16.9 Change of Variables in Multiple Integrals

Recall: For sinlge variable, we change variables x to u in an integral by the formula (substitution rule)

$$\int_a^b f(x)dx = \int_c^d f(x(u))\frac{dx}{du}du$$

where $x = x(u)$, $dx = \frac{dx}{du}du$, and the interval changes from $[a, b]$ to $[c, d] = [x^{-1}(a), x^{-1}(b)]$.

Why do we do change of variables?

1. We get a simpler integrand.

2. In addition to converting the integrand into something simpler it will often also transform the region into one that is much easier to deal with.

notation: We call the equations that define the change of variables a **transformation**.

Example Determine the new region that we get by applying the given transformation to the region R.

(a) R is the ellipse $x^2 + \frac{y^2}{36} = 1$ and the transformation is $x = \frac{u}{2}$, $y = 3v$.

(b) R is the region bounded by $y = -x + 4$, $y = x + 1$, and $y = x/3 - 4/3$ and the transformation is $x = \frac{1}{2}(u + v)$, $y = \frac{1}{2}(u - v)$

Soln:

(a) Plug the transformation into the equation for the ellipse.

$$(\frac{u}{2})^2 + \frac{(3v)^2}{36} = 1$$
$$\frac{u^2}{4} + \frac{9v^2}{36} = 1$$
$$u^2 + v^2 = 4$$

After the transformation we had a disk of radius 2 in the uv-plane.

(b)

Plugging in the transformation gives:

$$y = -x + 4 \Rightarrow \frac{1}{2}(u - v) = -\frac{1}{2}(u + v) \Rightarrow u = 4$$

$$y = x + 1 \Rightarrow \frac{1}{2}(u - v) = \frac{1}{2}(u + v) + 1 \Rightarrow v = -1$$

$$y = x/3 - 4/3 \Rightarrow \frac{1}{2}(u - v) = \frac{1}{3}\frac{1}{2}(u + v) - 4/3 \Rightarrow v = \frac{u}{2} + 2$$

See Fig. 1 and Fig. 2 for the original and the transformed region.

Note: We can not always expect to transform a specific type of region (a triangle for example) into the same kind of region.

Definition

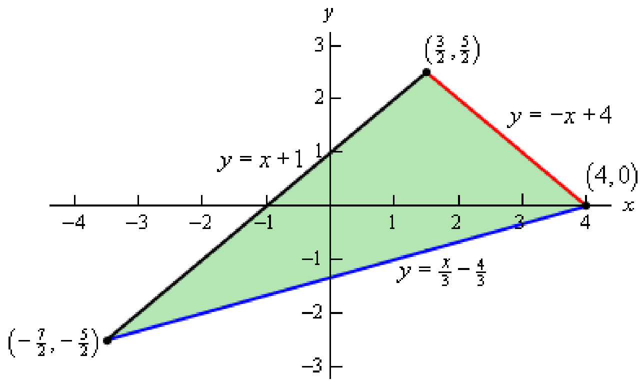

Figure 1:

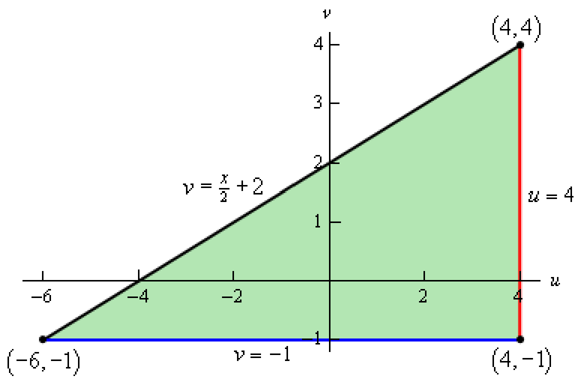

Figure 2:

The **Jacobian** of the transformation $x = g(u, v)$, $y = h(u, v)$ is:

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = g_u h_v - g_v h_u$$

Change of Variables for a Double Integral

Assume we want to integrate $f(x, y)$ over the region R in the xy-plane. Under the transformation $x = g(u, v)$, $y = h(u, v)$, S is the region R transformed into the uv-plane, and the integral becomes

$$\iint_R f(x, y) dA = \iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv$$

Note: 1. The $dudv$ on the right side of the above formula is just an indication that the right side integral is an integral in terms of u and v variables. The real oder of integration depends on the set-up of the problem.

2. If we look just at the differentials in the above formula we can also say that

$$dA = \left| \frac{\partial(x, y)}{\partial(x, y)} \right| dudv$$

.

