Supplementary material 2

Theorem 4.6.7.1 The necessary condition that the sequence generated by SOR method converges to the solution of $Ax = b$ is that $0 < w < 2$.

Proof:

$\det(1/wD + L)^{-1} = w^n \prod_{i=1}^n 1/a_{ii}$; $\det((1 - 1/w)D + U) = (1 - 1/w)^n \prod_{i=1}^n a_{ii}$. $\det(G_s) = (-1)^n \det(1/wD + L)^{-1} \det((1 - 1/w)D + U) = (1 - w)^n$.

Let the eigenvalues of matrix A be $\lambda_1, ..., \lambda_n$. $\lambda_1 \cdot ... \cdot \lambda_n = (1 - w)^n$, and $\lambda_1 + ... + \lambda_n =$ Trace(A). Trace$(A) = (a_{11} + ... + a_{nn})$.

Therefore $\rho(G_s) \geq |1 - w|$. In order to converge, $\rho(G_s) < 1$. So $|1 - w| < 1$, $0 < w < 2$.

Theorem 4.6.7.2 If matrix A is diagonally dominant, SOR converges if $0 < w \leq 1$.

Proof:

Assume SOR does not converge. Then there exists at least one eigenvalue $|\mu| \geq 1$. $\det(\mu I - G_s) = 0$.

$\mu I - G_s = \mu I + (1/wD + L)^{-1}[(1 - 1/w)D + U] = \mu(1/wD + L)^{-1}[(1/\mu(1 - 1/w) + 1/w)D + L + 1/\mu U]$. Denote $B = (1/\mu(1 - 1/w) + 1/w)D + L + 1/\mu U$.

$\det(\mu I - G_s) = \mu^n \det(1/wD + L)^{-1} \det B$.

$\mu \neq 0$, and $\det(1/wD + L)^{-1} \neq 0$.

Let $B = (b_{ij})$. $b_{ii} = (1/\mu(1 - 1/w) + 1/w)a_{ii}$. When $i > j$, $b_{ij} = a_{ij}$; when $i < j$, $b_{ij} = 1/\mu a_{ij}$. Since $|\mu| \geq 1, 0 < w \leq 1$, $1/\mu(1 - 1/w) + 1/w = 1 + (1 - 1/\mu)(1/w - 1) \geq 1$. $|b_{ii}| \geq |a_{ii}|$, and $a_{ij} \geq |b_{ij}|, i \neq j$. Because A is diagonally dominant, B is diagonally dominant as well, and $\det B \neq 0$. Therefore, $\det(\mu I - G_s) \neq 0$. It is a contradiction.