Problem 1

Soll: \[ G = \begin{pmatrix} 0 & 0.5 & -0.5 \\ -1 & 0 & -1 \\ 0.5 & 0.5 & 0 \end{pmatrix} \]

Solve \( 0 = \det(\lambda I - G) = \lambda(\lambda^2 + 1.25) \), \( \lambda_1 = 0, \lambda_2 = \sqrt{1.25}i, \lambda_3 = -\sqrt{1.25}i \).

According to Theorem 4.6.5 (necessary and sufficient conditions), \( \rho(G) = \sqrt{1.25} > 1 \), the Jacobian iterative method does not converge.

Problem 2

Soll:
\[ x = (0.3, 1.6, 1.4)^T. \]

Problem 3

Soll: Let \( f(x_i) = 1, i = 0, 1, \ldots, n \).

\[ p(x) = \sum l_k(x)f(x_k) - 1 \] is a polynomial of degree \( n \). It has at most \( n \) zeros. Since \( p(x_i) = 0, i = 0, 1, \ldots, n \), it has \( n + 1 \) zeros. Then \( p(x) \equiv 0 \). \( \Rightarrow \sum_{k=0}^{n} l_k(x) \equiv 1. \)

Problem 4

Soll: Denote \( p_L(x) = \sum_{i=0}^{n} f(x_i)l_i(x) \), and \( p_N(x) = \sum_{i=0}^{n} f[x_0,x_1,...,x_i]\prod_{j=0}^{i-1}(x-x_j). \) \( p_L(x) = p_N(x) \)

by uniqueness (Theorem 6.1.1).

\[ f[x_0,x_1,...,x_n] \] and \( \sum_{i=0}^{n} f(x_i)\prod_{j=0,j\neq i}^{n}(x_i-x_j)^{-1} \) are coefficients of \( x^n \) of \( p_N(x) \) and \( p_L(x) \) respectively. Since \( p_L(x) = p_N(x) \), \( f[x_0,x_1,...,x_n] = \sum_{i=0}^{n} f(x_i)\prod_{j=0,j\neq i}^{n}(x_i-x_j)^{-1} \).

Problem 5

Soll: Let \( f = \sum a_iu_i, a_i = \langle f,u_i \rangle \) and \( g = \sum b_ju_j, b_j = \langle g,u_j \rangle \). Then \( \langle f,g \rangle = \)
\[ \sum_i a_i u_i, \sum_j b_j u_j = \sum_i \sum_j a_i b_j < u_i, u_j > = \sum_i a_i b_i = \sum_i < f, u_i > < g, u_j >. \]

**Problem 6**

Sln:
\[ < f - p, f - p > = < f - p, f > - < f - p, p > = < f, f > - < p, f >, \] since \( < f - p, p > = 0 \) (\( p(x) \) is the best approximation).

Therefore, \( < f - p, f - p > = < f, f > - < \sum_{j=0}^n a_j \psi_j, f > = < f, f > - \sum_{j=0}^n a_j < \psi_j, f >. \)

**Problem 7**

Sln:

a. \( P_n(f + g) = \sum_i < f + g, u_i > u_i = \sum_i < f, u_i > + \sum_i < g, u_i > u_i = P_n f + P_n g. \)

b. \( P_n^2 f = \sum_j < f, u_j > u_j = \sum_j < f, u_j > u_j = P_n f. \)

c. \( < P_n f, g >= < \sum_i < f, u_i > u_i, g >= \sum_i < f, u_i > < u_i, g >= \sum_i < f, u_i < u_i, g >= < f, \sum_i u_i < u_i, g > >= < f, P_n g >. \)

**Problem 8**

Sln:

By the least-square approximation, \( a = 12(\pi^2 - 10)/\pi^3, b = -60(\pi^2 - 12)/\pi^4, c = 60(\pi^2 - 12)/\pi^5. \)

**Problem 9**

Sln:

\( A^* A \) is also Hermitian, and it has a complete set of \( n \) orthonormal eigenvectors, \( u_1, ..., u_n, \) such that \( u_i^* u_j = \delta_{i,j}, A^* A u_s = \lambda_s u_s. \) The eigenvalues are real.

\[ \Rightarrow \lambda_s = u_s^* A^* A u_s \geq 0. \]

Pick \( y = \sum \alpha_s u_s, ||y||_2 = 1, \) such that \( ||A||_2 = ||Ay||_2. \) We also have \( ||y||_2^2 = \sum |\alpha_s|^2 = 1. \)

\[ ||A||_2^2 = \sum_i \sum_j \alpha_i^* u_i^* A \sum_s \alpha_s u_s = \sum_i \sum_j \alpha_i^* u_i^* \sum_s \alpha_s \lambda_s u_s = \sum_s \lambda_s |\alpha_s|^2 \leq \max_s \lambda_s \sum_i |\alpha_i|^2 = \max_s \lambda_s = \rho(A^* A). \]

Thus \( \sqrt{\rho(A^* A)} \) is an upper bound of \( ||A||_2. \) Let \( y = u_s, \) where \( \lambda_s = \rho(A^* A), \) we get \( ||A u_s||_2 = (u_s A^* A u_s)^{1/2} = \rho(A^* A). \) So \( ||A||_2 = \sqrt{\rho(A^* A)}. \)