Lecture 1 : Overview of PDEs of Relevance to Astrophysics

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http://www.nd.edu/~dbalsara/Numerical-PDE-Course

1.1) Introduction

Many Astrophysical problems are governed by *partial differential equations* (PDEs).

Analytical solutions to astrophysical PDEs, few and limited.

Very effective *numerical solution techniques* are now available.

Powerful computers make it possible to obtain solutions to large, real-world problems.

Algorithms make it happen. They apply to broad classes of astrophysical PDEs, not to a specific astrophysical PDE.

Learn general classes of algorithms and you can solve broad classes of PDEs in astrophysics. Emphasis on *theory and technique*.

Broad classes of PDEs of interest (with pedestrian introductions):

Hyperbolic PDEs : Enable information to propagate as waves.
Examples: Water waves, sound waves, oscillations in a solid structure and electromagnetic radiation.

Parabolic PDEs : Enable information to travel as diffusive processes.
Examples: heat transfer, mass diffusion in the ground, diffusion of photons out of the sun.

Elliptic PDEs : Don't have time variation, convey action at a distance.
Examples: Gravitational field, electrostatics.

We first study solution techniques for these PDEs piecemeal and then learn how to assemble them together for more complex PDEs.

Since we all have most direct experience with astrophysical fluids, we will use the *Navier Stokes equations* as our motivating example. Several other PDEs of astrophysical interest are also introduced in this lecture.

Meet the 4 PDEs Systems of V. High Importance in Astrophysics:-

Euler Equations:

Conservation form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho\, \mathrm{v}_i\right) = 0$$

$$\frac{\partial}{\partial t}\left(\rho \mathrm{v}_i\right) + \frac{\partial}{\partial x_j}\left(\rho \mathrm{v}_i \mathrm{v}_j + \mathrm{P}\,\delta_{ij}\right) = 0 \quad \Leftrightarrow$$

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial x_i}\left(\left(\varepsilon + \mathrm{P}\right) \mathrm{v}_i\right) = 0$$

$$\varepsilon = \mathrm{e} + \rho\, \mathbf{v}^2/2; \quad \text{EOS:-} \quad \mathrm{e} = \mathrm{P}/\left(\Gamma - 1\right)$$

Primitive form:

$$\frac{D\rho}{Dt} = -\rho\, \nabla \bullet \mathbf{v}$$

$$\rho \frac{D\, \mathrm{v}_i}{D\, t} + \frac{\partial\, \mathrm{P}}{\partial\, x_i} = 0$$

$$\frac{\mathrm{D\, e}}{\mathrm{D}\, t} = -\left(\mathrm{e} + \mathrm{P}\right) \nabla \bullet \mathbf{v}$$

Relativistic Hydrodynamics (RHD):-

$$\frac{\partial}{\partial t}\left(\rho\, \gamma\right) + \frac{\partial}{\partial x_i}\left(\rho\, \gamma\, \mathrm{v}_i\right) = 0$$

$$\frac{\partial}{\partial t}\left(\rho\, h\, \gamma^2\, \mathrm{v}_i\right) + \frac{\partial}{\partial x_j}\left(\rho\, h\, \gamma^2\, \mathrm{v}_i\, \mathrm{v}_j\right) = 0$$

$$\frac{\partial}{\partial t}\left(\rho\, h\, \gamma^2 - \mathrm{P}\right) + \frac{\partial}{\partial x_i}\left(\rho\, h\, \gamma^2\, \mathrm{v}_i\right) = 0$$

$$h = 1 + \mathrm{P}\,\Gamma/\left[\rho\, \left(\Gamma - 1\right)\right] \quad ; \quad \gamma = 1/\sqrt{1 - \vec{\mathrm{v}}^2}$$

Notice:-

1) Conservation form – conserved variables + fluxes. Better for numerics.

2) EOS is important for achieving closure of the equations.

3) Primitive form looks simpler – easier for analysis.

Magnetohydrodynamics (MHD):-

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho \, \mathrm{v}_i\right) = 0$$

$$\frac{\partial}{\partial t}\left(\rho \mathrm{v}_i\right) + \frac{\partial}{\partial x_j}\left(\rho \mathrm{v}_i \mathrm{v}_j + \left(\mathrm{P} + \mathbf{B}^2/8\pi\right)\delta_{ij} - \mathrm{B}_i \, \mathrm{B}_j/4\pi \right) = 0$$

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial x_i}\left(\left(\varepsilon + \mathrm{P} + \mathbf{B}^2/8\pi\right)\mathrm{v}_i - \mathrm{B}_i\left(\mathbf{v}\cdot\mathbf{B}\right)/4\pi\right) = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla\times\mathbf{E} \quad \text{with} \quad \mathbf{E} = -\frac{1}{c}\mathbf{v}\times\mathbf{B} \quad \text{and constraint} \quad \nabla \bullet \mathbf{B} = 0$$

EOS:- $\mathrm{e} = \dfrac{\mathrm{P}}{\Gamma - 1}$

$$\varepsilon = \mathrm{e} + \frac{1}{2}\rho\,\mathbf{v}^2 + \frac{\mathbf{B}^2}{8\pi}$$

Notice:-

1) Magnetic fields contribute pressure and tension terms – anisotropic.

2) Electromagnetic energy contributes to the total energy.

3) Magnetic field evolves according to Faraday's law – once divergence-free, always divergence-free. ➔ presence of an involution constraint.

4) Electric field given constitutively as a function of B-field and velocity. Consequence of MHD approximation.

Relativistic Magnetohydrodynamics (RMHD):-

$$\frac{\partial}{\partial t}(\rho\,\gamma)+\frac{\partial}{\partial x_i}(\rho\,\gamma\,\mathrm{v}_i)=0$$

$$\frac{\partial}{\partial t}\left(\rho\,h\,\gamma^2\,\mathrm{v}_i + (\mathbf{E}\times\mathbf{B})_i\right) + \frac{\partial}{\partial x_j}\left(\rho\,h\,\gamma^2\,\mathrm{v}_i\,\mathrm{v}_j - \mathrm{E}_i\,\mathrm{E}_j - \mathrm{B}_i\,\mathrm{B}_j + \left(\mathrm{P} + \frac{1}{2}\left(\mathbf{E}^2+\mathbf{B}^2\right)\right)\gamma\,\delta_{ij}\right) = 0$$

$$\frac{\partial}{\partial t}\left(\rho\,h\,\gamma^2 - \mathrm{P} + \frac{1}{2}\left(\mathbf{E}^2+\mathbf{B}^2\right)\right) + \frac{\partial}{\partial x_i}\left(\rho\,h\,\gamma^2\,\mathrm{v}_i + (\mathbf{E}\times\mathbf{B})_i\right)=0$$

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) \qquad ; \qquad \nabla\bullet\mathbf{B}=0 \qquad ; \qquad \mathbf{E}=-\mathbf{v}\times\mathbf{B}$$

$$h = 1 + \mathrm{P}\,\Gamma\Big/\left[\rho\,(\Gamma - 1)\right]$$

Observe the extensions from RHD to RMHD and from MHD to RMHD.

The ***fluid approximation*** holds for the above 4 PDEs if the length of the system L >> $l_{collisions}$ and the timescales over which we observe the system are T >> $\tau_{collisions}$.

1.2) The Euler Equations

Conservation form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho\, \mathrm{v}_i\right) = 0$$

$$\frac{\partial}{\partial t}\left(\rho \mathrm{v}_i\right) + \frac{\partial}{\partial x_j}\left(\rho \mathrm{v}_i \mathrm{v}_j + \mathrm{P}\,\delta_{ij}\right) = 0 \quad \Leftrightarrow$$

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial x_i}\left(\left(\varepsilon + \mathrm{P}\right) \mathrm{v}_i\right) = 0$$

Primitive form:

$$\frac{D\rho}{Dt} = -\rho\, \nabla \bullet \mathbf{v}$$

$$\rho \frac{D\,\mathrm{v}_i}{D\,t} + \frac{\partial\, \mathrm{P}}{\partial\, x_i} = 0$$

$$\frac{\mathrm{D\,e}}{\mathrm{D}\,t} = -\left(\mathrm{e} + \mathrm{P}\right) \nabla \bullet \mathbf{v}$$

EOS:- $\mathrm{e} = \frac{\mathrm{P}}{\Gamma - 1}$

$\varepsilon = \mathrm{e} + \rho\, \mathbf{v}^2/2$

Notice that we have 6 variables (density, 3 velocities, internal energy and pressure) but only 5 equations. This is called the *closure problem*. The *equation of state* helps close the system, i.e. provides the one extra equation needed to get the number of equations == number of unknowns.

Question: How do the dimensions of the fluxes relate to the dimensions of the conserved variables? Interpret your answer physically?

