ACMS40390: Numerical Analysis

Solution key for Practice Exam 2

Note: You need to show the works to get credits.

1. (a) Use proper two-point difference formula to determine the missing entry in the following table.

x	$f(x)$	$f'(x)$
0.5	1.9	
0.6		2.2

Soln:

(1) Use backward difference formula for $f'(0.6) = (f(0.6) - f(0.5))/(0.6 - 0.5)$ to get $f(0.6) = 2.12$.

(2) Once we have $f(0.6) = 2.12$, use forward difference formula to get $f'(0.5) = (f(0.6) - f(0.5))/(0.6 - 0.5)$

(Question: why $f'(0.6) = f'(0.5)$ numerically?)

(b) Use the most accurate three-point formula to determine $f'(0.6)$ based on the following table. Use the central difference formula to determine $f''(0.6)$.

x	$f(x)$	
0.5	1.9	
0.6	2.3	
0.7	1.5	

Soln: Use three-point central difference formula

$$f'(0.6) = (1.5 - 1.9)/(2.0 \times 0.1)$$

2. (a) Use the most accurate three-point formula to determine $f'(1.2)$ based on the values given in the table

x	$f(x)$	$f'(x)$
1.2	1.64024	
1.3	1.70470	
1.4	1.71277	

(b) If $f(x) = x sin(x) + x^2 \cos(x)$, compute the actual error in (a) and find the proper error bound.

Soln: (a) Use forward difference formula

$$f'(1.2) = \frac{1}{2\times 0.1}[-3f(1.2) + 4f(1.3) - f(1.4)]$$

(b) $f'(x) = sin(x) + 3x cos(x) - x^2 \sin(x)$

$f''(x) = 4cos(x) - 5x sin(x) - x^2 \cos(x)$

$f'''(x) = -9sin(x) - 7x cos(x) + x^2 \sin(x)$

Error bound $= \frac{0.01}{3}|f'''(\xi)| = \frac{0.01}{3}|-9sin(\xi) - 7\xi cos(\xi) + \xi^2 \sin(\xi)|$ for $\xi \in (1.2, 1.4)$.

Error bound $\leq \frac{0.01}{3}(9|sin(\xi)| + 7\xi|cos(\xi)| + \xi^2|\sin(\xi)|)$

$\leq \frac{0.01}{3}(9 + 7 \times 1.4 + 1.4^2)$

3. Find the degree of accuracy of the quadrature formula $\int_0^a f(x)dx = \frac{a}{2}f(0) + \frac{a}{2}f(a)$. Here $a > 0$.

Soln: k=0, $f(x) = 1$, $\int_0^a f(x)dx = \int_0^a 1dx = a$, and $\frac{a}{2}f(0) + \frac{a}{2}f(a) = a$

K=1, $f(x) = x$, $\int_0^a f(x)dx = \int_0^a xdx = \frac{a^2}{2}$, and $\frac{a}{2}f(0) + \frac{a}{2}f(a) = \frac{a}{2}(0) + \frac{a}{2}(a) = \frac{a^2}{2}$

K= 2, $f(x) = x^2$, $\int_0^a f(x)dx = \int_0^a x^2dx = \frac{a^3}{3}$

and $\frac{a}{2}f(0) + \frac{a}{2}f(a) = \frac{a}{2}(0) + \frac{a}{2}(a^2) = \frac{a^3}{2}$.

The degree of accuracy is 1.

4. Approximate the following integral using the Simpson's rule.
$\int_{-0.5}^{1.5} xln(x+2)dx.$

Soln: h=(1.5-(-0.5))/2.0=1.0. Let's define $f(x) = xln(x+2)$.

$$\int_{-0.5}^{1.5} xln(x+2)dx \approx \frac{1.0}{3}[f(-0.5) + 4f(0.5) + f(1.4)]$$

5. Find an error bound for $\int_{-0.5}^{1.5} xln(x+2)dx$ approximated by the Simpson's rule.
Soln: The error is given by $\frac{h^5}{90}f^{(4)}(\xi)$, where $\xi \in (-0.5, 1.5)$ (see Problem 4 for h and f(x)).

$$f^{(4)}(x) = \frac{2}{(x+2)^3} + \frac{12}{(x+2)^4}$$

$$\frac{h^5}{90}f^{(4)}(\xi) \leq \frac{h^5}{90}[\frac{2}{(1.5+2)^3} + \frac{12}{(1.5+2)^4}]$$

6. Use the composite Trapezoidal rule with $n = 3$ to compute $\int_0^3 \frac{2}{x^2+4}dx$
Soln: Let's define $f(x) = \frac{2}{x^2+4}$.
h = (3-0)/3=1.

$$\int_0^3 \frac{2}{x^2+4}dx \approx \frac{h}{2}[f(0) + 2(f(1.0) + f(2.0) + f(3.0)]$$

7. Determine the values of n and h required to approximate $\int_0^2 \frac{2}{x\ \ +4} dx$ to within 10^{-5} by using composite Simpson's rule.

