Numerical Analysis

Second Exam Solution

Fall 2006

Problem 1

Soln:

$$G = \begin{pmatrix} 0 & 0.5 & -0.5 \\ -1 & 0 & -1 \\ 0.5 & 0.5 & 0 \end{pmatrix}. \text{ Solve } 0 = \det(\lambda I - G) = \begin{pmatrix} \lambda & -0.5 & 0.5 \\ 1 & \lambda & 1 \\ -0.5 & -0.5 & \lambda \end{pmatrix} = \lambda(\lambda^2 + 1.25),$$

$\lambda_1 = 0, \lambda_2 = \sqrt{1.25}i, \lambda_3 = -\sqrt{1.25}i$.

According to Theorem 4.6.5 (necessary and sufficient conditions), $\rho(G) = \sqrt{1.25} > 1$, the Jacobian iterative method does not converge.

Problem 2

Soln:

$x = (0.3, 1.6, 1.4)^T$.

Problem 3

Soln:

Let $f(x_i) = 1$, $i = 0, 1, ..., n$.

$p(x) = \sum l_k(x) f(x_k) - 1$ is a polynomial of degree n. It has at most n zeros. Since $p(x_i) = 0$, $i = 0, 1, ..., n$, it has $n+1$ zeros. Then $p(x) \equiv 0$. $\Rightarrow \sum_{k=0}^{n} l_k(x) \equiv 1$.

Problem 4

Soln:

Denote $p_L(x) = \sum_{i=0}^{n} f(x_i) l_i(x)$, and $p_N(x) = \sum_{i=0}^{n} f[x_0, x_1, ..., x_i] \prod_{j=0}^{i-1}(x - x_j)$. $p_L(x) = p_N(x)$ by uniqueness (Theorem 6.1.1).

$f[x_0, x_1, ..., x_n]$ and $\sum_{i=0}^{n} f(x_i) \prod_{j=0, j\neq i}^{n} (x_i - x_j)^{-1}$ are coefficients of x^n of $p_N(x)$ and $p_L(x)$ respectively. Since $p_L(x) = p_N(x)$, $f[x_0, x_1, ..., x_n] = \sum_{i=0}^{n} f(x_i) \prod_{j=0, j\neq i}^{n} (x_i - x_j)^{-1}$.

Problem 5

Soln:

Let $f = \sum_i a_i u_i$, $a_i =< f, u_i >$ and $g = \sum_j b_j u_j$, $b_j =< g, u_j >$. Then $< f, g >=<$

$\sum_i a_i u_i, \sum_j b_j u_j >= \sum_i \sum_j a_i b_j < u_i, u_j >= \sum_i a_i b_i = \sum_i < f, u_i >< g, u_j >$.

Problem 6

Soln:

$< f - p, f - p >=< f - p, f > - < f - p, p >=< f, f > - < p, f >$, since $< f - p, p >= 0$ ($p(x)$ is the best approximation).

Therefore, $< f - p, f - p >=< f, f > - < p, f >=< f, f > - < \sum_{j=0}^{n} a_j \psi_j, f >=< f, f > - \sum_{j=0}^{n} a_j < \psi_j, f >$.

Problem 7

Soln:

a. $P_n(f + g) = \sum_i < f + g, u_i > u_i = \sum_i < f, u_i > u_i + \sum_i < g, u_i > u_i = P_n f + P_n g$.

b. $P_n^2 f = \sum_j < \sum_i < f, u_i > u_i, u_j > u_j = \sum_j < f, u_j > u_j = P_n f$.

c. $< P_n f, g >=< \sum_i < f, u_i > u_i, g >= \sum_i < f, u_i >< u_i, g >= \sum_i < f, u_i < u_i, g >>=< f, \sum_i u_i < u_i, g >>=< f, P_n g >$.

Problem 8

Soln:

By the least-square approximation, $a = 12(\pi^2 - 10)/\pi^3, b = -60(\pi^2 - 12)/\pi^4, c = 60(\pi^2 - 12)/\pi^5$.

Problem 9

Soln:

A^*A is also Hermitian, and it has a complete set of n orthonormal eigenvectors, $u_1, ..., u_n$, such that $u_i^*, u_j = \delta_{i,j}$, $A^*Au_s = \lambda_s u_s$. The eigenvalues are real.

$\Rightarrow \lambda_s = u_s^* A^* A u_s \geq 0$.

Pick $y = \sum \alpha_s u_s$, $||y||_2 = 1$, such that $||A||_2 = ||Ay||_2$. We also have $||y||_2^2 = \sum |\alpha_s|^2 = 1$.

$||A||_2^2 = \sum_t \bar{\alpha}_t u_t^* A^* A \sum_s \alpha_s u_s = \sum_t \bar{\alpha}_t u_t^* \sum_s \alpha_s \lambda_s u_s = \sum_s \lambda_s |\alpha_s|^2 \leq \max_s \lambda_s \sum_t |\alpha_t|^2 = \max_s \lambda_s = \rho(A^*A)$.

Thus $\sqrt{\rho(A^*A)}$ is an upper bound of $||A||_2$. Let $y = u_s$, where $\lambda_s = \rho(A^*A)$, we get $||Au_s||_2 = (u_s A^* A u_s)^{1/2} = \rho(A^*A)$. So $||A||_2 = \sqrt{\rho(A^*A)}$.