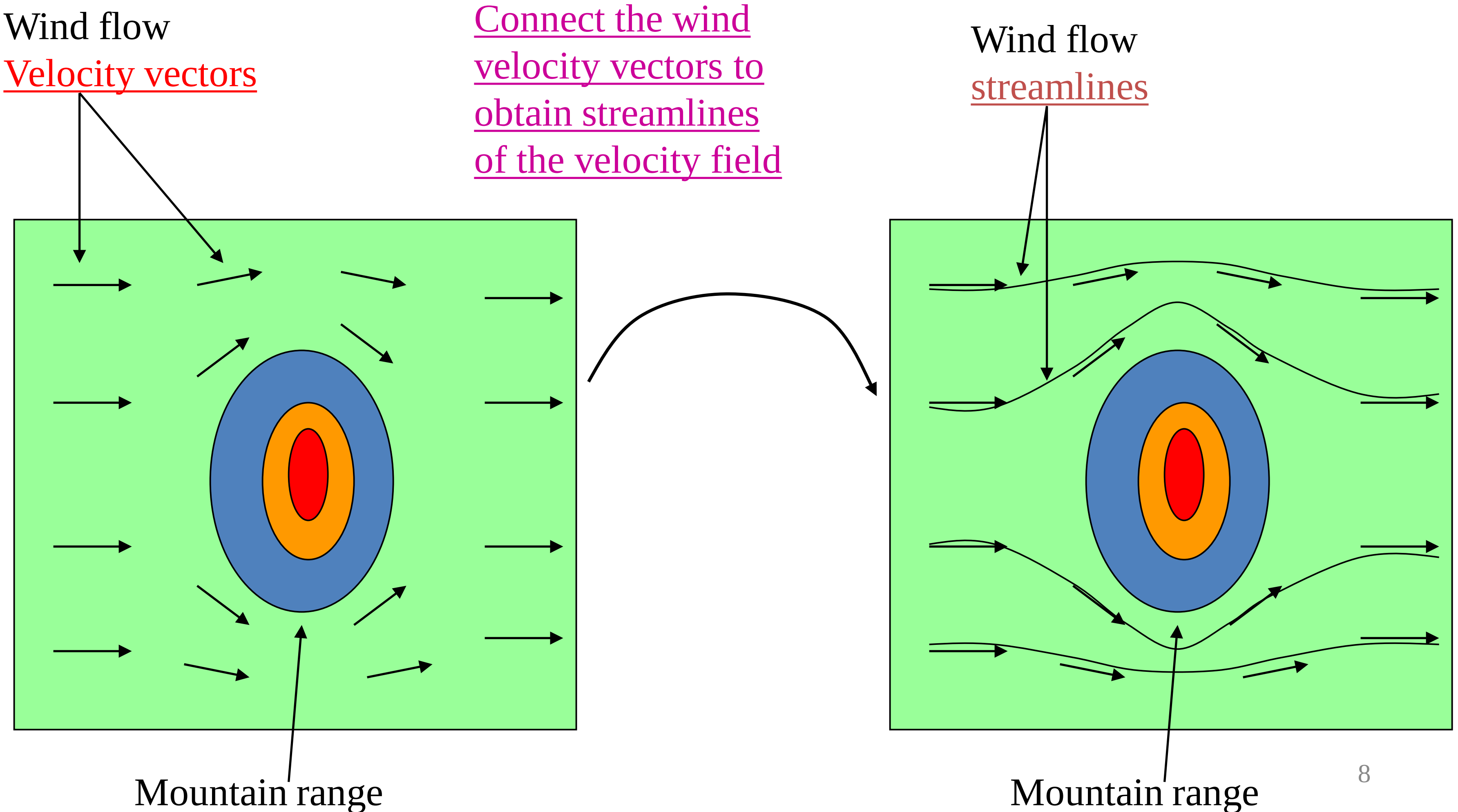
Primitive form is v. useful for analytic work; conserved form for computation (especially when discontinuities present). Question : Why?

The *Lagrangian derivative / material derivative*, shown above, has a form that occurs very often in fluid dynamics:-

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \bullet \nabla \quad \text{, for example:-} \quad \frac{D\rho}{Dt} = -\rho \nabla \bullet \mathbf{v}$$

It is very important to intuitively understand the *Lagrangian derivative* .

<u>Question</u>: What is it tracing?



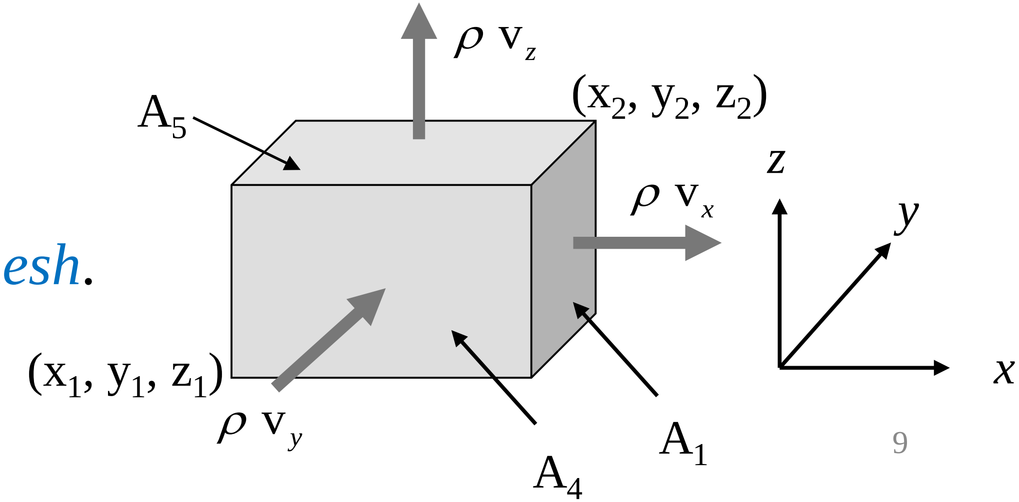
Notice that the fluid variables evolve in time in response to their own spatial gradients. This is often the case with most PDEs.

Question: So what makes the ***conservation form*** so special?
Answer: Gauss' Law.
Let's focus on the continuity equation and the figure below.

$$\iiint_V \left(\frac{\partial \rho}{\partial t} + \frac{\partial(\rho \mathrm{v}_x)}{\partial x} + \frac{\partial(\rho \mathrm{v}_y)}{\partial y} + \frac{\partial(\rho \mathrm{v}_z)}{\partial z} \right) dx\, dy\, dz = 0 \quad \Rightarrow$$

$$\frac{\partial}{\partial t}\iiint_V \rho\, dx\, dy\, dz + \iint_{A_1} \rho \mathrm{v}_x\, dy\, dz - \iint_{A_2} \rho \mathrm{v}_x\, dy\, dz + \iint_{A_3} \rho \mathrm{v}_y\, dx\, dz - \iint_{A_4} \rho \mathrm{v}_y\, dx\, dz$$
$$+ \iint_{A_5} \rho \mathrm{v}_z\, dx\, dy - \iint_{A_6} \rho \mathrm{v}_z\, dx\, dy = 0$$

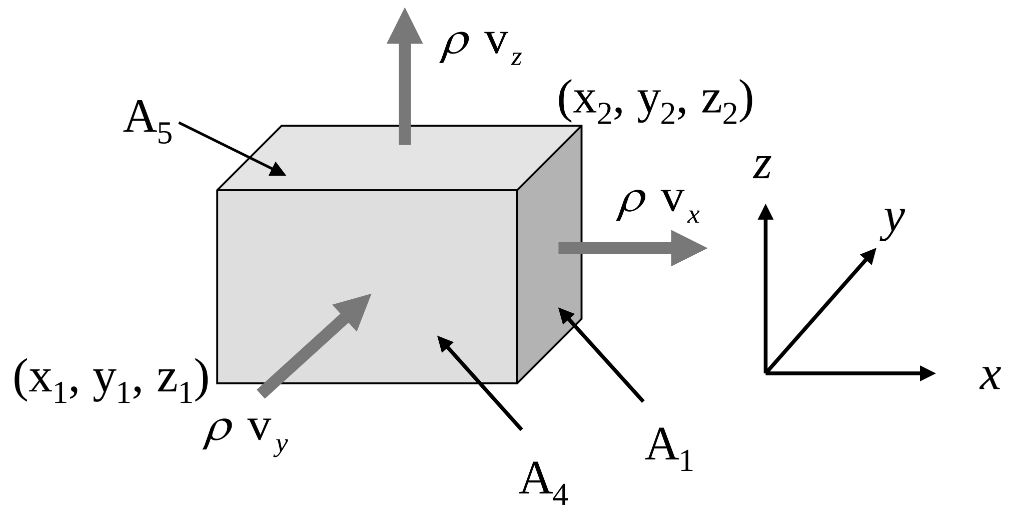
When ***discontinuities/shocks*** are present, we have no hope of predicting the flow structure inside a *zone* in our computational *mesh*. However, the ***conservation form remains valid***!



$$\iiint_V \left(\frac{\partial \rho}{\partial t} + \frac{\partial (\rho \, v_x)}{\partial x} + \frac{\partial (\rho \, v_y)}{\partial y} + \frac{\partial (\rho \, v_z)}{\partial z} \right) dx\, dy\, dz = 0$$

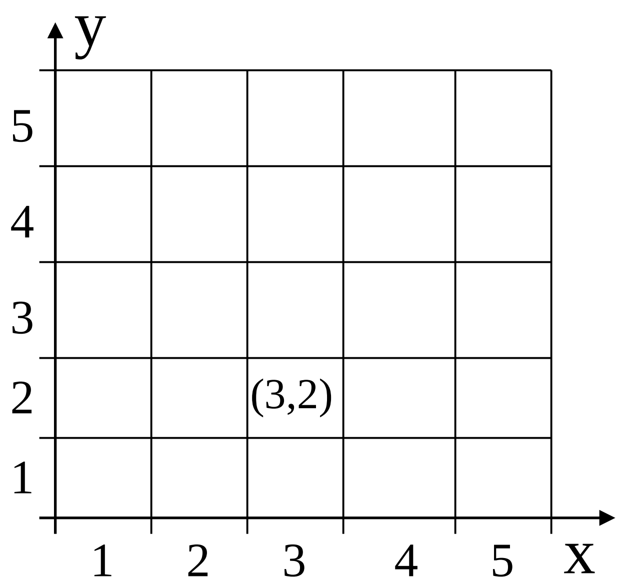
$$\iiint_V \frac{\partial \rho}{\partial t}\, dx\, dy\, dz =$$

$$\iiint_V \frac{\partial (\rho \, v_x)}{\partial x}\, dx\, dy\, dz =$$



$$\frac{\partial}{\partial t}\iiint_V \rho\, dx\, dy\, dz + \iint_{A_1} \rho\, v_x\, dy\, dz - \iint_{A_2} \rho\, v_x\, dy\, dz + \iint_{A_3} \rho\, v_y\, dx\, dz - \iint_{A_4} \rho\, v_y\, dx\, dz$$

$$+ \iint_{A_5} \rho\, v_z\, dx\, dy - \iint_{A_6} \rho\, v_z\, dx\, dy = 0$$



Introducing concepts of a mesh, zones and timestep.

