2.6.2) Stability Analysis for Time-Implicit and Semi-Implicit Linear Parabolic PDEs

Time-Implicit Scheme: $$\boxed{\frac{u_j^{n+1}-u_j^{n}}{\Delta t} = \sigma\left(\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{\Delta x^2}\right)} \Rightarrow$$

$$u_j^{n+1} - \mu\left(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\right) = u_j^{n}$$

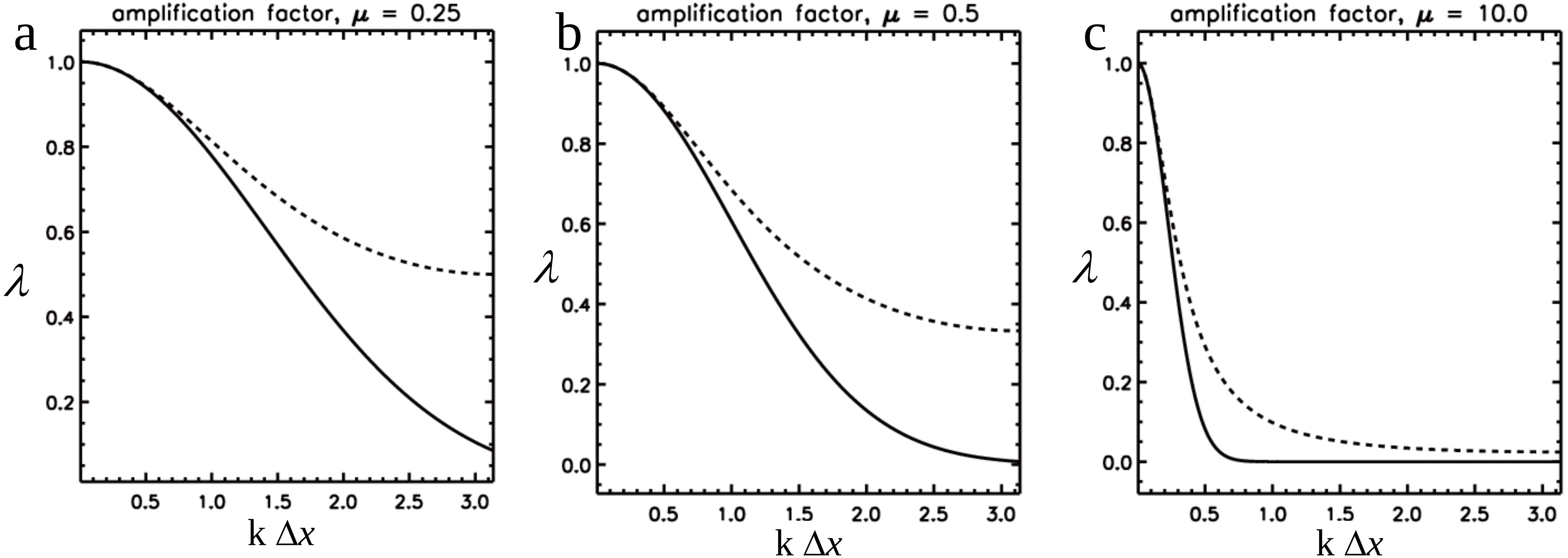
We get

$$\lambda_{FDA}(k) = \frac{U_k^{n+1}}{U_k^{n}} = \frac{1}{\left[1+4\,\mu\,\sin^2(k\,\Delta x/2)\right]}$$

← Question: What is this amplification factor telling you about stability?

The amplification factors are shown for $\mu = 0.25, 0.5, 10.0$.

Dashed curve : FDA Solid curve : PDE Question: What do you see?



$$\mathrm{u}_j^n = \mathrm{U}_\mathrm{k}^n \ \mathrm{e}^{i\ \mathrm{k}\, x_j} \quad ; \quad \mathrm{u}_j^{n+1} = \mathrm{U}_\mathrm{k}^{n+1} \ \mathrm{e}^{i\ \mathrm{k}\, x_j} \ \Rightarrow$$

