ACMS40390: Numerical Analysis

Practice Exam 1

Note: You need to show the works to get credits.

1. Find $\max_{a\ll x\ll b}|f(x)|$ for the function $f(x) = e^x + 3x$, on the interval $x \in [0,1]$.

Solution: Use the Extreme Value Theorem.

$f'(x) = e^x + 3.$

Since $e^x + 3 = 0$ has not solution, we only check end points of the interval.

$\max_{a\ll x\ll b}|f(x)| = f(1) = e + 3$

2. (a) Find the second Taylor polynomial $P_2(x)$ for the function $f(x) = xe^x + x$, about $x_0 = 0$.
(b) Find a bound for the error $|P_2(x) - f(x)|$ in using $P_2(x)$ to approximate $f(x)$ on the interval [0, 1].

Solution: (a) $p(x) = (1 + 1)x + \frac{(2)}{2!}x^2$

(b) Use the remainder term:

$|p(x) - f(x)| = |\frac{(3e^\xi + \xi e^\xi)}{3!}x^3| \ll |\frac{(3e^\xi + \xi e^\xi)}{6}|$ since $x \in [0,1]$.

$\left|\frac{(3e^\xi + \xi e^\xi)}{6}\right| \ll \frac{1}{6}|3e^\xi| + \frac{1}{6}|\xi e^\xi| \ll \frac{1}{6}|3e| + \frac{1}{6}|(1)(e)|$ since $\xi \in (0,1)$.

3. Use the 64-bit long real format to find the decimal equivalent of the machine number:
0 00000000011 110000000000000…00000

Solution: $c = 2 + 1 = 3$

$f = 1\left(\frac{1}{2}\right) + 1(\frac{1}{2})^2$

$s = 0$

Decimal number = $2^{3-1023}(1 + f) =$. ..

4. Use the three-digit rounding arithmetic to perform the calculation for $(-10\pi + 3)$.

Solution:

$fl(\pi) = 0.314 \times 10^1,\ fl(10) = 0.100 \times 10^2, fl(3) = 0.300 \times 10^1$,

$$fl(\pi)fl(10) = 0.314 \times 10^2$$

$$fl\big(-fl(\pi)fl(10) + fl(3)\big) = fl(-0.314 \times 10^2 + 0.300 \times 10^1) = fl(-0.284 \times 10^2)$$
$$= -0.284 \times 10^2$$

5. Suppose we are asked to solve the quadratic equation $1.002x^2 - 1000.0x + 0.25 = 0$ by four digit-rounding arithmetic. Let $a = 1.002; b = -1000.0;$ and $c = 0.25$. We are given the following four formulas as

(a) x_1 by $\frac{(-b+\sqrt{b^2-4ac})}{2a}$; or by $\frac{-2c}{(b+\sqrt{b^2-4ac})}$

(b) x_2 by $\frac{(-b-\sqrt{b^2-4ac})}{2a}$; or by $\frac{-2c}{(b-\sqrt{b^2-4ac})}$

Find the most accurate formulas to compute x_1 and x_2 and explain why (Note: you do not need to actually compute x_1 and x_2).

Solution: (a) x_1 by $\frac{(-b+\sqrt{b^2-4ac})}{2a}$; (b) x_2 by $\frac{-2c}{(b-\sqrt{b^2-4ac})}$. No subtraction of nearly equal numbers.

6. Find the rate of convergence for the sequence $\lim_{n\to\infty}\left(sin\frac{1}{n}\right)^4 = 0$

Solution: Since $sin\frac{1}{n} \ll \frac{1}{n}\ as\ n \to \infty$, $\left(sin\frac{1}{n}\right)^4 \ll (\frac{1}{n})^4$.

The rate of convergence is $O((\frac{1}{n})^4)$.

7. Use the bisection method to find P_2 for $f(x) = \sqrt{x} - cosx$ on the interval $x \in [0,1]$ (Note: P_2 is the approximate solution at the second iteration).

   Solution:

8. If we use the fixed-point iteration to find an approximation to the fixed point that is accurate to within 10^{-2} for $g(x) = \pi + 0.5\sin(x)$ on the interval $x \in [0,\pi]$, how many iterations do we need?

Solution: Use Corollary 2.4.

$g'(x) = 0.5\cos(x)$.

Since $|g'^{(x)}| \ll 0.5 \equiv k$,

we obtain $|p_n - p| \ll k^n \max\{p_0 - 0, \pi - p_0\} \ll k^n\pi \ll 10^{-2}$

Now we solve $k^n\pi \ll 10^{-2}$ for n.

9. Let $f(x) = -x^3 - cosx$. Use Newton's method to find P_2 with $P_0 = -1$ (Note: P_2 is the approximate solution at the second iteration).

Solution:

10. (a) Use the appropriate Lagrange interpolating polynomial of degree two $P_2(x)$ to approximate $f(0.22)$ if $f(0.1) = 0.665$; $f(0.2) = 0.8$; $f(0.3) = 1.8$; and $f(0.4) = 0.25$.

   (c) Suppose we know that $|f^{(3)}(x)| < 5$ on the interval contains 0.1, 0.2, 0.3, and 0.4. What is the error $|f(0.22) - P_2(0.22)|$ made by the interpolation?

   (Note: You do not need to find the actual value of $f(0.22)$, instead, you only need to write down the interpolating polynomial).

   Solution: We use $f(0.1) = 0.665$; $f(0.2) = 0.8$; $f(0.3) = 1.8$.

$$L_{2,0}(x) = \frac{(x - 0.2)(x - 0.3)}{(0.1 - 0.2)(0.1 - 0.3)}$$

$$L_{2,1}(x) = \frac{(x - 0.1)(x - 0.3)}{(0.2 - 0.1)(0.2 - 0.3)}$$

$$L_{2,2}(x) = \frac{(x - 0.1)(x - 0.2)}{(0.3 - 0.1)(0.3 - 0.2)}$$

$$P_2(x) = L_{2,0}(x)f(0.1) + L_{2,1}(x)f(0.2) + L_{2,2}(x)f(0.3)$$

Use Theorem 3.3 to estimate the error.

$$|f(0.22) - P_2(0.22)| = |\frac{f^{(3)}(\xi(x))}{6}(0.22 - 0.1)(0.22 - 0.2)(0.22 - 0.3)|$$

$$\ll 5|(0.22 - 0.1)(0.22 - 0.2)(0.22 - 0.3)|$$