Role of timestep in conveying information.

Value of conservation form on a mesh.

1.3) The Navier Stokes Equations

Conservation form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho \, v_i\right) = 0$$

$$\frac{\partial}{\partial t}\left(\rho v_i\right) + \frac{\partial}{\partial x_j}\left(\rho v_i v_j + P\,\delta_{ij} - \pi_{ij}\right) = 0$$

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial x_i}\left(\left(\varepsilon + P\right) v_i - v_j \, \pi_{ji} + F_i^{cond}\right) = 0$$

$\Leftrightarrow$

Primitive form:

$$\frac{D\rho}{Dt} = -\rho \, \nabla \bullet \mathbf{v}$$

$$\rho \frac{D\,v_i}{D\,t} + \frac{\partial P}{\partial x_i} - \frac{\partial}{\partial x_j}\pi_{ij} = 0$$

$$\frac{D\,e}{D\,t} = -\left(e + P\right) \nabla \bullet \mathbf{v} - \nabla \bullet \mathbf{F}^{cond} + \pi_{ij}\frac{\partial v_i}{\partial x_j}$$

As before, the equation of state helps close the system.

Notice though, that we now have extra terms for the viscosity and heat flux. Theory alone cannot specify these terms; experiments are needed. We have the following *stress-strain relation* for the *viscous stresses*:

$$\boxed{\pi_{ij} \equiv \mu \, D_{ij} \quad ; \quad D_{ij} \equiv \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\left(\nabla \bullet \mathbf{v}\right) \delta_{ij}}$$

For the *thermal conduction* we have:

$$\boxed{\mathbf{F}^{cond} \equiv -\kappa \, \nabla T}$$

parabolic terms

1.4) Classifying and Understanding PDEs

(Or How to read a PDE like a book)

1.4.1) Motivation

Let's start simple with scalar PDEs. Consider our first example:

$$\frac{\partial \rho}{\partial t} + \mathrm{a}\,\frac{\partial \rho}{\partial x} + \mathrm{b}\,\frac{\partial \rho}{\partial y} = 0 \quad \leftarrow$$ What does it remind you of? How does ρ move in 2d?

Analyzing a PDE is same as bringing out its character. Since the above PDE has a wave character, i.e. it is *hyperbolic*, we try harmonic modes:

$$\rho(x,y,t) = \rho_0 + \rho_1\, \mathrm{e}^{i\left(k_x x + k_y y - \omega t\right)}$$ Substitution in the PDE gives : $\omega = k_x\,\mathrm{a} + k_y\,\mathrm{b}$

$$\Rightarrow\ \rho(x,y,t) = \rho_0 + \rho_1\, \mathrm{e}^{i\left[k_x(x - \mathrm{a}\,t) + k_y(y - \mathrm{b}\,t)\right]} \quad \leftarrow$$ Propagating wave-like modes.

The harmonic modes take on a time-evolution that is purely multiplicative with no change in the amplitude. Such modes are called the ***eigenmodes*** of the PDE. "ω" is the ***eigenvalue*** associated with that eigenmode.

So here we started with a PDE that we knew to be hyperbolic and divined its mathematical character.

Substitute $\rho(x,y,t) = \rho_0 + \rho_1 \; e^{i\left(k_x x + k_y y - \omega t\right)}$ in $\dfrac{\partial \rho}{\partial t} + a \dfrac{\partial \rho}{\partial x} + b \dfrac{\partial \rho}{\partial y} = 0$

$$\rho(x,y,t) = \rho_0 + \rho_1 \; e^{i\left[k_x(x - a\,t) + k_y(y - b\,t)\right]} \text{ with } \omega = k_x\, a + k_y\, b$$

Consider the heat equation in 2d with constant coefficients:

$$\frac{\partial \mathrm{T}}{\partial t} = \kappa \left(\frac{\partial^2 \mathrm{T}}{\partial x^2} + \frac{\partial^2 \mathrm{T}}{\partial y^2} \right)$$

Again, let us put in harmonic modes to bring out the character of this well-known *parabolic* PDE:

$$\mathrm{T}(x, y, t) = \mathrm{T}_0 + \mathrm{T}_1 \, \mathrm{e}^{i \left(k_x x + k_y y - \omega t \right)} \quad \text{Substitution in the PDE gives} : \omega = -i \, \kappa \left(k_x^2 + k_y^2 \right)$$

$$\Rightarrow \mathrm{T}(x, y, t) = \mathrm{T}_0 + \mathrm{T}_1 \, \mathrm{e}^{i \left(k_x x + k_y y \right) - \kappa \left(k_x^2 + k_y^2 \right) t} \quad \leftarrow \text{ A } \textit{time-decaying} \text{ solution}$$

Such modes are called the *eigenmodes* of the PDE. "ω" is the *eigenvalue* associated with that eigenmode.
So here we started with a PDE that we knew to be parabolic and divined its mathematical character.

Now the ideas can be combined. -- A super simple chemo-taxis example:

$$\frac{\partial \rho}{\partial t} + \mathrm{a} \frac{\partial \rho}{\partial x} + \mathrm{b} \frac{\partial \rho}{\partial y} - \kappa \left(\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} \right) = \mathrm{s}$$

Substitute $\mathrm{T}(x, y, t) = \mathrm{T}_0 + \mathrm{T}_1 \, \mathrm{e}^{i\left(k_x x + k_y y - \omega t\right)}$ in $\dfrac{\partial \mathrm{T}}{\partial t} = \kappa \left(\dfrac{\partial^2 \mathrm{T}}{\partial x^2} + \dfrac{\partial^2 \mathrm{T}}{\partial y^2} \right)$

$\mathrm{T}(x, y, t) = \mathrm{T}_0 + \mathrm{T}_1 \, \mathrm{e}^{i\left(k_x x + k_y y\right) - \kappa\left(k_x^2 + k_y^2\right)t}$ with $\omega = -i \, \kappa \left(k_x^2 + k_y^2\right)$

1.4.2) Characteristic Analysis of the Euler Equations

Let us write the 1d Euler equations in a matrix form:

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ \mathrm{v_x} \\ \mathrm{P}\end{pmatrix} + \begin{pmatrix}\mathrm{v_x} & \rho & 0 \\ 0 & \mathrm{v_x} & \frac{1}{\rho} \\ 0 & \Gamma\ \mathrm{P} & \mathrm{v_x}\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\rho \\ \mathrm{v_x} \\ \mathrm{P}\end{pmatrix} = 0\,. \quad \text{Try the wave-like solution :} \begin{pmatrix}\rho(x,t) \\ \mathrm{v_x}(x,t) \\ \mathrm{P}(x,t)\end{pmatrix} = \begin{pmatrix}\rho_0 \\ \mathrm{v_{x0}} \\ \mathrm{P_0}\end{pmatrix} + \begin{pmatrix}\rho_1 \\ \mathrm{v_{x1}} \\ \mathrm{P_1}\end{pmatrix} e^{\,i\,(k\,x\,-\,\omega\,t)}$$

We think of the variables with subscript “0” as the constant values around which we introduce small fluctuations, i.e. the variables with subscript “1”. This has the advantage that it freezes the matrix. We get:

$$-\,i\;\omega\begin{pmatrix}\rho_1 \\ \mathrm{v_{x1}} \\ \mathrm{P_1}\end{pmatrix} + i\,\mathrm{k}\begin{pmatrix}\mathrm{v_{x0}} & \rho_0 & 0 \\ 0 & \mathrm{v_{x0}} & \frac{1}{\rho_0} \\ 0 & \Gamma\ \mathrm{P_0} & \mathrm{v_{x0}}\end{pmatrix}\begin{pmatrix}\rho_1 \\ \mathrm{v_{x1}} \\ \mathrm{P_1}\end{pmatrix} = 0\,.\ \text{Use } \lambda \equiv \omega/k \text{ to get :} \begin{pmatrix}\mathrm{v_{x0}}-\lambda & \rho_0 & 0 \\ 0 & \mathrm{v_{x0}}-\lambda & \frac{1}{\rho_0} \\ 0 & \Gamma\ \mathrm{P_0} & \mathrm{v_{x0}}-\lambda\end{pmatrix}\begin{pmatrix}\rho_1 \\ \mathrm{v_{x1}} \\ \mathrm{P_1}\end{pmatrix} = 0$$

Notice that for systems we have to analyze the above *characteristic matrix*. Its determinant yields the *characteristic equation* with solutions:

$$\boxed{\lambda^1 = \mathrm{v_{x0}} - \mathrm{c_0} \quad ; \quad \lambda^2 = \mathrm{v_{x0}} \quad ; \quad \lambda^3 = \mathrm{v_{x0}} + \mathrm{c_0} \quad ; \quad \mathrm{c_0} \equiv \sqrt{\frac{\Gamma\ \mathrm{P_0}}{\rho_0}} \leftarrow \text{speed of sound}}$$

$$\frac{\partial \rho}{\partial t} + \mathrm{v}_x \frac{\partial \rho}{\partial x} + \rho \frac{\partial \mathrm{v}_x}{\partial x} = 0 \quad ;$$

$$\frac{\partial \mathrm{v}_x}{\partial t} + \mathrm{v}_x \frac{\partial \mathrm{v}_x}{\partial x} + \frac{1}{\rho}\frac{\partial \mathrm{P}}{\partial x} = 0 \quad ;$$

$$\frac{\partial \mathrm{e}}{\partial t} + \mathrm{v}_x \frac{\partial \mathrm{e}}{\partial x} + \mathrm{e}\frac{\partial \mathrm{v}_x}{\partial x} + \mathrm{P}\frac{\partial \mathrm{v}_x}{\partial x} = 0 \quad \text{with } \mathrm{e} = \frac{\mathrm{P}}{\Gamma - 1}$$

