**Contiune on 16.7 Triple Integrals**

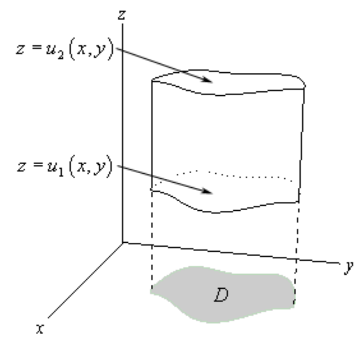


Figure 1:

$$\iiint_E f(x,y,z)dV = \iint_D \left[ \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz \right] dA$$

**Applications of Triple Integrals** Let $E$ be a solid region with a density function $\rho(x,y,z)$.

**Volume:** $V(E) = \iiint_E 1dV$

**Mass:** $m = \iiint_E \rho(x,y,z)dV$

**Moments about the coordinate planes:**

$$M_{xy} = \iiint_E z\rho(x,y,z)dV$$

$$M_{xz} = \iiint_E y\rho(x,y,z)dV$$

$$M_{yz} = \iiint_E x\rho(x,y,z)dV$$

**Center of mass:** $(\bar{x}, \bar{y}, \bar{z})$

$$\bar{x} = M_{yz}/m \quad , \quad \bar{y} = M_{xz}/m \quad , \quad \bar{z} = M_{xy}/m \quad .$$

**Remark:** The center of mass is just the weighted average of the coordinate functions over the solid region. If $\rho(x,y,z) = 1$, the mass of the solid equals its volume and the center of mass is also called the **centroid** of the solid.

**Example** Find the volume of the solid region $E$ between $y = 4 - x^2 - z^2$ and $y = x^2 + z^2$.

*Soln:* $E$ is described by $x^2 + z^2 \le y \le 4 - x^2 - z^2$ over a disk $D$ in the xz-plane whose radius is given by the intersection of the two surfaces: $y = 4 - x^2 - z^2$ and $y = x^2 + z^2$.

$4 - x^2 - z^2 = x^2 + z^2 \Rightarrow x^2 + z^2 = 2$. So the radius is $\sqrt{2}$.

Therefore

$$V(E) = \iiint_E 1dV = \iint_D \left[\int_{x^2+z^2}^{4-x^2-z^2} 1dy\right] dA = \iint_D 4 - 2(x^2+z^2)dA$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} (4-2r^2)rdrd\theta = \int_0^{2\pi} \left[2r^2 - \frac{1}{2}r^4\right]_0^{\sqrt{2}} = 4\pi$$

**Example** Find the mass of the solid region bounded by the sheet $z = 1 - x^2$ and the planes $z = 0, y = -1, y = 1$ with a density function $\rho(x,y,z) = z(y+2)$.

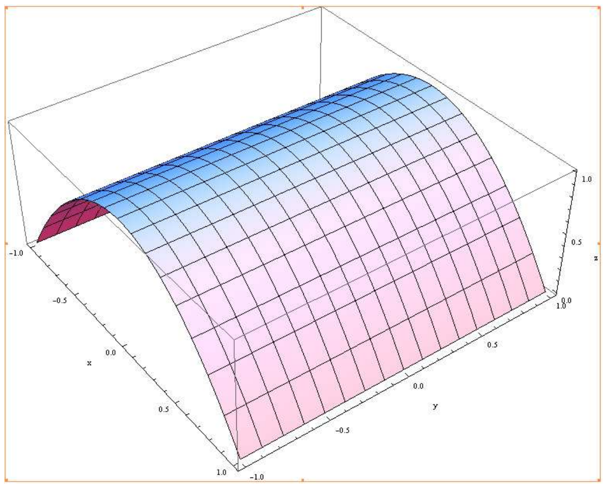

Figure 2:

*Soln:* The top surface of the solid is $z = 1 - x^2$ and the bottom surface is $z = 0$ over the region $D$ in the xy-plane which is bounded by the other equations in the xy-plane and the intersection of the top and bottom surfaces.

