16.9 Change of Variables in Multiple Integrals

Recall: For single variable, we change variables $x$ to $u$ in an integral by the formula (substitution rule)

$$\int_a^b f(x)dx = \int_c^d f(x(u))\frac{dx}{du}du$$

where $x = x(u)$, $dx = \frac{dx}{du}du$, and the interval changes from $[a, b]$ to $[c, d] = [x^{-1}(a), x^{-1}(b)]$.

Why do we do change of variables?
1. We get a simpler integrand.
2. In addition to converting the integrand into something simpler it will often also transform the region into one that is much easier to deal with.

notation: We call the equations that define the change of variables a transformation.

Example Determine the new region that we get by applying the given transformation to the region $R$.

(a) $R$ is the ellipse $x^2 + \frac{y^2}{36} = 1$ and the transformation is $x = \frac{u}{2}$, $y = 3v$.
(b) $R$ is the region bounded by $y = -x + 4$, $y = x + 1$, and $y = x/3 - 4/3$ and the transformation is $x = \frac{1}{2}(u + v)$, $y = \frac{1}{2}(u - v)$

Soln:
(a) Plug the transformation into the equation for the ellipse.

$$\left(\frac{u}{2}\right)^2 + \frac{(3v)^2}{36} = 1$$

$$\frac{u^2}{4} + \frac{9v^2}{36} = 1$$

$$u^2 + v^2 = 4$$

After the transformation we had a disk of radius 2 in the $uv$-plane.

(b)
Plugging in the transformation gives:

$$y = -x + 4 \Rightarrow \frac{1}{2}(u - v) = -\frac{1}{2}(u + v) \Rightarrow u = 4$$

$$y = x + 1 \Rightarrow \frac{1}{2}(u - v) = \frac{1}{2}(u + v) + 1 \Rightarrow v = -1$$

$$y = x/3 - 4/3 \Rightarrow \frac{1}{2}(u - v) = \frac{1}{3}\frac{1}{2}(u + v) - 4/3 \Rightarrow v = \frac{u}{2} + 2$$

See Fig. 1 and Fig. 2 for the original and the transformed region.

Note: We can not always expect to transform a specific type of region (a triangle for example) into the same kind of region.

Definition
The Jacobian of the transformation $x = g(u, v), y = h(u, v)$ is:

$$\frac{\partial (x, y)}{\partial (u, v)} = \det \left[ \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right] = g_u h_v - g_v h_u$$

Change of Variables for a Double Integral

Assume we want to integrate $f(x, y)$ over the region $R$ in the $xy$-plane. Under the transformation $x = g(u, v), y = h(u, v)$, $S$ is the region $R$ transformed into the $uv$-plane, and the integral becomes

$$\int \int_R f(x, y) dA = \int \int_S f(g(u, v), h(u, v)) \left| \frac{\partial (x, y)}{\partial (u, v)} \right| dudv$$

**Note:**

1. The $dudv$ on the right side of the above formula is just an indication that the right side integral is an integral in terms of $u$ and $v$ variables. The real order of integration depends on the set-up of the problem.

2. If we look just at the differentials in the above formula we can also say that

$$dA = \left| \frac{\partial (x, y)}{\partial (x, y)} \right| dudv$$
3. Here we take the absolute value of the Jacobian. The one dimensional formula is just the derivative \( \frac{dx}{du} \).

**Example** Show that when changing to polar coordinates we have \( dA = r\,dr\,d\theta \)

**Soln:**
The transformation here is \( x = r \cos(\theta), \ y = r \sin(\theta) \).

\[
\begin{vmatrix}
\frac{\partial(x, y)}{\partial(r, \theta)}
\end{vmatrix}
= \begin{vmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{vmatrix}
= \begin{vmatrix}
\cos(\theta) & -r \sin(\theta) \\
\sin(\theta) & r \cos(\theta)
\end{vmatrix}
= r(\cos^2(\theta) + \sin^2(\theta)) = r
\]

So we have \( dA = \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| \, dr\,d\theta = |r|dr\,d\theta = r\,dr\,d\theta. \)

**Example** Evaluate \( \iint_R x + y\,dA \) where \( R \) is the trapezoidal region with vertices given by \( (0, 0), \ (5, 0), \ (5/2, 5/2) \) and \( (5/2, -5/2) \) using the transformation \( x = 2u + 3v \) and \( y = 2u - 3v \)

**Soln:**

![Figure 3:](image)

Plugging in the transformation gives:

\[
\begin{align*}
y &= x \Rightarrow v = 0 \\
y &= -x \Rightarrow u = 0 \\
y &= -x + 5 \Rightarrow u = 5/4 \\
y &= x - 5 \Rightarrow v = 5/6
\end{align*}
\]

Therefore the region \( S \) in \( uv \)-plane is then a rectangle whose sides are given \( u = 0, \ v = 0, \ u = 5/4 \) and \( v = 5/6 \).
The Jacobian
\[
\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{bmatrix} 2 & 3 \\ 2 & -3 \end{bmatrix} = -6 - 6 = -12
\]

\[
\iint_R x + y dA = \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} [(2u + 3v) + (2u - 3v)] | - 12 | dudv = \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} 48ududv = \int_0^{\frac{\pi}{4}} 24u^2|_0^{\frac{\pi}{4}} dv = \int_0^{\frac{\pi}{4}} 75/2dv = 125/4
\]

**Example** Compute \( \iint_R y^2 dA \) where \( R \) is the region bounded by \( xy = 1, \ xy = 2, \ xy^2 = 1 \) and \( xy^2 = 2 \)

**Soln:**

![Figure 4](image)

The curves intersect in 4 points:

\[
1 = xy = xy^2 \Rightarrow (1, 1)
\]

\[
1 = xy = xy^2/2 \Rightarrow (1/2, 2)
\]

\[
2 = xy = xy^2 \Rightarrow (2, 1)
\]

\[
1 = xy/2 = xy^2 \Rightarrow (4, 1/2)
\]

We choose a transformation \( u = xy \) and \( v = xy^2 \) to transform \( R \) into a new region \( S \) by \( 1 \leq u \leq 2 \) and \( 1 \leq v \leq 2 \).