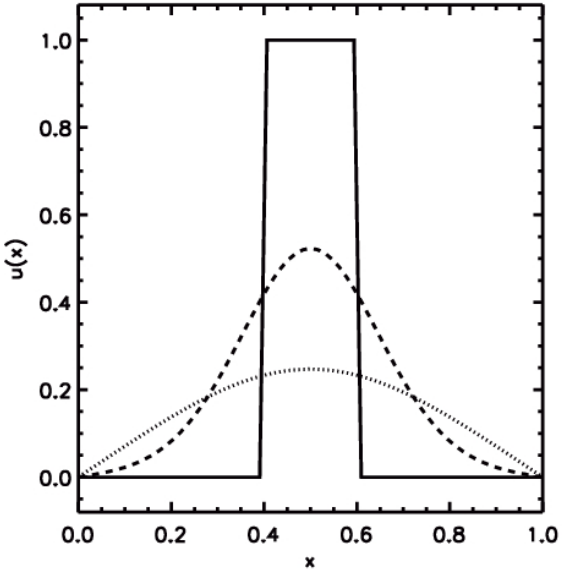
$$\mathrm{u}_{j+1}^n = \mathrm{U}_\mathrm{k}^n \ \mathrm{e}^{i\ \mathrm{k}\, x_j + i\ \mathrm{k}\, \Delta x} = \mathrm{U}_\mathrm{k}^n \ \mathrm{e}^{i\ \mathrm{k}\, x_j} \mathrm{e}^{i\ \mathrm{k}\, \Delta x} \text{ and } \mathrm{u}_{j-1}^n = \mathrm{U}_\mathrm{k}^n \ \mathrm{e}^{i\ \mathrm{k}\, x_j - i\ \mathrm{k}\, \Delta x} = \mathrm{U}_\mathrm{k}^n \ \mathrm{e}^{i\ \mathrm{k}\, x_j} \mathrm{e}^{-i\ \mathrm{k}\, \Delta x}$$

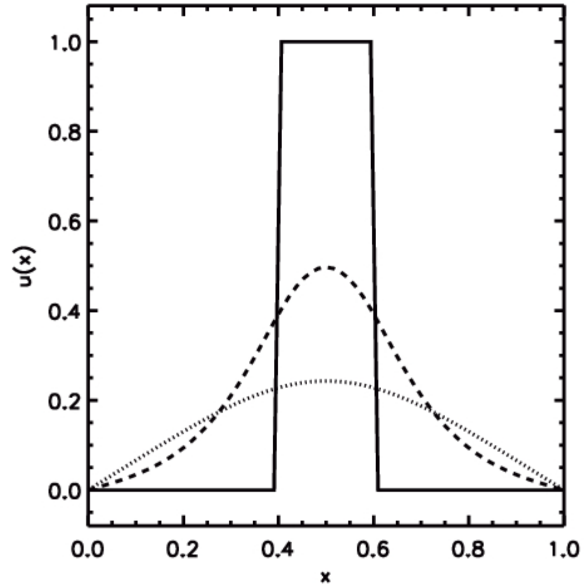
$$\mathrm{u}_j^{n+1} - \mu \left(\mathrm{u}_{j+1}^{n+1} - 2\mathrm{u}_j^{n+1} + \mathrm{u}_{j-1}^{n+1} \right) = \mathrm{u}_j^n \quad \Rightarrow$$

$$\lambda_{\mathrm{FDA}}(\mathrm{k}) = \frac{\mathrm{U}_\mathrm{k}^{n+1}}{\mathrm{U}_\mathrm{k}^n} = \frac{1}{\left[1 + 4\,\mu\, \sin^2\left(\mathrm{k}\, \Delta x / 2 \right) \right]}$$

Evolution of Square Pulse:

$\mu = 6.55$





$\mu = 32.75$

Semi-Implicit Scheme (Crank-Nicholson, α=1/2):

$$\boxed{\frac{u_j^{n+1} - u_j^n}{\Delta t} = \alpha\ \sigma \left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \right) + (1-\alpha)\ \sigma \left(\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} \right) \quad (0 \le \alpha \le 1)} \Rightarrow$$