3. Here we take the absolute value of the Jacobian. The one dimensional formula is just the derivative $\frac{dx}{du}$

Example Show that when changing to polar coordinates we have $dA = rdrd\theta$
Soln:
The transformation here is $x = r\cos(\theta)$, $y = r\sin(\theta)$.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}$$

$$= \det\begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix} = r(\cos^2(\theta) + \sin^2(\theta)) = r$$

So we have $dA = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| drd\theta = |r|drd\theta = rdrd\theta$.

Example Evaluate $\iint_R x + ydA$ where R is the trapezoidal region with vertices given by $(0,0)$, $(5,0)$, $(5/2, 5/2)$ and $(5/2, -5/2)$ using the transformation $x = 2u+3v$ and $y = 2u-3v$
Soln:

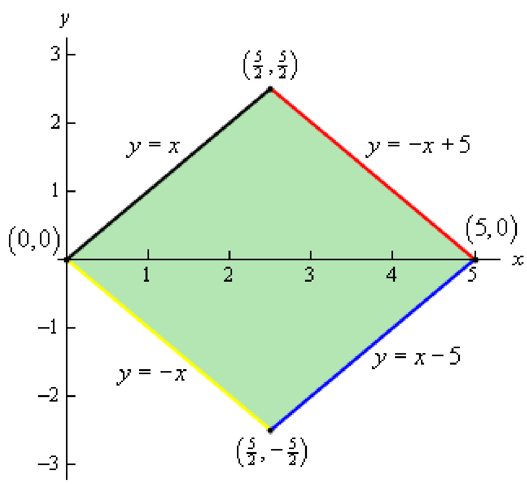

Figure 3:

Plugging in the transformation gives:

$$y = x \Rightarrow v = 0$$

$$y = -x \Rightarrow u = 0$$

$$y = -x + 5 \Rightarrow u = 5/4$$

$$y = x - 5 \Rightarrow v = 5/6$$

Therefore the region S in uv-plane is then a rectangle whose sides are given $u = 0$, $v = 0$, $u = 5/4$ and $v = 5/6$.

The Jacobian
$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} 2 & 3 \\ 2 & -3 \end{bmatrix} = -6-6 = -12$$

$$\iint_R x+y dA = \int_0^{\frac{5}{6}}\int_0^{\frac{5}{4}} ((2u+3v)+(2u-3v))|-12|dudv$$
$$= \int_0^{\frac{5}{6}}\int_0^{\frac{5}{4}} 48ududv = \int_0^{\frac{5}{6}} 24u^2|_0^{\frac{5}{4}} dv =$$
$$\int_0^{\frac{5}{6}} 75/2dv = 125/4$$

Example Compute $\iint_R y^2 dA$ where R is the region bounded by $xy=1$, $xy=2$, $xy^2=1$ and $xy^2=2$

Soln:

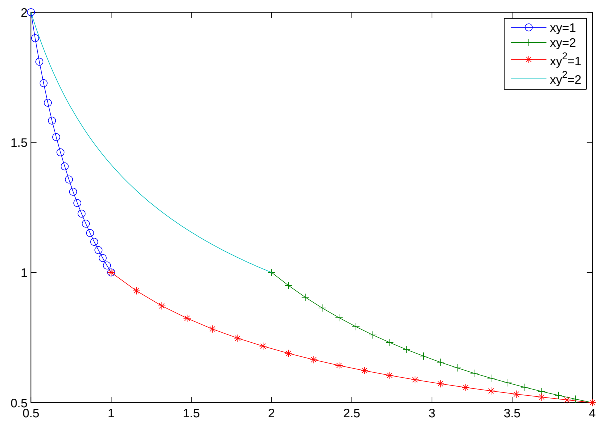

Figure 4:

The curves intersect in 4 points:

$$1 = xy = xy^2 \Rightarrow (1,1)$$
$$1 = xy = xy^2/2 \Rightarrow (1/2,2)$$
$$2 = xy = xy^2 \Rightarrow (2,1)$$
$$1 = xy/2 = xy^2 \Rightarrow (4,1/2)$$

We choose a transformation $u = xy$ and $v = xy^2$ to transform R into a new region S by $1 \le u \le 2$ and $1 \le v \le 2$.

Now we solve for x and y to compute the Jacobian:

$$u^2/v = x, \qquad v/u = y$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} 2u/v & -u^2/v^2 \\ -v/u^2 & 1/u \end{bmatrix} = 1/v$$

$$\iint_R y^2 dA = \int_1^2 \int_1^2 \frac{v^2}{u^2} \cdot \frac{1}{v} du dv = [-1/u]_1^2 \cdot [1/2v^2]_1^2 = 3/4$$

Note: In $\int_1^2 \int_1^2 \frac{v^2}{u^2} \cdot \frac{1}{v} du dv$, we dropped the absolute value sign for Jacobian $\frac{1}{v}$, since $\frac{1}{v}$ is positive in the region we were integrating over.

Example $\iint_R y^2 dA$ where R is the region in the first quadrant bounded by $x^2 - y^2 = 1$, $x^2 - y^2 = 4$, $y = 0$ and $y = (3/5)x$.

Soln:

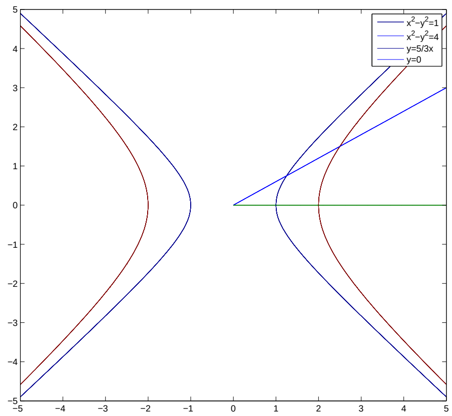

Figure 5:

We choose new variable to transform R into a simpler region. Let $u = x^2 - y^2 = (x-y)(x+y)$. Then two of the boundary curves for the new region S are $u = 1$ and $u = 4$. The integrand $e^{x^2-y^2}$ is also simplified to e^u.

We choose v so that we could easily solve for x and y. Let $v = x + y$, then $u/v = x - y$.

$$v + u/v = 2x \quad \text{and} \quad v - u/v = 2y$$

The boundaries $y = 0$ and $y = 3/5x$ becomes:

$$y = 0 \Rightarrow v - u/v = 0 \Rightarrow u = v^2$$

$$y = 3/5x \Rightarrow v - u/v = (3/5)(v + u/v) \Rightarrow u = (1/4)v^2$$

The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} (1/2)v & -(1-u/v^2)/2 \\ -(1/2)v & (1+u/v^2)/2 \end{bmatrix} = (1+u/v^2)/(4v) + (1-u/v^2)/(4v) = \frac{1}{2v}$$

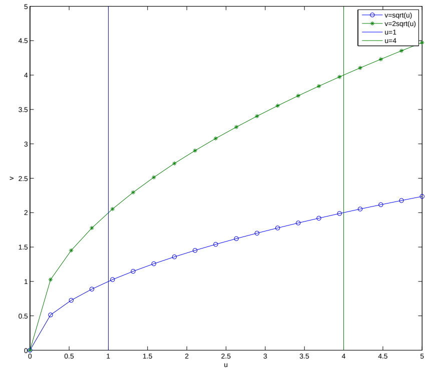

Figure 6:

$$\iint_R e^{x^2-y^2} dA = \iint_S e^u \frac{1}{2v} dA' = \int_1^4 \int_{\sqrt{u}}^{2\sqrt{u}} e^u \frac{1}{2v} dv du =$$

$$\int_1^4 \frac{e^u}{2}[\ln(2\sqrt{u}) - \ln(\sqrt{u})]du = \int_1^4 \frac{e^u}{2} \ln(2) du = \frac{\ln(2)}{2}(e^4 - e)$$

Note: In $\iint_S e^u \frac{1}{2v} dA'$, we dropped the absolute value sign for Jacobian $\frac{1}{2v}$, since $\frac{1}{2v}$ is positive in the region we were integrating over.

Triple Integrals

We start with a region R and use the transformation $x = g(u, v, w)$, $y = h(u, v, w)$, $z = k(u, v, w)$, and to transform the region R into the new region S.

The **Jabobian** is:

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \det \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix}$$

The integral under this transformation is:

$$\iiint_R f(x, y, z) dV = \iiint_S f(g(u, v, w), h(u, v, w), k(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dudvdw$$

Note: 1. $dudvdw$ on the right hand side of the above formula is just an indication that the right hand side integral is an integral in terms of u, v and w variables. The real oder of integration depends on the set-up of the problem.

2. As with double integrals,

$$dV = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dudvdw$$

Example If $x = \rho \sin(\phi)\cos(\theta)$, $y = \rho \sin(\phi)\sin(\theta)$, and $z = \rho\cos(\phi)$, then $\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho^2 \sin(\phi)$.

Example Find the volume V of the solid ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$

Soln:

We choose new variables $u = x/a$, $v = y/b$, $w = z/c$ and transform the ellipsoid into a sphere F: $u^2 + v^2 + w^2 \leq 1$.

The Jacobian is:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = abc$$

$$V = \iiint_E 1 dV = \iiint_F |\frac{\partial(x,y,z)}{\partial(u,v,w)}| dV' = \iiint_F abc dV' = abc\frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi abc$$