Soln: The error of the composite Simpson's rule $\frac{(2-0)h^4}{180} f^{(4)}(\xi) \leq 10^{-5}$. Here $\xi \in (0,2)$. $f(x) = \frac{2}{x\ \ +4}$

Use the approach similar to the one for solving Problem 5 to get the upper bound for $\frac{(2-0)h^4}{180} f^{(4)}(\xi)$. This upper bound $\leq 10^{-5}$ to solve for n and h (remember h =(2-0)/ n).

8. (a) Following table lists roots $x_i, i = 1, \dots, n$ of the n^{th} Legendre polynomial Solution and

$$c_i = \int_{-1}^{1} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j} dx$$

n	x_i	c_i
2	0.5773502692	1.0
	-0.5773502692	1.0
3	0.7745966692	0.55555556
	0.0	0.88888889
	-0.7745966692	0.55555556

Use Gaussian quadrature with $n = 3$ to compute $\int_0^{\pi} \frac{2}{\cos(x)\ \ +4} dx$.

(b) What's the degree of precision of a Gaussian quadrature with $n = 4$?

Soln: (a) $\int_0^{\pi} \frac{2}{\cos(x)\ \ +4} dx = (\pi/2) \int_{-1}^{1} \frac{2}{\cos(\frac{\pi t+\pi}{2})\ \ +4} dt$ by change variable.

$$\left(\frac{\pi}{2}\right)\int_{-1}^{1}\frac{2}{\cos\left(\frac{\pi t+\pi}{2}\right)+4}dt$$

$$=\left(\frac{\pi}{2}\right)\left(0.55555556\times\frac{2}{\cos\left(\frac{\pi\times 0.7745966692+\pi}{2}\right)+4}+0.88888889\times\frac{2}{\cos\left(\frac{\pi\times 0.0+\pi}{2}\right)+4}+0.55555556\times\frac{2}{\cos\left(\frac{-\pi\times 0.7745966692+\pi}{2}\right)+4}\right)$$

(b) By Theorem 4.7, degree of accuracy is 2n-1 = 7.

9. Show the following initial value problem has a unique solution.

$$y'=\frac{ty+y}{ty+2},\qquad 2\le t\le 4,\qquad y(2)=5$$

Soln: By Theorem 5.4, we need to show $f(t,y)=\frac{ty+y}{ty+2}$ satisfies Lipschitz condition in y.

$\frac{\partial f}{\partial y}=\frac{t+2}{(ty+2)^2}$. However $f(t,y)$ **does not** satisfy Lipschitz condition on $D=\{(t,y)|2\le t\le 4,-\infty<y<\infty\}$.

(To solve this problem, we need to solve for the solution analytically, which is beyond the requirement of this class).

10. Use Euler's method to approximate the solution for the following initial value problem.

$$y' = \frac{e^t + y}{ty + 20}, \qquad 1 \le t \le 1.75, \qquad y(1) = 2, \qquad with \;\; h = 0.25$$

Soln: $w_0 = 2$

$w_{i+1} = w_i + h(\frac{e^{t_i} + w_i}{t_i w_i + 20})$, $t_i = 1.0 + ih$ for $i = 0,1,2.$

11. Derive the difference equation of Taylor's method of order three to solve the following initial value problem. $y' = y - t^2 + 1, \;\; 0 \le t \le 2, \; y(0) = 0.5$

Soln: Let's define $f(t, y) = y - t^2 + 1$

$$f'(t, y) = y - t^2 - 2t + 1$$
$$f''(t, y) = y - t^2 - 2t - 1$$
$$T^{(3)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \frac{h^2}{6} f''(t_i, w_i)$$
$$T^{(3)}(t_i, w_i) = w_i - t_i{}^2 + 1 + \frac{h}{2}(w_i - t_i{}^2 - 2t_i + 1) + \frac{h^2}{6}(w_i - t_i{}^2 - 2t_i - 1)$$

Taylor method of order three: $w_0 = 0.5$

$w_{i+1} = w_i + hT^{(3)}(t_i, w_i)$, $t_i = ih$ for $i = 0,1, \dots N.$