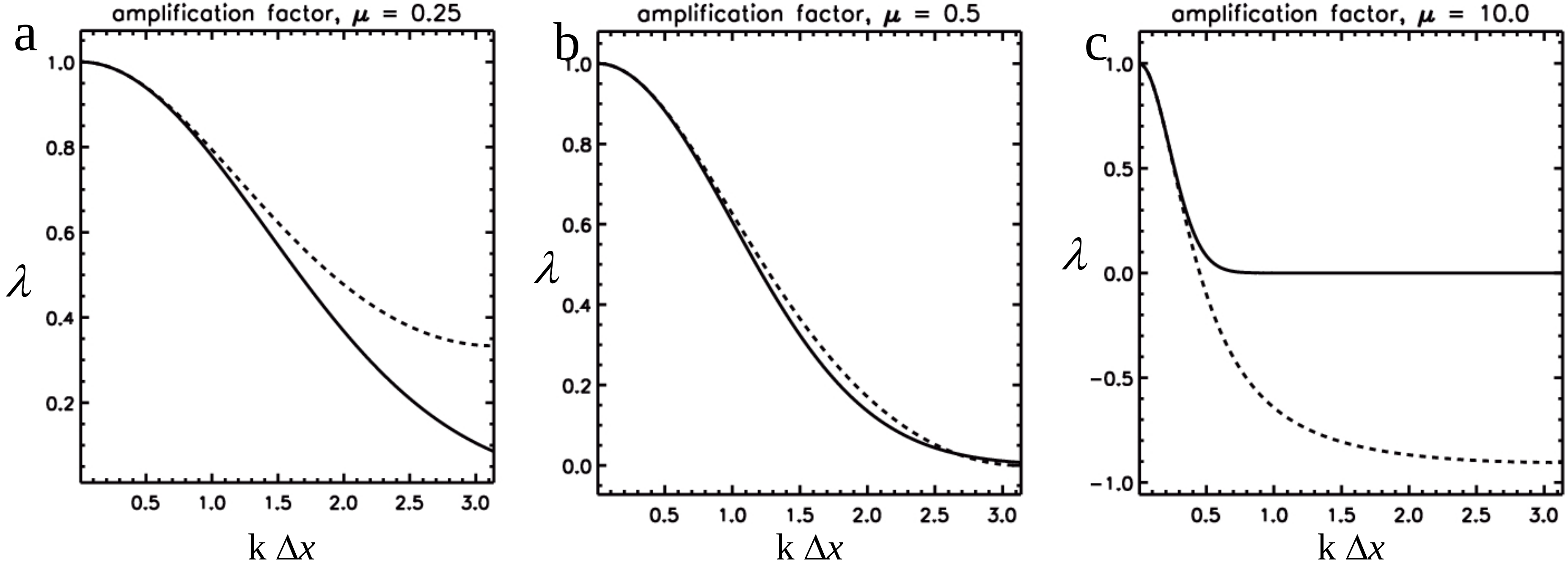
$$u_j^{n+1} - \mu(1-\alpha)\left(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\right) = u_j^n + \mu\alpha\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

We get

$$\lambda_{\mathrm{FDA}}(k) = \frac{U_k^{n+1}}{U_k^n} = \frac{\left[1 - 4\ \mu\ \alpha\ \sin^2\ (\, k\ \Delta x\ /\ 2)\right]}{\left[1 + 4\ \mu\ (1-\alpha)\ \sin^2\ (\, k\ \Delta x\ /\ 2)\right]}$$