Now we solve for \( x \) and \( y \) to compute the Jacobian:

\[
u^2/v = x, \quad v/u = y
\]
\[ \frac{\partial(x, y)}{\partial(u, v)} = \det \begin{bmatrix} 2u/v & -u^2/v^2 \\ -v/u^2 & 1/u \end{bmatrix} = 1/v \]

\[ \iint_R y^2 dA = \int_1^2 \int_1^2 \frac{v^2}{u^2} \cdot \frac{1}{v} dudv = \left[-\frac{1}{u}\right]^2_1 \cdot \left[\frac{1}{2v^2}\right]^2_1 = 3/4 \]

Note: In \( \int_1^2 \int_1^2 \frac{v^2}{u^2} \cdot \frac{1}{v} dudv \), we dropped the absolute value sign for Jacobian \( \frac{1}{v} \), since \( \frac{1}{v} \) is positive in the region we were integrating over.

**Example** \( \iint_R y^2 dA \) where \( R \) is the region in the first quadrant bounded by \( x^2 - y^2 = 1 \), \( x^2 - y^2 = 4 \), \( y = 0 \) and \( y = (3/5)x \).

**Soln:**

![Figure 5](image)

We choose new variable to transform \( R \) into a simpler region. Let \( u = x^2 - y^2 = (x - y)(x + y) \). Then two of the boundary curves for the new region \( S \) are \( u = 1 \) and \( u = 4 \). The integrand \( e^{x^2-y^2} \) is also simplified to \( e^u \).

We choose \( v \) so that we could easily solve for \( x \) and \( y \). Let \( v = x + y \), then \( u/v = x - y \).

\[ v + u/v = 2x \quad \text{and} \quad v - u/v = 2y \]

The boundaries \( y = 0 \) and \( y = 3/5x \) becomes:

\[ y = 0 \Rightarrow v - u/v = 0 \Rightarrow u = v^2 \]

\[ y = 3/5x \Rightarrow v - u/v = (3/5)(v + u/v) \Rightarrow u = (1/4)v^2 \]

The Jacobian is

\[ \frac{\partial(x, y)}{\partial(u, v)} = \det \begin{bmatrix} (1/2)v & -(1 - u/v^2)/2 \\ -(1/2)v & (1 + u/v^2)/2 \end{bmatrix} = (1 + u/v^2)/(4v) + (1 - u/v^2)/(4v) = \frac{1}{2v} \]
Figure 6:

\[
\int \int_{R} e^{x^2-y^2} \, dA = \int \int_{S} e^{u} \frac{1}{2v} \, dA' = \int_{1}^{4} \int_{\sqrt{u}}^{2\sqrt{u}} e^{u} \frac{1}{2v} \, dv \, du = \\
\int_{1}^{4} \frac{e^{u}}{2} \ln(2\sqrt{u}) - \ln(\sqrt{u}) \, du = \int_{1}^{4} \frac{e^{u}}{2} \ln(2) \, du = \frac{\ln(2)}{2} (e^{4} - e)
\]

Note: In \(\int \int_{S} e^{\frac{u}{2v}} \, dA'\), we dropped the absolute value sign for Jacobian \(\frac{1}{2v}\), since \(\frac{1}{2v}\) is positive in the region we were integrating over.

**Triple Integrals**

We start with a region \(R\) and use the transformation \(x = g(u, v, w), \ y = h(u, v, w), \ z = k(u, v, w)\), and to transform the region \(R\) into the new region \(S\).

The Jacobian is:

\[
\frac{\partial (x, y, z)}{\partial (u, v, w)} = \begin{vmatrix}
    x_u & x_v & x_w \\
    y_u & y_v & y_w \\
    z_u & z_v & z_w
\end{vmatrix}
\]

The integral under this transformation is:

\[
\int \int \int_{R} f(x, y, z) \, dV = \int \int \int_{S} f(g(u, v, w), h(u, v, w), k(u, v, w)) \left| \frac{\partial (x, y, z)}{\partial (u, v, w)} \right| du \, dv \, dw
\]

**Note:**
1. \(du \, dv \, dw\) on the right hand side of the above formula is just an indication that the right hand side integral is an integral in terms of \(u, v \) and \(w\) variables. The real order of integration depends on the set-up of the problem.
2. As with double integrals,

\[
dV = \left| \frac{\partial (x, y, z)}{\partial (u, v, w)} \right| du \, dv \, dw
\]
Example If $x = \rho \sin(\phi) \cos(\theta)$, $y = \rho \sin(\phi) \sin(\theta)$, and $z = \rho \cos(\phi)$, then $\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho^2 \sin(\phi)$.

Example Find the volume $V$ of the solid ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$

Soln: We choose new variables $u = x/a$, $v = y/b$, $w = z/c$ and transform the ellipsoid into a sphere $F$: $u^2 + v^2 + w^2 \leq 1$.

The Jacobian is:

$$
\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{bmatrix}
a & 0 & 0 \\
0 & b & 0 \\
0 & 0 & c \\
\end{bmatrix} = abc
$$

$$
V = \iiint_E 1\,dV = \iiint_F \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dV' = \iiint_F abc dV' = abc \frac{4}{3} \pi (1)^3 = \frac{4}{3} \pi abc
$$