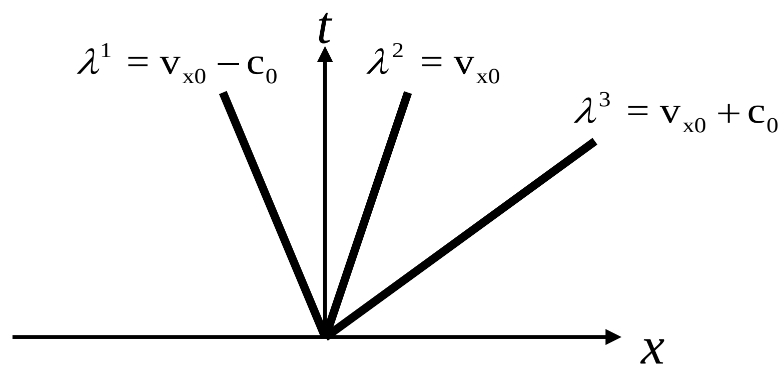
$$\longrightarrow \quad \frac{\partial}{\partial t}\begin{pmatrix} \rho \\ \mathrm{v_x} \\ \mathrm{P} \end{pmatrix} + \begin{pmatrix} \mathrm{v_x} & \rho & 0 \\ 0 & \mathrm{v_x} & \frac{1}{\rho} \\ 0 & \Gamma\,\mathrm{P} & \mathrm{v_x} \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} \rho \\ \mathrm{v_x} \\ \mathrm{P} \end{pmatrix} = 0$$

Equations *linearized* about a constant state: $(\rho_0,\ v_{x0},\ P_0)$

$$- i\ \omega \begin{pmatrix} \rho_1 \\ v_{x1} \\ P_1 \end{pmatrix} + i\ k \begin{pmatrix} v_{x0} & \rho_0 & 0 \\ 0 & v_{x0} & \dfrac{1}{\rho_0} \\ 0 & \Gamma\ P_0 & v_{x0} \end{pmatrix} \begin{pmatrix} \rho_1 \\ v_{x1} \\ P_1 \end{pmatrix} = 0 \quad \text{with } \lambda \equiv \omega / k$$

$$\lambda^1 = v_{x0} - c_0 \quad ; \quad \lambda^2 = v_{x0} \quad ; \quad \lambda^3 = v_{x0} + c_0 \quad ; \quad c_0 \equiv \sqrt{\frac{\Gamma\ P_0}{\rho_0}}$$

We can use a *space-time diagram* to trace the waves. The lines are called ***characteristic curves***.



Question: Identify sub-sonic and supersonic flow situations by drawing space-time diagrams for them. Are the characteristic curves always straight lines?

Analysis of the ***right eigenvectors*** yields a lot of further insight:

$$\boxed{r^1 = \begin{pmatrix} \rho_0 \\ -c_0 \\ \rho_0\, c_0^2 \end{pmatrix} \quad ; \quad r^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad ; \quad r^3 = \begin{pmatrix} \rho_0 \\ c_0 \\ \rho_0\, c_0^2 \end{pmatrix}}$$

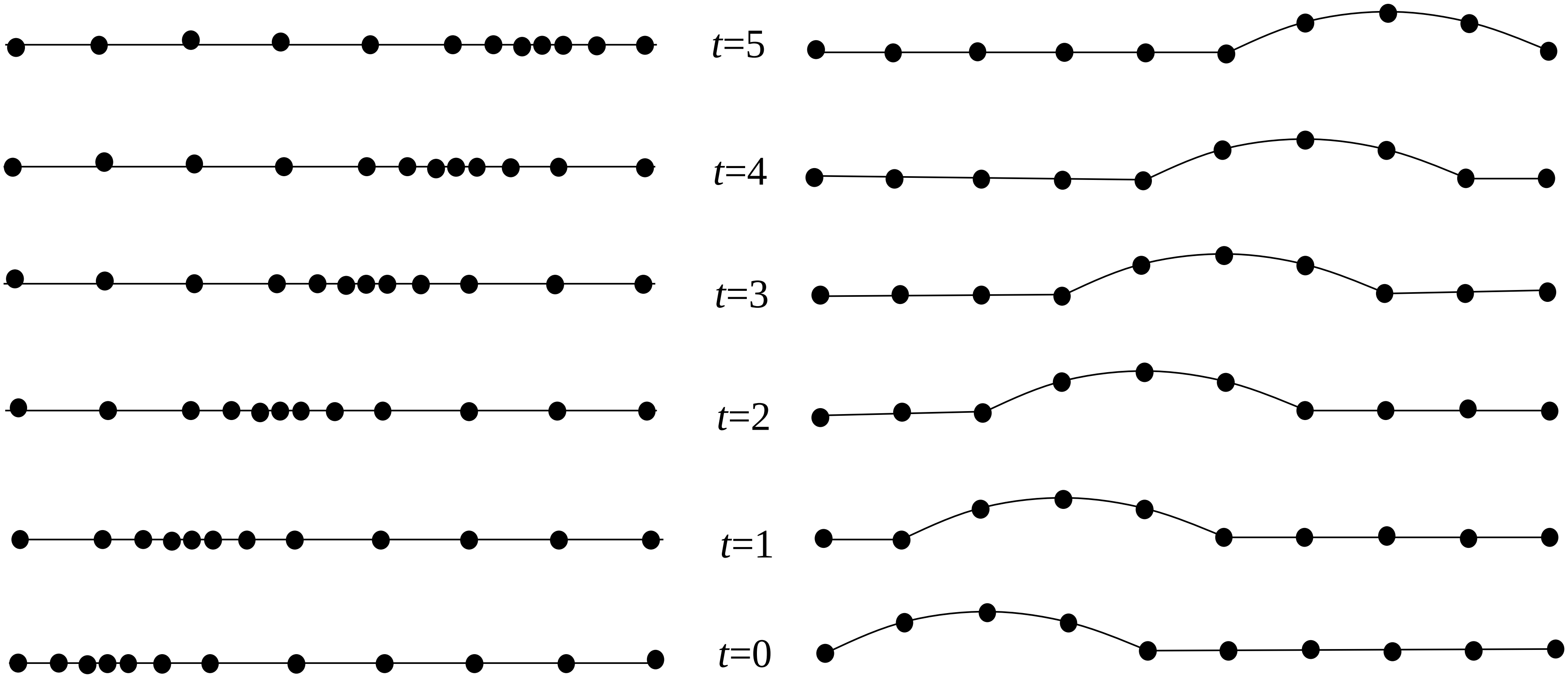
Observe from r^3 that the fluctuations have to have the ratios:- $\rho_1 : v_{x1} : P_1 = \rho_0 : c_0 : \rho_0\, c_0^2$

I.e. the ratios are preset and all these fluctuations produce *compressions* in the velocity and pressure fluctuations $\Rightarrow$ these are *right-going sound waves* (see the eigenvalue λ^3).

Observe from r^2 that the fluctuations have to have the ratios:- $\rho_1 : v_{x1} : P_1 = 1:0:0$

I.e. no pressure or velocity fluctuations; only changes in the density $\Rightarrow$ these are *entropy waves* (see the eigenvalue λ^2). They are *advected* with the fluid velocity.

Wave propagation can be via *longitudinal* or *transverse* fluctuations.
Question: Which one is which in the figure below? Which one represents sound waves?



The right eigenvectors form a *complete* basis set in a 3-d vector space.➔
There exists an *orthonormal* set of ***left eigenvectors***. They are:

$$\boxed{l^1 = \begin{pmatrix} 0 & \dfrac{-1}{2\,c_0} & \dfrac{1}{2\,\rho\,c_0^2} \end{pmatrix} \;;\; l^2 = \begin{pmatrix} 1 & 0 & -\dfrac{1}{c_0^2} \end{pmatrix} \;;\; l^3 = \begin{pmatrix} 0 & \dfrac{1}{2\,c_0} & \dfrac{1}{2\,\rho\,c_0^2} \end{pmatrix}}$$

$$\begin{pmatrix}\rho_1 & v_{x1} & P_1\end{pmatrix}\begin{pmatrix} v_{x0}-\lambda & \rho_0 & 0 \\ 0 & v_{x0}-\lambda & \dfrac{1}{\rho_0} \\ 0 & \Gamma\ P_0 & v_{x0}-\lambda \end{pmatrix} = 0$$

$$\lambda^1 = v_{x0} - c_0 \quad ; \quad \lambda^2 = v_{x0} \quad ; \quad \lambda^3 = v_{x0} + c_0 \quad ;$$

$$l^1 = \begin{pmatrix} 0 & \dfrac{-1}{2\ c_0} & \dfrac{1}{2\ \rho\ c_0^2} \end{pmatrix} \quad ; \quad l^2 = \begin{pmatrix} 1 & 0 & -\dfrac{1}{c_0^2} \end{pmatrix} \quad ; \quad l^3 = \begin{pmatrix} 0 & \dfrac{1}{2\ c_0} & \dfrac{1}{2\ \rho\ c_0^2} \end{pmatrix}$$

How is all this machinery of eigenvalues and eigenvectors useful?