The intersection gives $1 - x^2 = 0 \Rightarrow x = \pm 1$. Therefore $D$ is a square $[-1,1] \times [-1,1]$.

$$m = \iiint_E \rho(x,y,z)dV = \iiint_E z(y+2)dV = \iint_D \left[\int_0^{1-x^2} z(y+2)dz\right] dA$$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_0^{1-x^2} z(y+2)dzdxdy = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}(1-x^2)^2(y+2)dxdy =$$

$$\frac{8}{15}\int_{-1}^{1}(y+2)dy = 32/15$$

**Example** Find the centroid of the solid above the paraboloid $z = x^2 + y^2$ and below the plane $z = 4$.

*Soln:* The top surface of the solid is $z = 4$ and the bottom surface is $z = x^2 + y^2$ over the region $D$ defined in the xy-plane by the intersection of the top and bottom surfaces.

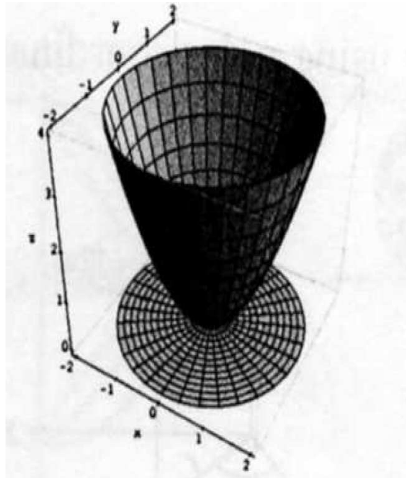

Figure 3:

The intersection gives $4 = x^2 + y^2$. Therefore $D$ is a disk of radius 2.

By the symmetry principle, $\bar{x} = \bar{y} = 0$. We only compute $\bar{z}$:

$$m = \iiint_E 1dV = \iint_D \left[\int_{x^2+y^2}^{4} 1dz\right] dA = \iint_D 4 - (x^2+y^2)dA = \int_0^{2\pi}\int_0^2 (4-r^2)rdrd\theta = 8\pi$$

$$M_{xy} = \iiint_E zdV = \iint_D \left[\int_{x^2+y^2}^{4} zdz\right] dA = \iint_D 8 - \frac{1}{2}(x^2+y^2)^2 dA =$$

$$\int_0^{2\pi}\int_0^2 (8 - \frac{1}{2}r^4)rdrd\theta = \int_0^{2\pi} [4r^2 - \frac{1}{12}r^6]_0^2 d\theta = 64\pi/3.$$

Therefore $\bar{z} = M_{xy}/m = 8/3$ and the centroid is $(0, 0, 8/3)$.

**16.8 Triple Integrals in Cylindrical and Spherical Coordinates**

**1. Triple Integrals in Cylindrical Coordinates**

A point in space can be located by using polar coordinates $r, \theta$ in the xy-plane and $z$ in the vertical direction.

Some equations in cylindrical coordinates (plug in $x = r\cos(\theta), y = r\sin(\theta)$):

Cylinder: $x^2 + y^2 = a^2 \Rightarrow r^2 = a^2 \Rightarrow r = a$;
Sphere: $x^2 + y^2 + z^2 = a^2 \Rightarrow r^2 + z^2 = a^2$;
Cone: $z^2 = a^2(x^2 + y^2) \Rightarrow z = ar$;
Paraboloid: $z = a(x^2 + y^2) \Rightarrow z = ar^2$.

**The formula for triple integration in cylindrical coordinates:**

If a solid $E$ is the region between $z = u_2(x, y)$ and $z = u_1(x, y)$ over a domain $D$ in the xy-plane, which is described in polar coordinates by $\alpha \le \theta \le \beta$, $h_1(\theta) \le r \le h_2(\theta)$, we plug

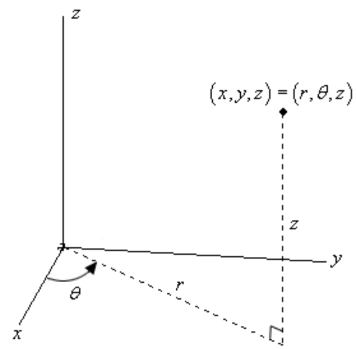


Figure 4:

in $x = r\cos(\theta), y = r\sin(\theta)$

$$\iiint_E f(x,y,z)dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz\right] dA =$$
$$\int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r dz dr d\theta$$

**Note:** $dV \to r dz dr d\theta$

**Example** Evaluate $\iiint_E zdV$ where $E$ is the portion of the solid sphere $x^2+y^2+z^2 \leq 9$ that is inside the cylinder $x^2+y^2 = 1$ and above the cone $x^2+y^2 = z^2$.

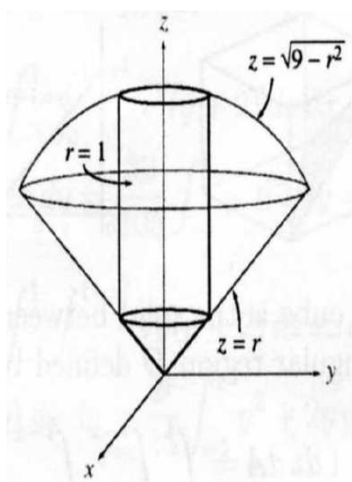


Figure 5:

*Soln:* The top surface is $z = u_2(x,y) = \sqrt{9-x^2-y^2} = \sqrt{9-r^2}$ and the bottom surface is $z = u_1(x,y) = \sqrt{x^2+y^2} = r$ over the region $D$ defined by the intersection of the top (or bottom) and the cylinder which is a disk $x^2+y^2 \le 1$ or $0 \le r \le 1$ in the xy-plane.

$$\iiint_E z dV = \iint_D \left[\int_r^{\sqrt{9-r^2}} z dz\right] dA = \int_0^{2\pi}\int_0^1\int_r^{\sqrt{9-r^2}} z r dz dr d\theta =$$

$$\int_0^{2\pi}\int_0^1 \frac{1}{2}[9-2r^2] r dr d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{2}[9r-2r^3] dr d\theta = \int_0^{2\pi}[9/4-1/4]d\theta = 4\pi$$

**Example** Find the volume of the portion of the sphere $x^2+y^2+z^2=4$ inside the cylinder $(y-1)^2+x^2=1$.

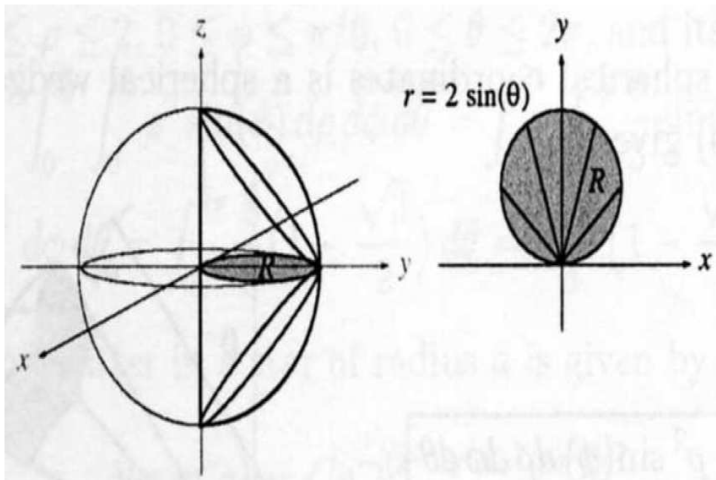


Figure 6:

*Soln:* The top surface is $z=\sqrt{4-x^2-y^2}=\sqrt{4-r^2}$ and the bottom is $z=-\sqrt{4-x^2-y^2}=-\sqrt{4-r^2}$ over the region $D$ defined by the cylinder equation in the xy-plane. So rewrite the cylinder equation $x^2+(y-1)^2=1$ as $x^2+y^2-2y+1=1 \Rightarrow r^2=2r\sin(\theta) \Rightarrow r=2\sin(\theta)$.

$$V(E)=\iiint_E 1 dV = \iint_D \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} 1 dz dA = \int_0^{\pi}\int_0^{2\sin(\theta)}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} 1 r dz dr d\theta =$$

$$\int_0^{\pi}\int_0^{2\sin(\theta)} 2r\sqrt{4-r^2} dr d\theta \text{ (by substitution u} = 4-\text{r}^2) =$$

$$\int_0^{\pi} -\frac{2}{3}[(4-4\sin^2(\theta))^{3/2}-(4)^{3/2}]d\theta \text{ (use identity } 1=\cos^2(\theta)+\sin^2(\theta)) =$$

$$\int_0^{\pi}\frac{16}{3}[1-|\cos(\theta)|^3]d\theta = \int_0^{\pi/2}\frac{16}{3}[1-\cos^3(\theta)]d\theta + \int_{\pi/2}^{\pi}\frac{16}{3}[1+\cos^3(\theta)]d\theta =$$

$$\int_0^{\pi/2}\frac{16}{3}[1-(1-\sin^2\theta)\cos\theta]d\theta + \int_{\pi/2}^{\pi}\frac{16}{3}[1+(1-\sin^2\theta)\cos\theta]d\theta =$$

$$16/3[(\theta-\sin\theta+\sin^3\theta/3)|_0^{\pi/2} + (\theta+\sin\theta-\sin^3\theta/3)|_{\pi/2}^{\pi}] = 16\pi/3-64/9$$

**2. Triple Integrals in Spherical Coordinates**

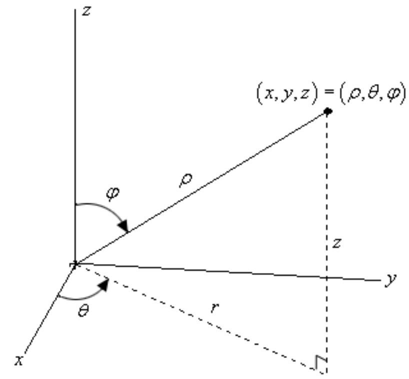


Figure 7:

In spherical coordinates, a point is located in space by longitude, latitude, and radial distance.

Longitude: $0 \leq \theta \leq 2\pi$;
Latitude: $0 \leq \phi \leq \pi$;
Radial distance: $\rho = \sqrt{x^2 + y^2 + z^2}$.
From $r = \rho \sin(\phi)$

$$x = r\cos(\theta) = \rho \sin(\phi)\cos(\theta)$$

$$y = r\sin(\theta) = \rho \sin(\phi)\sin(\theta)$$

$$z = \rho \cos(\phi)$$

Some equations in spherical coordinates:

Sphere: $x^2 + y^2 + z^2 = a^2 \Rightarrow \rho = a$
Cone: $z^2 = a^2(x^2 + y^2) \Rightarrow \cos^2(\phi) = a^2 \sin^2(\phi)$
Cylinder: $x^2 + y^2 = a^2 \Rightarrow r = a$ or $\rho \sin(\phi) = a$

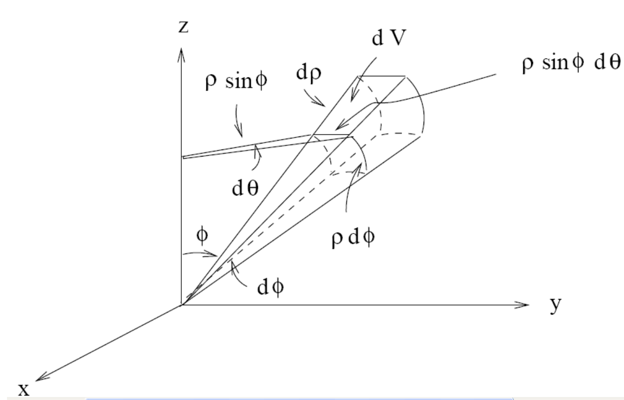


Figure 8: Spherical wedge element

The volume element in spherical coordinates is a spherical wedge with sides $d\rho, \rho d\phi, r d\theta$. Replacing $r$ with $\rho \sin(\phi)$ gives:

$$dV = \rho^2 \sin(\phi) d\rho d\phi d\theta$$

For our integrals we are going to restrict $E$ down to a spherical wedge. This will mean $a \leq \rho \leq b$, $\alpha \leq \theta \leq \beta$, $c \leq \phi \leq d$,

$$\iiint_E f(x,y,z)dV = \int_\alpha^\beta \int_c^d \int_a^b f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\rho^2 \sin(\phi) d\rho d\phi d\theta$$

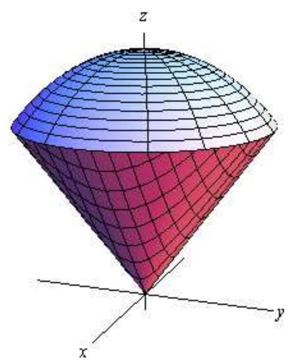


Figure 9: One example of the sphere wedge, the lower limit for both $\rho$ and $\phi$ are 0

**The more general formula for triple integration in spherical coordinates:**
If a solid $E$ is the region between $g_1(\theta,\phi) \leq \rho \leq g_2(\theta,\phi)$, $\alpha \leq \theta \leq \beta$, $c \leq \phi \leq d$, then

$$\iiint_E f(x,y,z)dV = \int_\alpha^\beta \int_c^d \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\rho^2 \sin(\phi) d\rho d\phi d\theta$$

**Example** Find the volume of the solid region above the cone $z^2 = 3(x^2+y^2)$ $(z \geq 0)$ and below the sphere $x^2+y^2+z^2 = 4$.

*Soln:* The sphere $x^2+y^2+z^2 = 4$ in spherical coordinates is $\rho = 2$. The cone $z^2 = 3(x^2+y^2)$ $(z \geq 0)$ in spherical coordinates is $z = \sqrt{3(x^2+y^2)} = \sqrt{3}r \Rightarrow \rho\cos(\phi) = \sqrt{3}\rho\sin(\phi) \Rightarrow \tan(\phi) = 1/\sqrt{3} \Rightarrow \phi = \pi/6$.

Thus $E$ is defined by $0 \leq \rho \leq 2$, $0 \leq \phi \leq \pi/6$, $0 \leq \theta \leq 2\pi$.

$$V(E) = \iiint_E 1dV = \int_0^{2\pi} \int_0^{\pi/6} \int_0^2 \rho^2 \sin(\phi) d\rho d\phi d\theta =$$

$$\int_0^{2\pi} \int_0^{\pi/6} \frac{8}{3} \sin(\phi) d\phi d\theta = \frac{16\pi}{3}(1 - \frac{\sqrt{3}}{2})$$

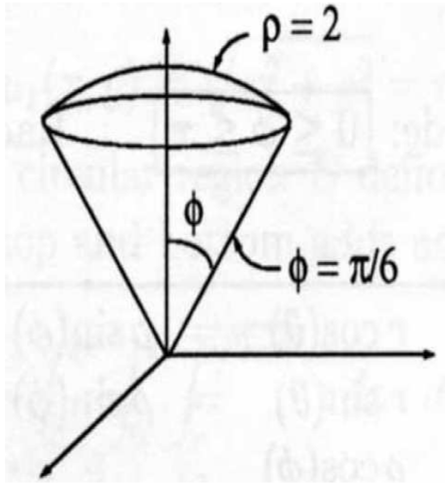


Figure 10:

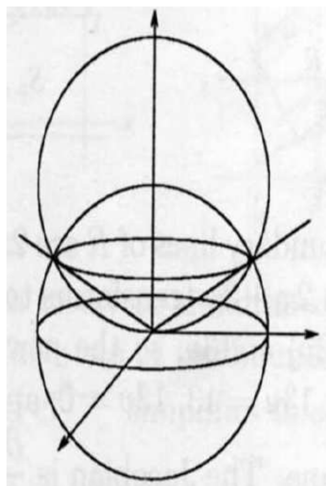

Figure 11:

**Example** Find the centroid of the solid region $E$ lying inside the sphere $x^2+y^2+z^2 = 2z$ and outside the sphere $x^2 + y^2 + z^2 = 1$ *Soln:* By the symmetry principle, the centroid lies on the z axis. Thus we only need to compute $\bar{z}$

The top surface is $x^2 + y^2 + z^2 = 2z \Rightarrow \rho^2 = 2\rho\cos(\phi)$ or $\rho = 2\cos(\phi)$. The bottom surface is $x^2 + y^2 + z^2 = 1 \Rightarrow \rho = 1$. They intersect at $2\cos(\phi) = 1 \Rightarrow \phi = \pi/3$.

$$m = \iiint_E 1dV = \int_0^{2\pi}\int_0^{\pi/3}\int_1^{2\cos(\phi)} \rho^2\sin(\phi)d\rho d\phi d\theta =$$

$$\int_0^{2\pi}\int_0^{\pi/3}\frac{8}{3}\cos^3(\phi)\sin(\phi)d\phi d\theta - \int_0^{2\pi}\int_0^{\pi/3}\frac{1}{3}\sin(\phi)d\phi d\theta = \frac{11\pi}{12}$$

$$\bar{z} = M_{xy}/m = \frac{12}{11\pi}\iiint_E z dV = \frac{12}{11\pi}\int_0^{2\pi}\int_0^{\pi/3}\int_1^{2\cos(\phi)} \rho\cos(\phi)\rho^2\sin(\phi)d\rho d\phi d\theta =$$

$$\frac{12}{11\pi}\left[\int_0^{2\pi}\int_0^{\pi/3} 4\cos^5(\phi)\sin(\phi)d\phi d\theta - \int_0^{2\pi}\int_0^{\pi/3} 1/4\cos(\phi)\sin(\phi)d\phi d\theta\right] =$$

$$\frac{12}{11\pi}\left[\frac{-4}{6}\cos^6(\phi)|_0^{\pi/3} - \frac{1}{4}\frac{\sin^2(\phi)}{2}|_0^{\pi/3}\right]$$

$$\frac{12}{11\pi}[9\pi/8] \simeq 1.2$$

**Example** Convert $\int_0^3\int_0^{\sqrt{9-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} x^2+y^2+z^2 dzdxdy$ into spherical coordinates.
*Soln:* We first write down the limits of the variables:

$$0 \le y \le 3$$

$$0 \le x \le \sqrt{9-y^2}$$

$$\sqrt{x^2+y^2} \le z \le \sqrt{18-x^2-y^2}$$

.

The range for $x$ tells us that we have a portion of the right half of a disk of radius 3 centered at the origin. Since $y \ge 0$, we will have the quarter disk in the first quadrant. Therefore since $D$ is in the first quadrant the region, $E$, must be in the first octant and this in turn tells us that we have the following range for $\theta$

$$0 \le \theta \le \pi/2$$

.

Now, lets see what the range for $z$ tells us. The lower bound, $z = \sqrt{x^2+y^2}$ is the upper half of a cone $z^2 = x^2+y^2$. The upper bound, $z = \sqrt{18-x^2-y^2}$ is the upper half of the sphere $x^2+y^2+z^2 = 18$. So the range for $\rho$

$$0 \le \rho \le \sqrt{18}$$

Now we try to find the range for $\phi$. We can get it from the equation of the cone. In spherical coordinates, the equation of the cone is $1 = \tan(\phi)$, which gives $\phi = \pi/4$. We have the range for $\phi$

$$0 \le \phi \le \pi/4$$

.

Thus

$$\int_0^3\int_0^{\sqrt{9-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} x^2+y^2+z^2 dzdxdy = \int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\sqrt{18}} \rho^4\sin(\phi)d\rho d\theta d\phi$$