left(t_i + \frac{h}{3}, w_i + \frac{h}{3} f(t_i, w_i)\right))]$$

for each $i = 0, 1, 2, \cdots, N - 1$.

Runge-Kutta Method of order four

$$w_0 = \beta$$
$$k_1 = hf(t_i, w_i)$$
$$k_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1\right)$$
$$k_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2\right)$$
$$k_4 = hf(t_{i+1}, w_i + k_3)$$

$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$
for each $i = 0, 1, 2, \cdots, N-1.$

Example 3. Use the Runge-Kutta method of order 4 with $N = 10, h = 0.2,\ t_i = 0.2i$ and $w_0 = 0.5$ to solve the IVP

$$y' = y - t^2 + 1, \qquad 0 \le t \le 2, \quad y(0) = 0.5.$$