Imagine you have a mean fluid state and you impose a small Gaussian pulse, i.e. a perturbation. How do you predict its time-evolution?

$$\begin{pmatrix} \rho(x,t=0) \\ \mathrm{v_x}(x,t=0) \\ \mathrm{P}(x,t=0) \end{pmatrix} = \begin{pmatrix} \rho_0 \\ \mathrm{v_{x0}} \\ \mathrm{P_0} \end{pmatrix} + \begin{pmatrix} \rho_1 \\ \mathrm{v_{x1}} \\ \mathrm{P_1} \end{pmatrix} \mathrm{e}^{-x^2} = \mathrm{V_0} + \mathrm{V_1}\,\mathrm{e}^{-x^2}$$

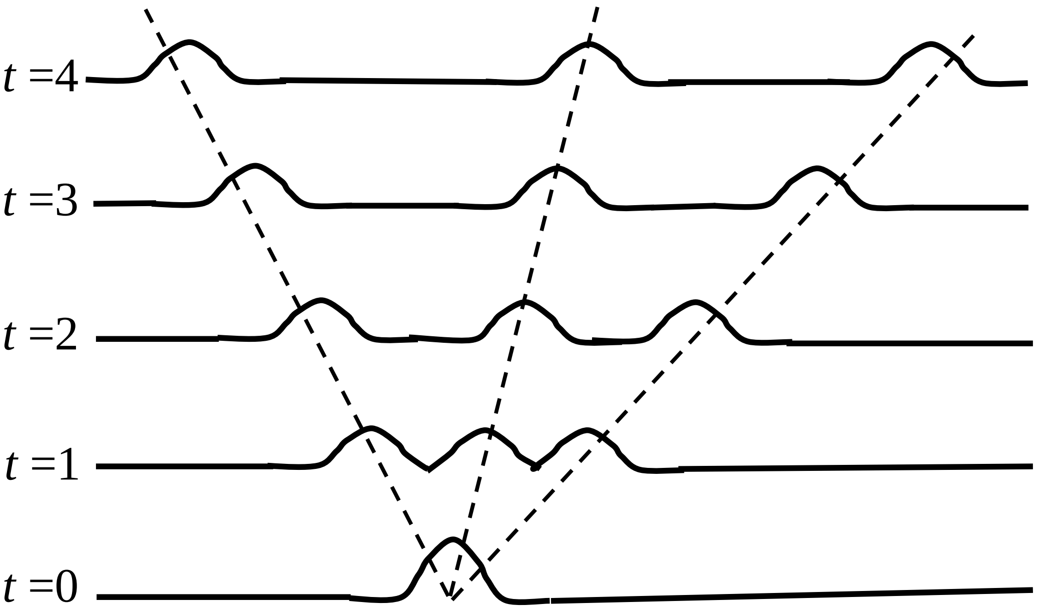
We know that each right eigenvector is a pure wave that travels with a preset speed. So use the left eigenvectors to find out what fraction of the initial perturbation contributes to each of the waves. This is given to us by the ***eigenweights***:

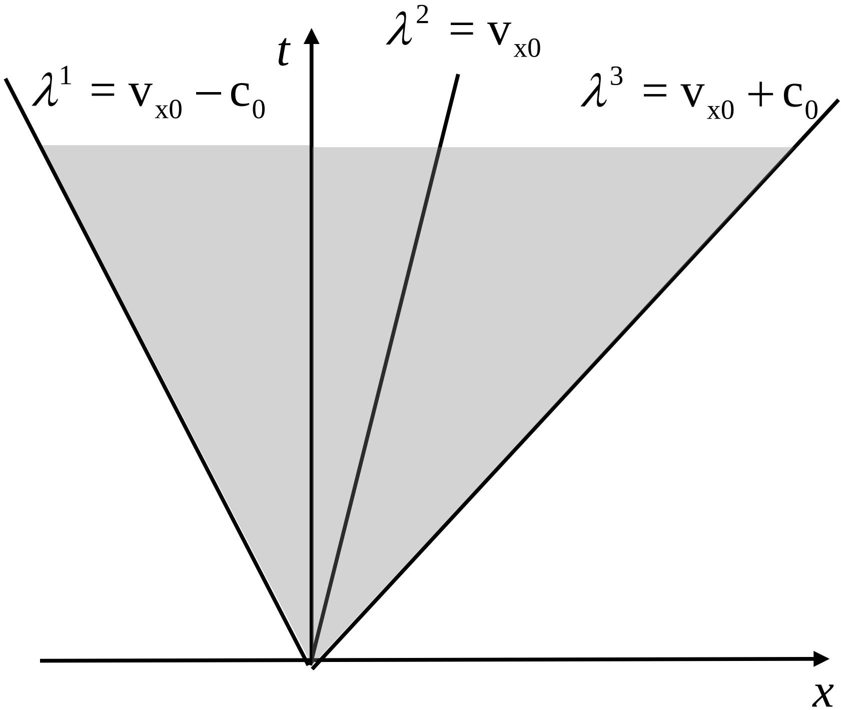
i.e. in order to write $\mathrm{V_1} = \alpha^1 r^1 + \alpha^2 r^2 + \alpha^3 r^3$ we need to find α^1, α^2, and α^3.

$$\alpha^1 = \left(l^1 \bullet \mathrm{V_1}\right) \quad ; \quad \alpha^2 = \left(l^2 \bullet \mathrm{V_1}\right) \quad ; \quad \alpha^3 = \left(l^3 \bullet \mathrm{V_1}\right)$$

The time-evolution of the Gaussian perturbation is then given by:

$$\begin{pmatrix} \rho(x,t) \\ \mathrm{v_x}(x,t) \\ \mathrm{P}(x,t) \end{pmatrix} = \begin{pmatrix} \rho_0 \\ \mathrm{v_{x0}} \\ \mathrm{P_0} \end{pmatrix} + \alpha^1 \begin{pmatrix} \rho_0 \\ -\mathrm{c_0} \\ \rho_0\,\mathrm{c_0^2} \end{pmatrix} \mathrm{e}^{-(x-\lambda_1 t)^2} + \alpha^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathrm{e}^{-(x-\lambda_2 t)^2} + \alpha^3 \begin{pmatrix} \rho_0 \\ \mathrm{c_0} \\ \rho_0\,\mathrm{c_0^2} \end{pmatrix} \mathrm{e}^{-(x-\lambda_3 t)^2}$$

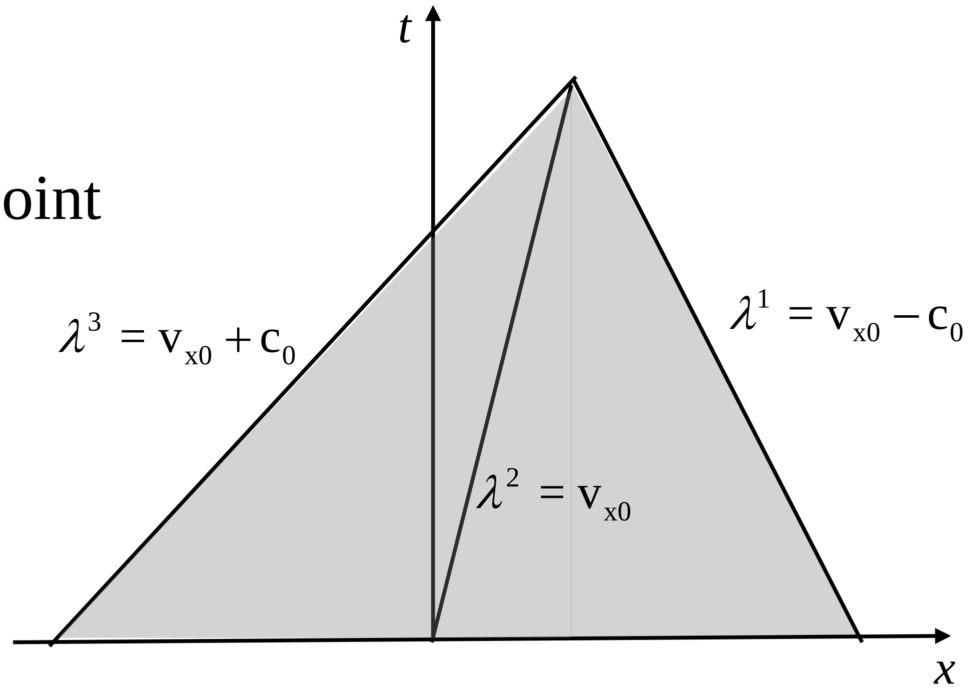




The tiny initial Gaussian pulse propagates away as three tinier Gaussian pulses with amplitudes given by the eigenweights and speeds given by the eigenvalues.

The shaded region in the space-time diagram shows the ***range of influence***. I.e. it gives us the portions of space-time that get influenced by the initial perturbation.
Notice that the extremal wave speeds, λ^1 and λ^3 , determine the range of influence.

Say that the x-axis is seeded with small fluctuations at t=0.
Question: If we pick a space-time point (x,t) with t>0, which points on the original x-axis will influence its evolution?



Realize that information travels at a finite speed in a hyperbolic system.

Since the characteristics curves are straight lines, in our linearized system, it is easy to find that domain by propagating the characteristics backward. Again, the extremal wave speeds, λ^1 and λ^3 , are very useful in identifying this domain.

The shaded region shows the ***domain of dependence*** in space-time.

1.4.3) Generalized Definition of a Hyperbolic PDE

Many, though not all, hyperbolic systems can be written in conservation form:

$$\mathrm{U}_t + \mathrm{F}\left(\mathrm{U}\right)_x + \mathrm{G}\left(\mathrm{U}\right)_y + \mathrm{H}\left(\mathrm{U}\right)_z = \mathrm{S}\left(\mathrm{U}\right)$$

"U" is the vector of "*M*" conserved variables and "F", "G" and "H" are the flux vectors. "S" is the vector of source terms. One can then obtain the characteristic matrices "A", "B" and "C", which are all $M \times M$:

$$\mathrm{U}_t + \mathrm{A(U)}\,\mathrm{U}_x + \mathrm{B(U)}\,\mathrm{U}_y + \mathrm{C(U)}\,\mathrm{U}_z = \mathrm{S}\left(\mathrm{U}\right) \quad \left(\text{In component form } \mathrm{A(U)}_{i,j} \equiv \partial \mathrm{F}_i\left(\mathrm{U}\right) / \partial \mathrm{U}_j\right)$$

$$\text{with } \mathrm{A(U)} \equiv \frac{\partial \mathrm{F}\left(\mathrm{U}\right)}{\partial \mathrm{U}} \;;\; \mathrm{B(U)} \equiv \frac{\partial \mathrm{G}\left(\mathrm{U}\right)}{\partial \mathrm{U}} \;;\; \mathrm{C(U)} \equiv \frac{\partial \mathrm{H}\left(\mathrm{U}\right)}{\partial \mathrm{U}}$$

The *hyperbolic property* then depends on the *eigenstructure of the characteristic matrices*.