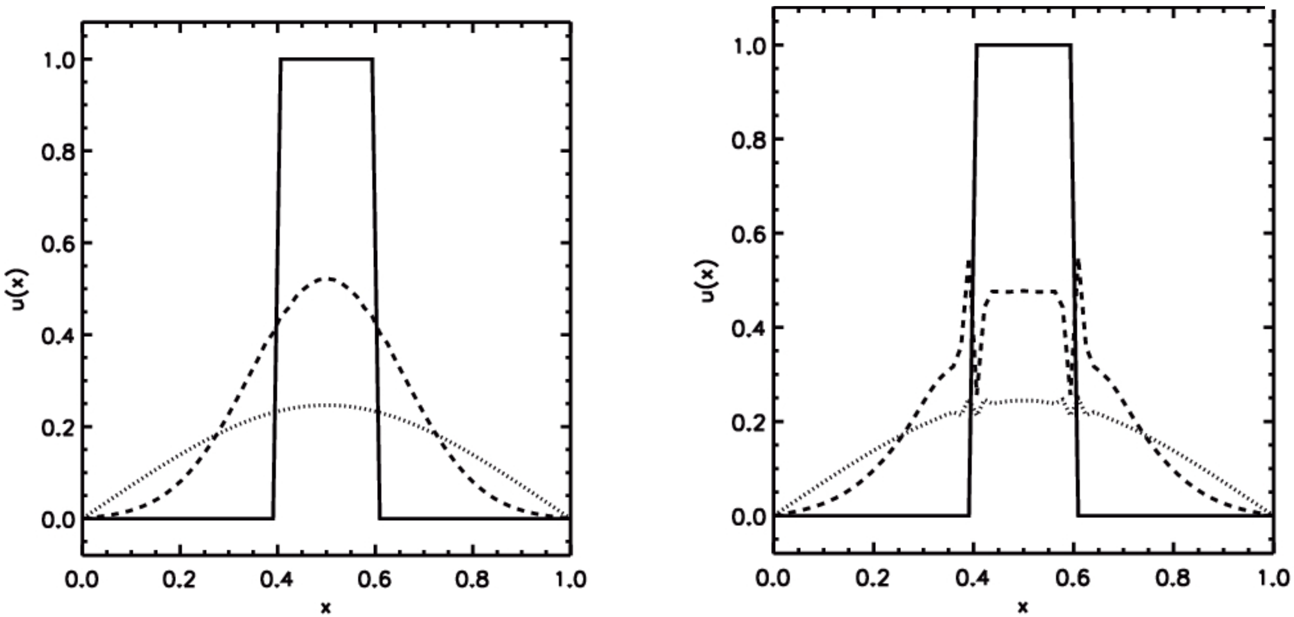
The amplification factors are shown for μ = 0.25, 0.5, 10.0. and α=1/2

Dashed curve : FDA   Solid curve : PDE   <u>Question</u>: What do you see?



Evolution of Square Pulse:

$\mu = 3.5$



$\mu = 10.0$

2.6.3) Stability Analysis of the Time-Implicit TR-BDF2 Method

The Crank-Nicholson scheme, despite being second order accurate, has the deficiency that it produces spurious oscillations.

Can one obtain a second order accurate scheme for parabolic problems that is free of these oscillations? Ans: If one is willing to invert the matrix twice, then the answer is yes!

One uses a **TR**apezoidal scheme for the first step which only takes us up to a time of $t^n + \Delta t/2$ from a time of t^n . This is written as:

$$u_j^{n+1/2} - \frac{\mu}{4}\left(u_{j+1}^{n+1/2} - 2u_j^{n+1/2} + u_{j-1}^{n+1/2}\right) = u_j^n + \frac{\mu}{4}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

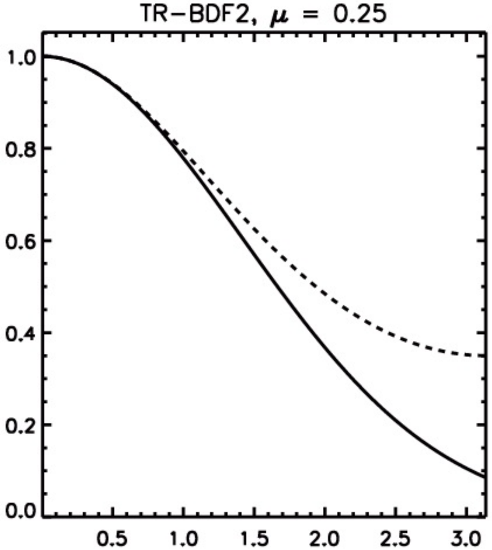
Using time levels t^n and $t^{n+1/2}$, we now use a **B**ackward **D**ifference **F**ormula of **2**nd order as:

$$u_j^{n+1} - \frac{\mu}{3}\left(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\right) = -\frac{1}{3}u_j^n + \frac{4}{3}u_j^{n+1/2}$$

Hence the name TR-BDF2. This scheme is also useful when stiff source terms are present in addition to the diffusion terms.

