Numerical Analysis

Second Exam

Fall 2006

Name:

To receive credit you must show your work.

Problems

1. (10 points) Consider the linear system $\begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ Can it be solved by the Jacobi iterative method?

2. (15 points) Write a computer program to use SOR with $w = 1.25$ to solve $\begin{bmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ with $||x^{(k+1)} - x^{(k)}||_\infty < 10^{-5}$, $x = (x_1, x_2, x_3)^T$.

3. (10 points) Let $l_k(x) = \prod_{j=0, j\neq k}^{n} \frac{x - x_j}{x_k - x_j}$ $(k = 0, ..., n)$ be Lagrange polynomials on $x_0, ..., x_n$ distinct nodes. Prove that $\sum_{k=0}^{n} l_k(x) \equiv 1$.

4. (15 points) Let $l_i(x)$ be lagrange polynomials and $f[x_0, x_1, ..., x_i]$ be divided difference of $f \in C^n$ on $x_0, ..., x_n$ distinct nodes. Prove that $\sum_{i=0}^{n} f(x_i) l_i(x) = \sum_{i=0}^{n} f[x_0, x_1, ..., x_i] \prod_{j=0}^{i-1} (x - x_j)$, and $f[x_0, x_1, ..., x_n] = \sum_{i=0}^{n} f(x_i) \prod_{j=0, j\neq i}^{n} (x_i - x_j)^{-1}$.

5. (10 points) Prove the Parseval identity : $< f, g >= \sum_{i=1}^{n} < f, u_i >< g, u_i >$ which is valid if f and g are in the span of the orthonormal set $[u_1, u_2, ..., u_n]$.

6. (10 points) Let $p(x) = \sum_{j=0}^{n} a_j \psi_j(x)$ be the best approximation of $f(x) \in C[a, b]$ in the subspace $H = Span\{\psi_0, \psi_1, ..., \psi_n\}$. Prove $< f - p, f - p >=< f, f > - \sum_{j=0}^{n} a_j < \psi_j, f >$

7. (15 points) If $[u_1, ..., u_n]$ is an orthonormal system in an i. p. s. E, then a family of projection operators, P_n is defined by the equation $P_n f = \sum_{i=1}^{n} < f, u_i > u_i$. Prove that:

a. $P_n(f + g) = P_n f + P_n g$. ($P_n$ is linear.)

b. $P_n^2 = P_n$. (P_n is a projection.)

c. $< P_n f, g >=< f, P_n g >$. ($P_n$ is self-adjoint.)

8. (15 points) Find the coefficients a, b, c such that $\int_0^\pi (\sin(x) - a - bx - cx^2)^2 dx$ attains its

minimum.

Bonus:

9. (10 points) Let A be an $n \times n$ Hermitian matrix. The spectral radius of any square matrix A is defined by $\rho(A) = \max_s |\lambda_s(A)|$, where $\lambda_s(A)$ denotes the s^{th} eigenvalue of A. Prove that $||A||_2 = \sqrt{\rho(A^*A)}$.

10. (10 points) Write a computer program to compute the cubic spline function S that interpolates the function $\sin(x)$ at 21 equally spaced nodes x_i $i = 0, ..., 20$ in ther interval $[0, 2\pi]$, with $x_0 = 0$ and $x_20 = 2\pi$. Use the first boundary condition at the end points.