Often times, as with Euler eqns., it is easier to analyze the hyperbolic system in terms of primitive variables V : $\delta \mathrm{U} = \left(\frac{\partial \mathrm{U}}{\partial \mathrm{V}}\right) \delta \mathrm{V} \;;\; \delta \mathrm{V} = \left(\frac{\partial \mathrm{V}}{\partial \mathrm{U}}\right) \delta \mathrm{U}$

Any general hyperbolic system can then be written as:

$$V_t + A(V)\, V_x + B(V)\, V_y + C(V)\, V_z = S(V)$$

$$\text{with } A(V) \equiv \left(\frac{\partial V}{\partial U}\right) \frac{\partial F(U)}{\partial U} \left(\frac{\partial U}{\partial V}\right) \quad ; \quad S(V) \equiv \left(\frac{\partial V}{\partial U}\right) S(U) \text{ if it is in conservation form.}$$

The system is then said to be ***hyperbolic*** if each of the matrices “A”, “B” and “C” admit “M” *real eigenvalues* and a *complete* set of “M” *right eigenvectors*. This also ensures the existence of orthonormal left eigenvectors. I.e. solutions are wave-like when propagating in all directions.

Above definition gives us the useful properties that :

1) Waves can propagate in any direction.

2) Any small initial fluctuation can be evolved forward in time for at least a short amount of time. It enables us to do dynamics.

$$U_t + F(U)_x + G(U)_y + H(U)_z = S(U) \quad \rightarrow \quad U_t + A(U)\, U_x + B(U)\, U_y + C(U)\, U_z = S(U)$$

$$\delta U = \left(\frac{\partial U}{\partial V} \right) \delta V \; ; \; \delta V = \left(\frac{\partial V}{\partial U} \right) \delta U \quad \rightarrow \quad V_t + A(V)\, V_x + B(V)\, V_y + C(V)\, V_z = S(V)$$

Let us now focus on the 1d case. We assume that the characteristic matrix "A" is frozen about some constant state V_0 . For small fluctuations V_1 about that constant state, we have

$$\frac{\partial V_1}{\partial t} + A \; \frac{\partial V_1}{\partial x} = 0$$

where A admits an ordered set of M real eigenvalues: $\lambda^1 \leq \lambda^2 \leq \ldots \leq \lambda^M$

We have M left and right eigenvectors of A

so that: $\; A \, r^m = \lambda^m \; r^m \; ; \; l^m \; A = l^m \lambda^m \quad \forall \;\; m=1,..,M$

Let the left and right eigenvectors be orthonormalized w.r.t. each other.

When the eigenvalues are degenerate we use Gram-Schmidt orthonormalization to obtain linearly independent eigenvectors.

The next three steps are purely formal but yields a very compact and useful notation that is used over and over in this field:

1) Let "R" be a matrix of right eigenvectors whose m^{th} column is given by r^m .

2) Let "L" be the matrix of left eigenvectors whose m^{th} row is given by l^m . We want $L\,R = I$, *i.e.* L is left inverse of R.

3) Define $\Lambda \equiv \text{diag}\left\{\lambda^1, \lambda^2, \ldots, \lambda^M\right\}$.

We then have:

$$\boxed{A\,R = R\,\Lambda \quad ; \quad L\,A = \Lambda\,L \quad ; \quad L\,A\,R = \Lambda \quad ; \quad A = R\,\Lambda\,L}$$

$$R = \begin{bmatrix} \boxed{r^1} & \boxed{r^2} & \cdots & \boxed{r^M} \end{bmatrix} \; ; \; L = \begin{bmatrix} \boxed{l^1} \\ \boxed{l^2} \\ \vdots \\ \boxed{l^M} \end{bmatrix} \; ; \; \Lambda = \begin{bmatrix} \lambda^1 & 0 & \bullet & 0 \\ 0 & \lambda^2 & \bullet & 0 \\ \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \lambda^M \end{bmatrix}$$

Fig. schematically shows the structure of the matrices R, L and Λ .

The whole purpose of the formal build-up so far is so that we can ***do dynamics***. I.e. , given a constant state V_0 and a small initial fluctuation $V_1(x)$ about it, we wish to predict the time-evolution of $V_1(x)$.

Left-multiply the evolution equation to get:

$$L\,\frac{\partial \mathrm{V}_1}{\partial t} + \left(L\,\mathrm{A}\,R\right)\,L\,\frac{\partial \mathrm{V}_1}{\partial x} = 0 \qquad \Rightarrow \qquad W_t + \Lambda\,W_x = 0 \quad \text{with} \quad W \equiv L\mathrm{V}_1$$

For the m^{th} component of W we have: $\boxed{w_t^m + \lambda^m w_x^m = 0 \quad \text{for } m = 1, \ldots, M}$

Now say that we start with initial conditions $\mathrm{V}_0 + \mathrm{V}_1(x)$ at $t = 0$.

The fluctuation in the m^{th} eigenweight at t=0 is given by: $w^m(x) \equiv l^m \bullet \mathrm{V}_1(x)$

At a later time, $t > 0$, in light of its evolution equation, the eigenweight is given by: $w^m\left(x - \lambda^m t\right)$

The time-dependent solution for $t > 0$ is given by:

$$\boxed{\mathrm{V}(x,t) = \mathrm{V}_0 + \sum_{m=1}^{M} w^m\left(x - \lambda^m t\right)\, r^m}$$

<u>V. Imp. Question</u>:Why is the material in this Sub-section so very useful?

$$\frac{\partial \mathrm{V}_1}{\partial t} + \mathrm{A}\,\frac{\partial \mathrm{V}_1}{\partial x} = 0 \qquad \rightarrow \qquad W_t + \Lambda\, W_x = 0 \quad \text{with} \quad W \equiv L\mathrm{V}_1$$

$$w_t^m + \lambda^m w_x^m = 0 \quad \text{for } m = 1, \ldots, M \qquad \text{with} \quad w^m\left(x\right) \equiv l^m \bullet \mathrm{V}_1\left(x\right) \text{ at } t = 0$$

1.5) Maxwell's Equations

Useful in non-linear optics, designing fiber optic cables. Also needed for designing stealth technology.

Written as:

$$\frac{\partial \mathbf{B}}{\partial t} + c \, \nabla \times \mathbf{E} = 0 \qquad \leftarrow \text{ Faraday's Law}$$

$$\frac{\partial \mathbf{D}}{\partial t} - c \, \nabla \times \mathbf{H} = - 4 \pi \mathbf{J} \qquad \leftarrow \text{ Generalized Ampere's Law}$$

$$\nabla \bullet \mathbf{D} = 4 \pi \rho \qquad \leftarrow \text{ Gauss's Law}$$

$$\nabla \bullet \mathbf{B} = 0 \qquad \leftarrow \text{ Divergence-Free Constraint}$$

Closure only obtained by constitutive relationships:-

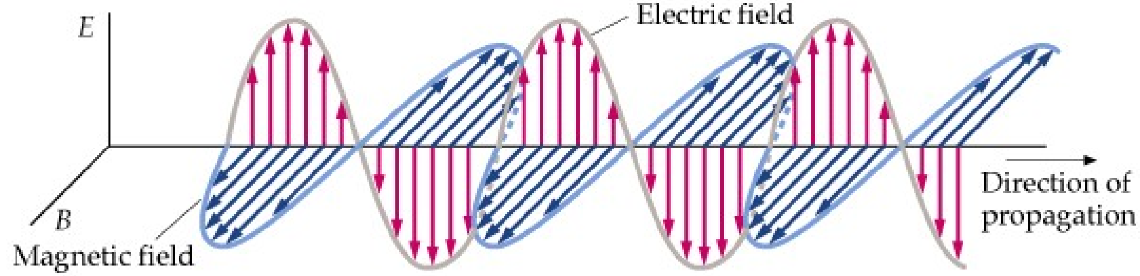
Relate magnetic induction to magnetic field : $\mathbf{B} = \mu \, \mathbf{H}$

Relate displacement vector to the electric field : $\mathbf{D} = \varepsilon \, \mathbf{E}$

Assume simple, linear scalar relations (material media require tensorial relationships).

$$\frac{\partial \mathbf{H}}{\partial t} + \frac{c}{\mu} \nabla \times \mathbf{E} = 0$$

$$\frac{\partial \mathbf{E}}{\partial t} - \frac{c}{\varepsilon} \nabla \times \mathbf{H} = - \frac{4\pi}{\varepsilon} \mathbf{J}$$



$$\nabla \bullet \mathbf{E} = \frac{4\pi}{\varepsilon} \rho$$

$\nabla \bullet \mathbf{B} = 0 \quad \leftarrow$ Constraint can be important to the fidelity of the solution process

Methods have also been developed to sweep any magnetic divergence off the mesh.

For now, we write the linear equations in a form that it designed to reveal the characteristic matrix:

$$\frac{\partial}{\partial t}\begin{pmatrix} E_x \\ E_y \\ E_z \\ B_x \\ B_y \\ B_z \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c/\varepsilon \\ 0 & 0 & 0 & 0 & -c/\varepsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -c/\mu & 0 & 0 & 0 \\ 0 & c/\mu & 0 & 0 & 0 & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} E_x \\ E_y \\ E_z \\ B_x \\ B_y \\ B_z \end{pmatrix} = 0$$

Question: What are the eigenvectors of this system telling us?

1.6) The Magnetohydrodynamic Equations

Very useful in terrestrial fusion expts., interiors of stars, solar wind, magnetospheres of stars and planets. Needed: large system + lo resistivity

When gas it hot enough, it becomes partially/fully ionized. Charged particles gyrate around magnetic fields ➔ *matter and field are tightly coupled*. Magnetohydrodynamics (***MHD***) is the simplest approximation.

Viewed over large enough length scales, plasma is neutral ➔ local charge imbalances are v. rapidly neutralized in plasma’s rest frame.