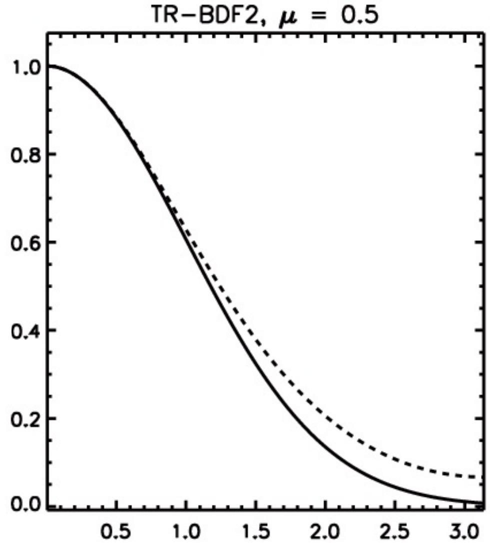
The amplification factors are shown for $\mu = 0.25, 0.5, 10.0.$ and TR-BDF2
Dashed curve : FDA   Solid curve : PDE   Question:What do you see?Compare with C-N.

λ



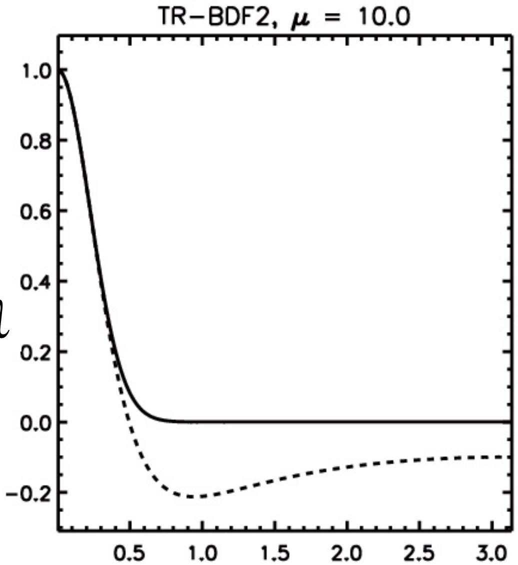
k Δx

λ



k Δx

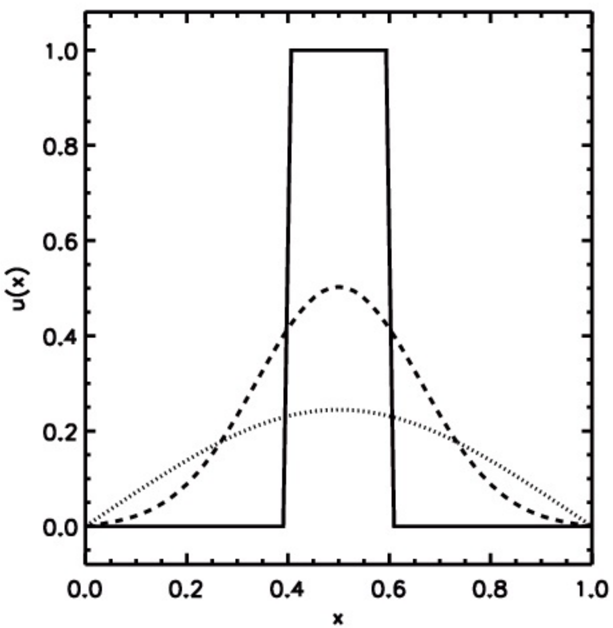
λ

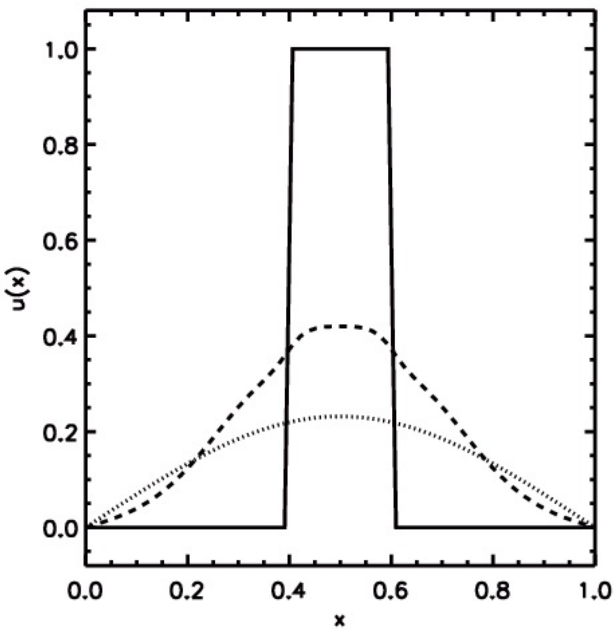


k Δx

Evolution of
Square Pulse:

$\mu = 6.55$





$\mu = 32.75$

2.6.4) Boundary Conditions for Parabolic Equations

Our *parabolic FDA* looks very much like the elliptic *Poisson equation*.

