2.1

1. $$\int_0^1 \int_0^1 \sqrt{x+y}\, dx\, dy = \int_0^1 \left[ \frac{2}{3}(x+y)^{\frac{3}{2}} \Big|_{x=0}^{x=1} \right] dy =$$

$$= \frac{2}{3}\int_0^1 \left[(1+y)^{\frac{3}{2}} - y^{\frac{3}{2}}\right] dy = \frac{2}{3} \cdot \left( \frac{2}{5} \cdot (1+y)^{\frac{5}{2}} - \frac{2}{5} y^{\frac{5}{2}} \right) \Bigg|_{y=0}^{y=1}$$

$$= \frac{4}{15}\left(2^{\frac{5}{2}} - 1 - 1 + 0\right) = \frac{4}{15}\left(2^{\frac{5}{2}} - 2\right)$$

---

2. $$\nabla f(1,2) = \left(f_x(1,2),\ f_y(1,2)\right) = \pi(2,1)$$

$f_x(x,y) = \pi y \cos \pi x y \qquad f_x(1,2) = 2\pi$

$f_y(x,y) = \pi x \cos \pi x y \qquad f_y(1,2) = \pi$

---

3. Distance to the origin function

$$f(x,y) = \sqrt{x^2+y^2}$$

It's easier to find the point(s) that realise the minimum for

$$g(x,y) = x^2+y^2$$

and plug these points in $f$.

2.1

3. For $g(x,y)=x^2+y^2$ with constraint

$h(x,y)=x^2+xy+y^2=6$

use lagrange multipliers.

$$\nabla g = \lambda \nabla h \Rightarrow (2x, 2y) = \lambda(2x+y, 2y+x)$$

$$\begin{cases} 2x = \lambda(2x+y) & (1) \\ 2y = \lambda(2y+x) & (2) \end{cases}$$

If $\lambda=0$ then $x=y=0$ which doesn't satisfy $x^2+xy+y^2=6$

So $\lambda \neq 0$ and dividing (1) by (2) we get:

$$\frac{x}{y} = \frac{2x+y}{2y+x} \Rightarrow 2xy+x^2 = 2xy+y^2 \Rightarrow x^2=y^2$$

$\Rightarrow x = \pm y$. Since $x^2+xy+y^2=6$ one has two possibilities

I $x=y \Rightarrow 3x^2=6 \Rightarrow x^2=2 \Rightarrow x=y=\pm\sqrt{2}$

which plugged into $f$ gives $\sqrt{4}=2$.

II $x=-y \Rightarrow x^2=6 \Rightarrow x=\pm\sqrt{6} \Rightarrow \begin{cases} x=\sqrt{6} \quad y=-\sqrt{6} \\ \text{or} \\ x=-\sqrt{6} \quad y=\sqrt{6} \end{cases}$

Plug this values into $f$ and you get $2\sqrt{3}$.

So the answer is 2.

2.1

4. $x^2 - y^2 - \ln(x+z) = 0$

The eq of the tangent plane at $(1,1,0)$ is given by $\nabla F(1,1,0) \cdot (x-1, y-1, z-0) = 0$

$F_x(1,1,0) = 1 \quad F_y(1,1,0) = -2 \quad F_z(1,1,0) = -1.$

So the eq is

$$x - 2y - z + 1 = 0$$

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5. The average value of $f$ is.

$$f_{ave} = \frac{1}{A(R)} \iint_R f(x,y)\, dA$$

where $R$ is the region between the circles. and $A(R)$ is its area

$$A(R) = \underbrace{4\pi}_{\text{area of the bigger circle}} - \underbrace{\pi}_{\text{area of the smaller circle}} = 3\pi$$

$$\iint_R f(x,y)\, dA \underset{\text{in polar coordinates}}{=} \int_0^{2\pi} \int_1^2 \frac{1}{\rho} \cdot \rho \, d\rho\, d\theta =$$

$= \int_0^{2\pi} \int_1^2 1 \, d\rho\, d\theta = 2\pi.$ So $f_{ave} = \frac{2\pi}{3\pi} = \frac{2}{3}$

$\underline{2.1}$

6. $V_{cylinder} = \frac{\pi d^2}{4} \cdot h$ d is the diameter
h is the height

$$V'(t) = \frac{\pi \cdot 2d'(t)\, d(t)}{4} h(t) + \frac{\pi d^2(t)}{4} \cdot h'(t)$$

When $d(t) = 2$ $d'(t) = -0.2$

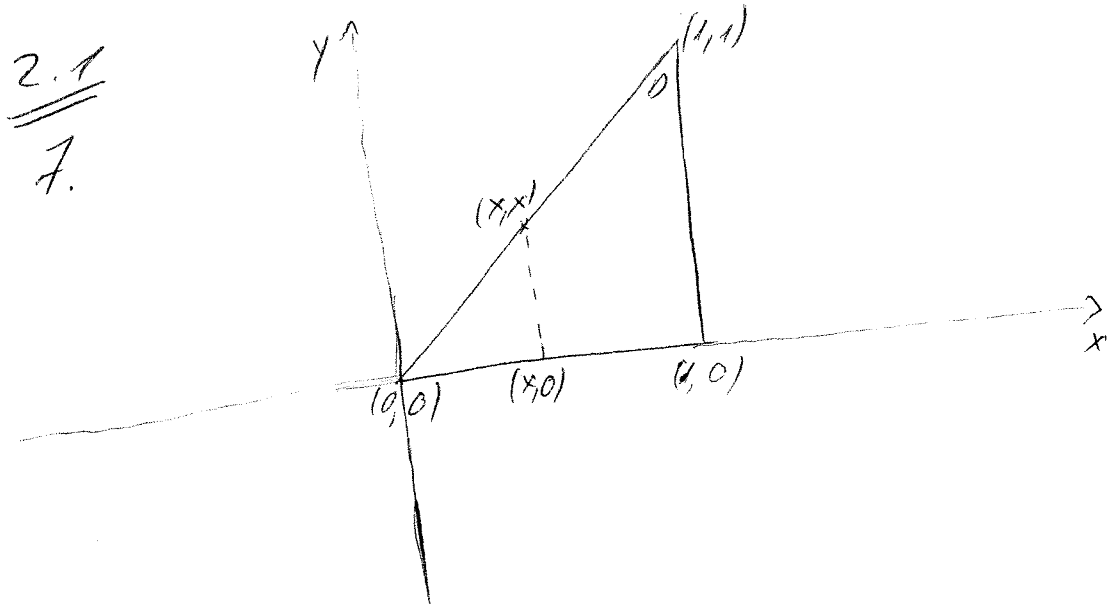
$h(t) = 4$ $h'(t) = -0.3$

~~I~~ we get

$$V'(t) = \pi(-0.8 - 0.3) = -1.1\pi.$$

2.1

7.




The eq of the line that goes thru (0,0) and (1,1) is $y=x$. So in a vertical section $y$ is going to vary between 0 and $x$.

$$\iint_D x^2y^3\,dA = \int_0^1\int_0^x x^2y^3\,dy\,dx = \int_0^1\left[x^2\frac{y^4}{4}\Big|_{y=0}^{y=x}\right]dx$$

$$= \int_0^1 \frac{x^6}{4}\,dx = \frac{1}{4}\,\frac{x^7}{7}\Big|_{x=0}^{x=1} = \frac{1}{28}$$

---

8. The maximum rate of change is given by

$$|\nabla f| = \sqrt{f_x^2+f_y^2+f_z^2}$$

$f_x = -(1-z)e^{-x}$ $\qquad f_x(0,1,0) = -1.$

$f_y = 2y$ $\qquad f_y(0,1,0) = 2$

$f_z = -e^{-x}$ $\qquad f_z(0,1,0) = -1$

So $|\nabla f(0,1,0)| = \sqrt{6}$

$\frac{2.1}{9}$ We need to solve the equation

~~$f(x,y,z) = x(y+z)$~~

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

where $g = xyz \qquad h = x^2+z^2$

together with $xyz=1 \qquad x^2+z^2=4$.

$$\nabla f = (y+z,\ x,\ x)$$
$$\nabla g = (yz,\ xz,\ xy)$$
$$\nabla h = (2x,\ 0,\ 2z)$$

So the system of equations that needs to be solved is:

$$y+z = \lambda yz + 2\mu x$$
$$x = \lambda xz \cancel{+ 2\mu xy}$$
$$x = \lambda xy + 2\mu z$$
$$xyz = 1$$
$$x^2+z^2 = 4$$

2.1

10. $f_x = 3x^2 - 3y^2$

$f_y = -6xy + 6$

$f_x(1,1) = f_y(1,1) = 0$ so $(1,1)$ is a critical point

$f_{xx} = 6x \quad f_{xy} = -6y \quad f_{yy} = -6x$

So $f_{xx}(1,1) = 6 \quad f_{xy}(1,1) = -6 \quad f_{yy}(1,1) = -6$

$f_{xx} f_{yy} - f_{xy}^2 = -36 - 36 = -72$

So $(1,1)$ is a saddle point.

---

11. The critical points of $f$

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 8(2x-1)y = 0 \\ 8(x^2 - x) + 6y = 0 \end{cases}$$

From the first eq. one gets that either $y=0$ or $x = \frac{1}{2}$. Now $y=0$ gives points on the boundary so it doesn't count.

$x = \frac{1}{2}$ plugged into the second eq. gives $y = \frac{1}{3}$

$f\left(\frac{1}{2}, \frac{1}{3}\right) = 8 \cdot \left(-\frac{1}{4}\right) \cdot \frac{1}{3} + 3 \cdot \frac{1}{9} = -\frac{2}{3} + \frac{1}{3} = -$

Now we have to check the values on the boundary.

2.1

0.11 The boundary of the rectangle $\begin{matrix} 0 \le x \le 1 \\ 0 \le y \le 1 \end{matrix}$ has

4 pieces

I $x = 0$ $\Rightarrow$ $f(0,y) = 3y^2$ whose min in the interval $[0,1]$ is $0$ and max is $f(0,1) = 3$.

II $y = 0$ $\Rightarrow$ $f(x,0) = 0$

III $x = 1$ $\Rightarrow$ $f(1,y) = 3y^2$ again with a min $= 0$ and a max $= 3$

IV $y = 1$ $\Rightarrow$ $f(x,1) = 8(x^2 - x) + 3$ and we need to find min & max for $g(x) = 8(x^2 - x) + 3$ when $x$ varies between $0$ and $1$.

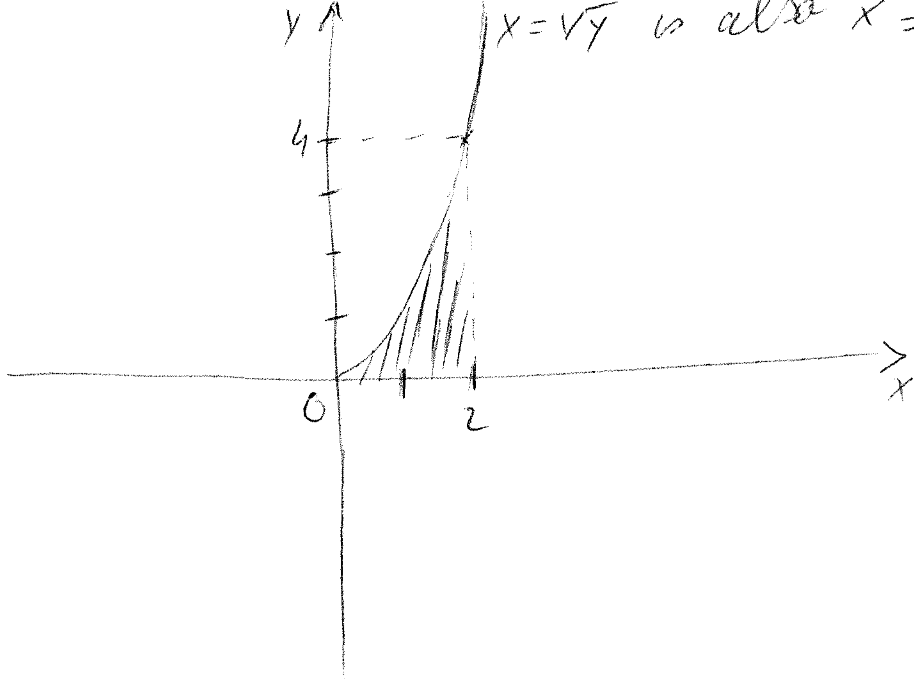
$g'(x) = 8(2x - 1)$ $\quad$ $g'(x) = 0$ gives $8(2x-1) = 0$ and so $x = \frac{1}{2}$.

$g\left(\frac{1}{2}\right) = -\frac{2}{[illegible]} + 3 = 1$.

$g(0) = g(1) = 3$

We conclude that the min is $-\frac{1}{3}$ and the max is $3$.

2.1 / 12




Notice that $x$ varies between $\sqrt{y}$ and $2$ on a horizontal section. But $x=\sqrt{y}$ is the same as $x^2=y$ so the region of integration is the region in the picture.

$$\int_0^4 \int_{\sqrt{y}}^2 \frac{4y^3+1}{x^6+1}\,dx\,dy = \int_0^2 \int_0^{x^2} \frac{4y^3+1}{x^6+1}\,dy\,dx$$

$$= \int_0^2 \left( \frac{1}{x^6+1} \int_0^{x^2} (4y^3+1)\,dy \right) dx = \int_0^2 \left[ \frac{1}{x^6+1} \cdot \left( (y^4+y) \Big|_{y=0}^{y=x^2} \right) \right]$$

$$= \int_0^2 \left( \frac{1}{x^6+1} \cdot (x^8+x^2) \right) dx = \int_0^2 \frac{1}{x^6+1}\, x^2(x^6+1)\,dx$$

$$= \int_0^2 x^2\,dx = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3}$$