Viewed over large enough time scales, i.e. longer than plasma waves ➔

$$\partial \mathbf{D}/\partial t = 0$$

The B-field is locked in the plasma ➔ Lorenz-force acts in momentum eqn:

$$\rho \frac{D\,\mathbf{v}}{D\,t} = -\nabla P + \frac{1}{c}\left(\mathbf{J}\times\mathbf{B}\right) + \overleftrightarrow{\nabla}\pi \quad \leftarrow$$

Recall: current needed to sustain B-field!

We also need an evolutionary equation for B – field :

$$\frac{\partial \mathbf{B}}{\partial t} + c\,\nabla\times\mathbf{E} = 0 \quad \leftarrow \quad \text{Faraday's Law;} \qquad \mathbf{J} = \frac{c}{4\,\pi}\,\nabla\times\mathbf{B} \quad \leftarrow \text{ Ampere's Law}$$

In the fluid's (primed) rest frame we have Ohm's law: $\mathbf{J}^{/} = \sigma\ \mathbf{E}^{/}$

Lorenz Transform to the fluid's rest frame to get : $\mathbf{J}^{/} = \mathbf{J} \quad ; \quad \mathbf{E}^{/} = \mathbf{E} + \frac{1}{c}\left(\mathbf{v}\times\mathbf{B}\right)$

Ohm's Law in the Eulerian frame of reference then gives:

$$\mathbf{J}^{/} = \sigma\ \mathbf{E}^{/} \quad\Rightarrow\quad \mathbf{J} = \sigma\left(\mathbf{E} + \frac{1}{c}\,\mathbf{v}\times\mathbf{B}\right) \quad\Rightarrow\quad \mathbf{E} = -\frac{1}{c}\,\mathbf{v}\times\mathbf{B} + \frac{c}{4\pi\sigma}\,\nabla\times\mathbf{B}$$

This gives us our *evolutionary equation for the B - field* :

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = \nabla\times\left(\mathbf{v}\times\mathbf{B}\right) + \frac{c^2}{4\pi\sigma}\,\nabla^2\mathbf{B}}$$

<u>Question</u>: Interpret these 2 terms

Write the Lorenz force as : $\frac{1}{c}\left(\mathbf{J}\times\mathbf{B}\right) = -\frac{1}{4\pi}\,\mathbf{B}\times\left(\nabla\times\mathbf{B}\right) = -\nabla\left(\frac{\mathbf{B}^2}{8\pi}\right) + \frac{1}{4\pi}\,\left(\mathbf{B}\bullet\nabla\right)\mathbf{B}$

The momentum equation then gives : $$\boxed{\rho \frac{D\,\mathbf{v}}{D\,t} = -\nabla\left(\mathrm{P} + \frac{\mathbf{B}^2}{8\pi}\right) + \frac{1}{4\pi}\left(\mathbf{B}\bullet\nabla\right)\mathbf{B} + \overleftrightarrow{\nabla}\pi}$$

The full set of ideal MHD equations in Conservation form can then be written as:

$$\boxed{\begin{aligned}
&\frac{\partial\,\rho}{\partial\,t} + \frac{\partial}{\partial\,x_i}\left(\rho\,\mathrm{v}_i\right) = 0 \\
&\frac{\partial}{\partial\,t}\left(\rho \mathrm{v}_i\right) + \frac{\partial}{\partial\,x_j}\left(\rho \mathrm{v}_i \mathrm{v}_j + \left(\mathrm{P} + \mathbf{B}^2/8\pi\right)\delta_{ij} - \mathrm{B}_i\,\mathrm{B}_j/4\pi\right) = 0 \\
&\frac{\partial\,\varepsilon}{\partial\,t} + \frac{\partial}{\partial\,x_i}\left(\left(\varepsilon + \mathrm{P} + \mathbf{B}^2/8\pi\right)\mathrm{v}_i - \mathrm{B}_i\left(\mathbf{v}\cdot\mathbf{B}\right)/4\pi\right) = 0 \\
&\frac{\partial \mathbf{B}}{\partial t} = \nabla\times\left(\mathbf{v}\times\mathbf{B}\right) \qquad \text{with } \nabla\bullet\mathbf{B} = 0
\end{aligned}}$$

with $\varepsilon = \mathrm{e} + \frac{1}{2}\rho\,\mathbf{v}^2 + \frac{\mathbf{B}^2}{8\pi}$ and $\mathrm{e} \equiv \frac{\mathrm{P}}{\left(\Gamma - 1\right)}$

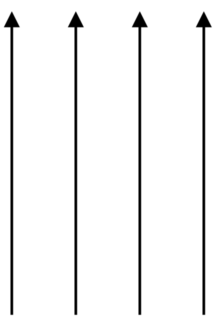
<u>Question:</u> Can you spot the magnetic energy density and Poynting flux on this page?

Flux Freezing Approximation for Ideal MHD

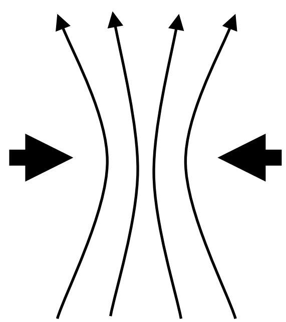
The magnetic fields are frozen into the fluid and move with the fluid in the ideal MHD limit.

Thus if the fluid is compressed transversely to the magnetic field direction, the magnetic fields get squished too. This gives us an extra magnetic pressure term. Hint: Think of $B^2 / 8\pi$ to see how it works.

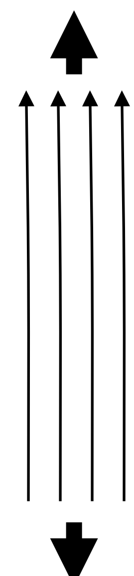
If the fluid is pulled longitudinal to the magnetic field direction, the magnetic fields also produce extra magnetic tensional forces, just like a bunch of rubber bands that are pulled on end.



Original field configuration



Magnetic pressure



Magnetic Tension

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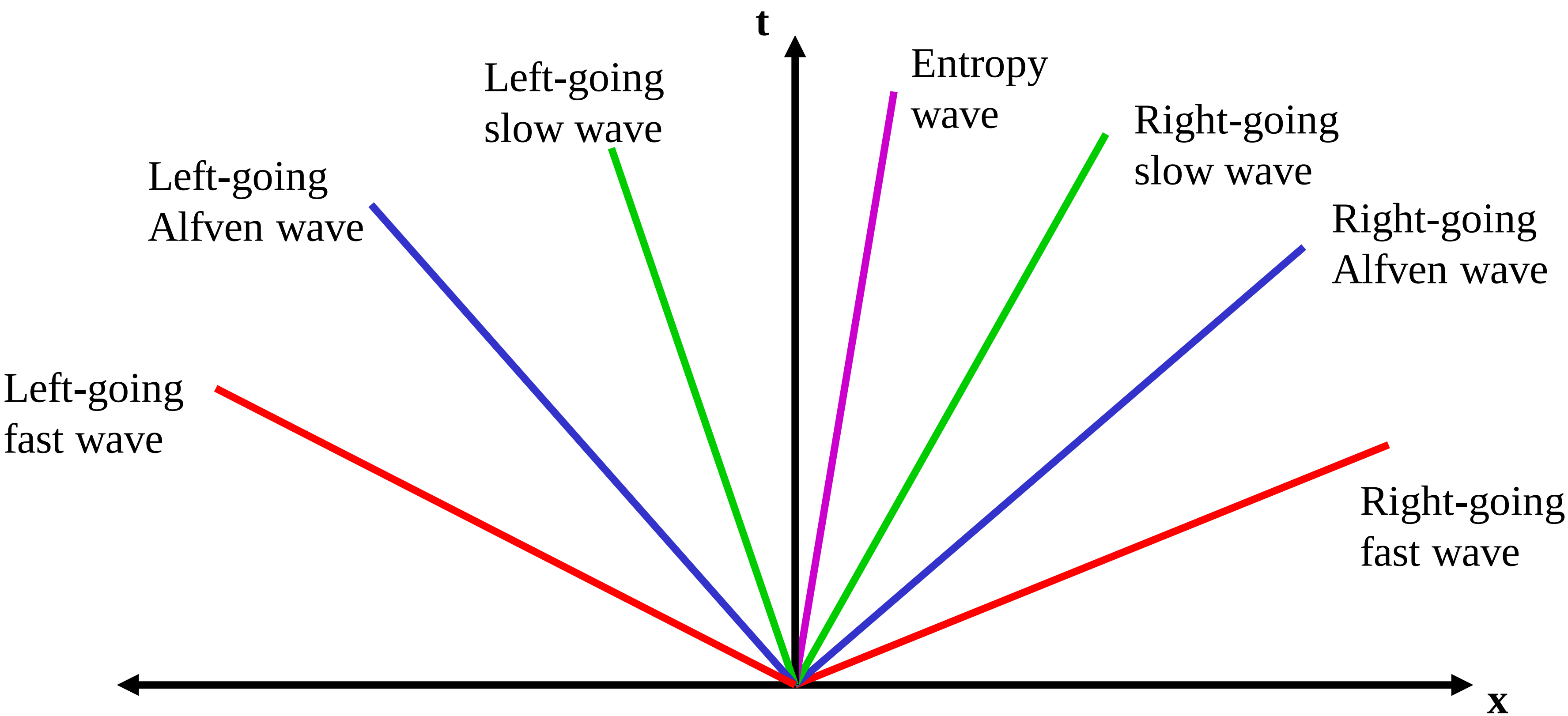
The equations of MHD have seven propagating wave families.

The Riemann problem (I.e. the cell-break problem) tells us how these families propagate at discontinuities. This eminently physical procedure is used to build some of the most robust and accurate numerical schemes for MHD.



Space-Time Diagram for MHD Waves

1.7) Flux Limited Diffusion (FLD) Radiation Hydrodynamics

It is tempting to build a "hydrodynamic" approximation for photons interacting with atoms. Question: What are the deficiencies in that? (Hint: Compare $\sigma_{Couloumb}$ to $\sigma_{Thompson}$.)

The Flux Limited Diffusion approximation does *not* solve this problem. It does, however, make it possible to arrive at a more tractable set of equations that can be solved.

Works best in the *optically thick regime*. Provides gracious breakdown in the *optically thin regime*. Question: What do these two regimes mean?

For some problems, we only care for the optically thick regime. FLD is ok in such situations because it assumes that photons *diffuse* through matter.

Mathematically: Assert that the *radiation energy density* E is the only variable of interest. Claim that *radiation flux* **F** and *radiation pressure tensor* **P** obtained from it. We will not derive, but give feel for equations.

$$\mathbf{F} = -\frac{c\lambda}{\kappa_{0R}} \nabla \mathrm{E} + \mathbf{v}\,\mathrm{E} + \mathbf{v} \bullet \mathbf{P}$$ $\leftarrow$ Interpret these terms ; λ is flux limiter.

Question: With κ_{0R} being the reciprocal of a mean free path, can you interpret c/κ_{0R}?

$$\mathbf{P} = \frac{\mathrm{E}}{2}\left[\left(1-\mathrm{R}_2\right)\mathbf{I} + \left(3\,\mathrm{R}_2 - 1\right)\mathbf{n}\otimes\mathbf{n}\right] \;\; ; \;\; \mathrm{R}_2 = \lambda + \lambda^2\,\mathrm{R}^2 \;\; ; \;\; \mathbf{n} = -\frac{\nabla \mathrm{E}}{\left|\nabla \mathrm{E}\right|} \;\; ; \;\; \mathrm{R} = \frac{\left|\nabla\,\mathrm{E}\right|}{\kappa_{0R}\,\mathrm{E}}$$

The final equations are:

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho\,\mathrm{v}_i\right) = 0$$

$$\frac{\partial}{\partial t}\left(\rho \mathrm{v}_i\right) + \frac{\partial}{\partial x_j}\left(\rho \mathrm{v}_i \mathrm{v}_j + \mathrm{P}\,\delta_{ij}\right) = -\lambda\,\frac{\partial\,\mathrm{E}}{\partial x_i}$$

$$\frac{\partial\,\varepsilon}{\partial t} + \frac{\partial}{\partial x_i}\left(\left(\varepsilon + \mathrm{P}\right)\mathrm{v}_i\right) = -\kappa_{0P}\left(4\pi\,\mathrm{B} - c\,\mathrm{E}\right) + \lambda\left(2\frac{\kappa_{0P}}{\kappa_{0R}} - 1\right)\mathbf{v}\bullet\nabla\mathrm{E} - \frac{3-\mathrm{R}_2}{2}\,\kappa_{0P}\,\frac{\mathbf{v}^2}{c}\,\mathrm{E}$$

$$\frac{\partial\,\mathrm{E}}{\partial t} + \frac{\partial}{\partial x_i}\left(\frac{3-\mathrm{R}_2}{2}\,\mathrm{E}\,\mathrm{v}_i\right) = \nabla\bullet\left(\frac{c\lambda}{\kappa_{0R}}\,\nabla\,\mathrm{E}\right)$$

$$+\,\kappa_{0P}\left(4\pi\,\mathrm{B} - c\,\mathrm{E}\right) - \lambda\left(2\frac{\kappa_{0P}}{\kappa_{0R}} - 1\right)\mathbf{v}\bullet\nabla\mathrm{E} + \frac{3-\mathrm{R}_2}{2}\,\kappa_{0P}\,\frac{\mathbf{v}^2}{c}\,\mathrm{E}$$

Question: Interpret the RHS terms. Interpret all the terms in the radiation energy equation.

1.8) Radiative Transfer

When the medium is not optically thick, photons do not propagate diffusively. We then have to treat the propagation of photons more carefully.

Photons can be absorbed or emitted by matter. (Give examples where this happens) They also scatter off the matter. (Question: Give terrestrial and stellar examples of scattering.)

In such situations, at each location "**x**", we study the propagation of photons in each direction "**Ω**".

The amount of radiant energy (in a frequency range ν to ν + dν) propagating per unit time through an infinitesimal area *dA* that is orthogonal to **Ω** is given by the *radiation intensity*: $I\left(\mathbf{x},\mathbf{\Omega},\nu,t\right)\ d\mathbf{\Omega}\ dA\ d\nu$

Question: How is this analogous to a distribution function for gas particles? Hint: for a photon, E = h ν = p c.

The *radiative transfer equation* for photons is the analogue of the collisional Boltzmann equation:

$$\frac{1}{c}\frac{\partial}{\partial t} I\left(\mathbf{x},\mathbf{\Omega},\nu,t\right) + \mathbf{\Omega} \bullet \nabla I\left(\mathbf{x},\mathbf{\Omega},\nu,t\right) = \kappa\left(\mathbf{x},\nu,t\right) I_b\left(\mathrm{T}\left(\mathbf{x},t\right),\nu\right)$$
$$-\left(\kappa\left(\mathbf{x},\nu,t\right)+\sigma\left(\mathbf{x},\nu,t\right)\right) I\left(\mathbf{x},\mathbf{\Omega},\nu,t\right)$$
$$+\frac{\sigma\left(\mathbf{x},\nu,t\right)}{4\pi}\int \Phi\left(\mathbf{\Omega},\mathbf{\Omega}'\right) I\left(\mathbf{x},\mathbf{\Omega}',\nu,t\right) d\mathbf{\Omega}'$$

Note: This is not just one equation for a single $\mathbf{\Omega}$, but rather a set of equations for the *ordinates*, $\mathbf{\Omega}$, spanning *all* directions.

The left hand side just says that photons stream freely in the absence of matter (and, therefore, in the absence of collisions with matter).

The right hand side contains the effect of photon-matter interaction. Question: Interpret each of the terms. Why does I_b depend on the temperature of matter? κ is the absorption opacity and σ is the scattering opacity. How do those terms differ in the right hand side?

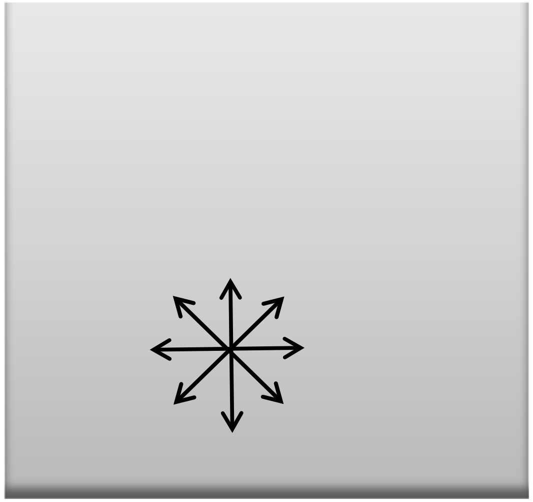
First simplification: Speed of light >> all other speeds. ➔ time-dependence can be dropped.

Second simplification: Solve for a small number of frequency bins – *picket fence approximation*. Alternatively, integrate over all frequencies – *gray approximation*.

Third simplification: Solve only for a small set of ordinate directions. For each integer "*N*" there are only *N (N + 2)* ordinates. Gives rise to an S_N method.

Fourth simplification: Oftentimes, the matter is assumed stationary.

$$
\begin{aligned}
\boldsymbol{\Omega}_i \bullet \nabla I\left(\mathbf{x},\boldsymbol{\Omega}_i,\nu\right) = &\ \kappa\left(\mathbf{x},\nu\right) I_b\left(\mathrm{T}\left(\mathbf{x}\right),\nu\right) \\
&-\left(\kappa\left(\mathbf{x},\nu\right)+\sigma\left(\mathbf{x},\nu\right)\right) I\left(\mathbf{x},\boldsymbol{\Omega}_i,\nu\right) \\
&+\frac{\sigma\left(\mathbf{x},\nu\right)}{4\pi} \sum_{j=1}^{N(N+2)} w_j\, \Phi\left(\boldsymbol{\Omega}_i,\boldsymbol{\Omega}_j\right) I\left(\mathbf{x},\boldsymbol{\Omega}_j,\nu\right)
\end{aligned}
$$



1.9) Relativistic Magnetohydrodynamics

(Rel-MHD) in Conservation form.(c=1)

1) Compare with non-relativistic case. Lab frame v/s rest frame.
2) Notice that the thermal energy and magnetic energy contribute to the inertia in the momentum equations. Similarly for Lorentz factor. Why?
3)The rest mass energy & magnetic energy contribute to the energy density.
4) Notice the promotion of the magnetic field to a 4-vector.
5) Notice that the induction equation remains unchanged owing to the fact that Maxwell's Equations are already Lorentz-invariant!

$$\frac{\partial}{\partial t}(\rho\,\gamma)+\frac{\partial}{\partial x_i}(\rho\,\gamma\,\mathrm{v}_i)=0$$

$$\frac{\partial}{\partial t}\left(\rho\,h\,\gamma^2\,\mathrm{v}_i + (\mathbf{E}\times\mathbf{B})_i\right) + \frac{\partial}{\partial x_j}\left(\rho\,h\,\gamma^2\,\mathrm{v}_i\,\mathrm{v}_j - \mathrm{E}_i\,\mathrm{E}_j - \mathrm{B}_i\,\mathrm{B}_j + \left(\mathrm{P} + \frac{1}{2}\left(\mathbf{E}^2+\mathbf{B}^2\right)\right)\gamma\,\delta_{ij}\right) = 0$$

$$\frac{\partial}{\partial t}\left(\rho\,h\,\gamma^2 - \mathrm{P} + \frac{1}{2}\left(\mathbf{E}^2+\mathbf{B}^2\right)\right) + \frac{\partial}{\partial x_i}\left(\rho\,h\,\gamma^2\,\mathrm{v}_i + (\mathbf{E}\times\mathbf{B})_i\right)=0$$

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) \quad ; \quad \nabla\bullet\mathbf{B}=0 \quad ; \quad \mathbf{E}=-\mathbf{v}\times\mathbf{B}$$

$$h = 1 + \mathrm{P}\,\Gamma\big/\left[\rho\,(\Gamma - 1)\right]$$