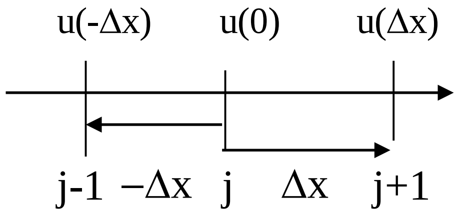
There is a theorem which states that for the Poisson problem we can either specify the value of the potential at the boundary or specify the gradient of the potential at the boundary. However, we can never specify the value of the potential and its gradient at a boundary.

For parabolic equations, the boundary conditions can change in time, but the same restrictions apply – we can't specify variable and its gradient at a boundary at any given time.

The boundary conditions we used in our previous example are called **Dirichlet** boundary conditions and consist of specifying the solution at the boundary of the domain.

Specifying the gradient gives us **Neumann** boundary conditions.

We may also specify a linear combination of the potential and its gradient, known as **mixed** boundary conditions.



$$a_l \; u_x + b_l \; u = c_l \quad ; \quad a_r \; u_x + b_r \; u = c_r$$

We may also require the boundary conditions to be **periodic**, which changes the dimension of the resulting matrix when implicit/semi-implicit formulations are used.

2.6.5) Introduction to Matrix Methods for Parabolic Equations

Consider the fully-implicit formulation on a 1d mesh. The mesh points are indexed from j=0 to j=J At the boundaries one can have the most general form of mixed boundary conditions by discretizing the boundary conditions as:

$$(b_l \Delta x - a_l)\, u_0^{n+1} + a_l \; u_1^{n+1} = c_l \Delta x \quad ; \quad -a_r u_{J-1}^{n+1} + (b_r \Delta x + a_r)\, u_J^{n+1} = c_r \Delta x$$

In the interior we have the FDA:

$$-\mu \; u_{j+1}^{n+1} + \left(1+2\mu\right) u_j^{n+1} - \mu \; u_{j-1}^{n+1} = u_j^n \quad \text{for} \quad j=1,..,J-1$$

Solve : $\mathrm{u}_j^{n+1} - \mu\left(\mathrm{u}_{j+1}^{n+1} - 2\mathrm{u}_j^{n+1} + \mathrm{u}_{j-1}^{n+1}\right) = \mathrm{u}_j^n \quad \text{for } 1 \le j \le J-1$

with boundary conditions : $\mathrm{a}_l \ \mathrm{u}_x + \mathrm{b}_l \ \mathrm{u} = \mathrm{c}_l$ at $j = 0$; $\mathrm{a}_r \ \mathrm{u}_x + \mathrm{b}_r \ \mathrm{u} = \mathrm{c}_r$ at $j = J$

The result is a ***banded sparse matrix*** with dimension $(J+1)\times(J+1)$:

$$\begin{bmatrix} b_l\Delta x - a_l & a_l & & & & \\ -\mu & (1+2\mu) & -\mu & & & \\ & -\mu & (1+2\mu) & -\mu & & \\ & & \cdots & \cdots & \cdots & \\ & & & -\mu & (1+2\mu) & -\mu \\ & & & & -a_r & b_r\Delta x + a_r \end{bmatrix} \begin{bmatrix} u_0^{n+1} \\ u_1^{n+1} \\ u_2^{n+1} \\ . \\ u_{J-1}^{n+1} \\ u_J^{n+1} \end{bmatrix} = \begin{bmatrix} c_l\Delta x \\ u_1^n \\ u_2^n \\ . \\ u_{J-1}^n \\ c_r\Delta x \end{bmatrix}$$

Such matrices also arise when discretizing elliptic and parabolic equations in multiple dimensions. For 2d problems we have the form: