Numerical PDE Techniques for Scientists and Engineers

By

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Please visit:-
http://www.nd.edu/~dbalsara/Numerical-PDE-Course

Chp 1 : Overview of PDEs of Relevance to Science and Engineering

By

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1.1) Introduction

Precise solutions needed for problems in science, engineering and applied math.

Many of these problems governed by *partial differential equations* (PDEs).

Analytical solutions to PDEs, few and limited.

Very effective *numerical solution techniques* are now available.

Powerful computers make it possible to obtain solutions to large, real-world problems.

Algorithms make it happen. They apply to broad classes of PDEs, not to a specific PDE. Learn general classes of algorithms and you can solve broad classes of PDEs. Emphasis on *theory and technique*.

Broad classes of PDEs of interest (with pedestrian introductions):

Hyperbolic PDEs : Enable information to propagate as waves.
Examples: Water waves, sound waves, oscillations in a solid structure and electromagnetic radiation.

Parabolic PDEs : Enable information to travel as diffusive processes.
Examples: heat transfer, mass diffusion in the ground, diffusion of photons out of the sun.

Elliptic PDEs : Don't have time variation, convey action at a distance.
Examples: Gravitational field, electrostatics.

We first study solution techniques for these PDEs piecemeal and then learn how to assemble them together for more complex PDEs.

Since we all have most direct experience with fluids, we will use the *Navier Stokes equations* as our motivating example. Several other PDEs of practical interest are also introduced in this chapter.

1.2) Derivation of the Fluid Equations

While this topic is well-known to some and not at all to others, it helps to bring everyone up to speed. *Navier Stokes equations* will serve as our *touchstone system* because it has all the ingredients that we want to study.

We do not dwell on details; we just provide as much intuitive background as is needed for understanding the nature of the PDEs.

Continuum behavior on the macroscopic scales arises from discrete molecular dynamics on the microscopic scales. Gives insight into PDEs.

1.2.1) Importance of Collisions

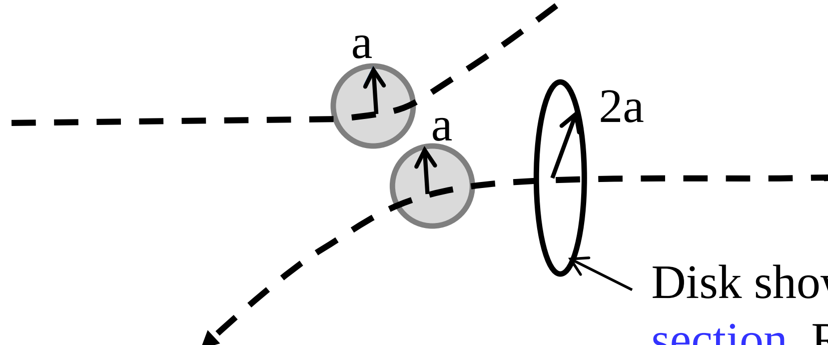
There are about 10^{19} molecules of air in a 1 cm^3 volume of air in this room. Simplifying assumption – single type of molecule.

Each molecule would minimally have a position vector **x** and a momentum vector **p** . Describe by a distribution function $f(\mathbf{x},\mathbf{p},t)$.
(Note $f(\mathbf{x},\mathbf{p},t)\, d^3\mathbf{p}$ has units of *number of particles per unit volume*.)
Problem: The *degrees of freedom* are too large;would soon overwhelm us.

Molecules make very frequent *collisions* with one another. The collisions *totally scramble the initial momentum* of the colliding particles.
Important Question: How quickly does this scrambling take place?

Fortunately, once collisions have acted, *statistical mechanics* enables us to make a profound simplification.

Let n be the number density of molecules and σ be their cross section. The *mean free path*, l, is the distance each molecule can coast before it collides with another: $l \sim 1/(n\sigma)$; $l \sim 10^{-4}$ cm for air in this room.
a~ 10^{-8} cm, $\sigma \sim 10^{-15}$ cm^2 for molecules.



Disk showing collision cross section. Radius of "spherical" molecule is "a" Area of disk is $\sigma=\pi\,(2a)^2$.

Particles in a gas move with a velocity that is comparable to the speed of sound in the gas.

Let the particle speed be w^T . For air in this room , $w^T \sim 3\times10^4$ cm/sec.

The time between collisions is given by $\tau \sim l/ w^T$. Again for air in this room we have $\tau \sim 3\times10^{-9}$ sec.

The ***fluid approximation*** then holds if the length of the system L >> l and the timescales over which we observe the system are T >> τ .

In all systems where the gas meets this scaling, we may easily use the fluid approximation. It is easily met for air in this room.

The fluid approximation then guarantees that the *local velocity distribution* of the molecules is almost *Gaussian* (recall statistical mechanics). This is a tremendous simplification in our distribution function : $f(\mathbf{x},\mathbf{p},t)$. The *temperature* and *density* then govern this distribution as just two independent parameters in the fluid's rest frame.

Mass conservation and the fact that molecules are neither created or destroyed enable us to write an expression for the *density*:

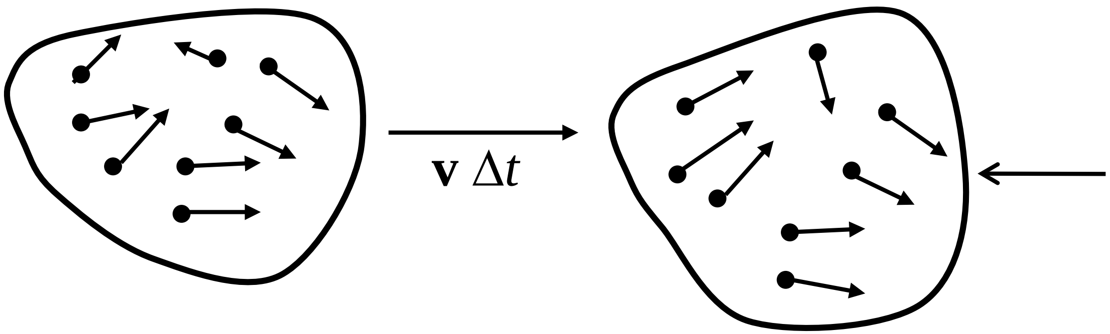
$$\rho(\mathbf{x},t) \equiv \int m\, f(\mathbf{x},\mathbf{p},t)\, d^3p$$

← Remember $f(\mathbf{x},\mathbf{p},t)\,d^3p$ has units of number density.

The frequent collisions ensure that molecules that form a parcel of gas do not stray too far from each other. This enables us to define a concept of bulk *fluid velocity,* **v** (check these equations for dimensional consistency):

$$\rho(\mathbf{x},t)\,\mathbf{v}(\mathbf{x},t) \equiv \int \mathbf{p}\, f(\mathbf{x},\mathbf{p},t)\, d^3p$$

Thus the *total velocity* **u** of a particle is made up of its *mean velocity* **v** plus its *fluctuation* from the mean, **w** . I.e. $\mathbf{u} = \mathbf{v} + \mathbf{w}$.



Frequent collisions keep particles bunched up.

Availability of control volume permits us to apply Newton's law to the *same* particles.

The energy of a single particle of mass m is then given by $\mathbf{p}^2/(2m)$.
Thus we write the *total energy density* as:

$$\mathcal{E}(\mathbf{x},t) \equiv \int \frac{\mathbf{p}^2}{2m} f(\mathbf{x},\mathbf{p},t)\, d^3p$$

Since the fluctuating velocity satisfies a *Maxwell-Boltzmann distribution* up to a reasonable approximation (though not valid for non-ideal effects), we can write the distribution function as:

$$f(\mathbf{x},\mathbf{p},t) = \frac{\rho(\mathbf{x},t)}{m} \frac{1}{\left(2\,\pi\, m\, k\, \mathrm{T}(\mathbf{x},t)\right)^{3/2}}\, \mathrm{e}^{-\frac{(\mathbf{p}-m\,\mathbf{v}(\mathbf{x},t))^2}{2\,m\,k\,\mathrm{T}(\mathbf{x},t)}}$$

Notice: we are in the local rest frame of the fluid.

The total energy density can now be shown to be a sum of the internal energy density and the kinetic energy density:

$$\mathcal{E}(\mathbf{x},t) = \frac{\mathrm{P}(\mathbf{x},t)}{\Gamma - 1} + \frac{1}{2}\rho(\mathbf{x},t)\,\mathbf{v}(\mathbf{x},t)^2$$

Thermodynamics then gives *equation of state* : $\mathrm{P} = \dfrac{R\rho\mathrm{T}}{\mu} \quad ; \quad \mathrm{e} = \dfrac{\mathrm{P}}{\Gamma - 1}$

1.2.2) The Boltzmann Equation

Without collisions, particles coast along their original trajectories, while responding to external forces **F** . The dynamics is quite uninteresting:

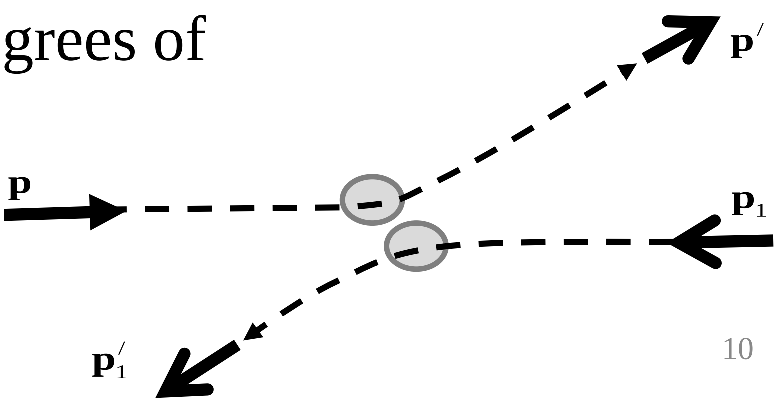
$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \bullet \frac{\partial f}{\partial \mathbf{x}} + \mathbf{F} \bullet \frac{\partial f}{\partial \mathbf{p}} = 0$$

← Write it out in component form to make sure you understand it.

With collisions, we introduce a very interesting dynamics:

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \bullet \frac{\partial f}{\partial \mathbf{x}} + \mathbf{F} \bullet \frac{\partial f}{\partial \mathbf{p}} = \left(\frac{\delta f}{\delta t}\right)_c$$

How a collision can scramble the phase space is constrained by momentum and energy conservation, consider the constraints in the diagram below. Question: How many degrees of freedom does the collision have?



Remaining d.o.f. governed by scattering cross-section : $\sigma\left(\mathbf{p},\mathbf{p}_1 \,|\, \mathbf{p}',\mathbf{p}_1'\right)$

Let incident particle be **p** and target particle be $\mathbf{p}_1$. *Number density* of particles that scatter ***out*** of phase space (**x**,**p**) *in a unit time* is given by:

$$\int \left(\frac{\delta f}{\delta t} \right)_{\text{c; sink}} \mathrm{d}^3\mathrm{p} = \int \underbrace{\frac{|\mathbf{p} - \mathbf{p}_1|}{m}}_{\text{cm/sec}} \underbrace{f(\mathbf{x}, \mathbf{p}_1, t)\, \mathrm{d}^3\mathrm{p}_1}_{\#/\text{cm}^3} \underbrace{\sigma(\mathbf{p}, \mathbf{p}_1 \,|\, \mathbf{p}', \mathbf{p}_1')\, \mathrm{d}\Omega}_{\text{cm}^2} \underbrace{f(\mathbf{x}, \mathbf{p}, t)\, \mathrm{d}^3\mathrm{p}}_{\#/\text{cm}^3}$$

Number density of particles that scatter ***into*** phase space (**x**,**p**) *per unit time* is given by:

$$\int \left(\frac{\delta f}{\delta t} \right)_{\text{c; source}} \mathrm{d}^3\mathrm{p} = \int \frac{|\mathbf{p}' - \mathbf{p}_1'|}{m} f(\mathbf{x}, \mathbf{p}_1', t)\, \mathrm{d}^3\mathrm{p}_1'\, \sigma(\mathbf{p}', \mathbf{p}_1' \,|\, \mathbf{p}, \mathbf{p}_1)\, \mathrm{d}\Omega\, f(\mathbf{x}, \mathbf{p}', t)\, \mathrm{d}^3\mathrm{p}'$$

Thus the total effect of the collision terms gives us the final ***Boltzmann equation*** (Question: Can you dimensionally analyze the RHS?)

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \bullet \frac{\partial f}{\partial \mathbf{x}} + \mathbf{F} \bullet \frac{\partial f}{\partial \mathbf{p}} =$$

$$\int \frac{|\mathbf{p} - \mathbf{p}_1|}{m} \sigma(\mathbf{p}, \mathbf{p}_1 \,|\, \mathbf{p}', \mathbf{p}_1') \left[f(\mathbf{x}, \mathbf{p}_1', t)\, f(\mathbf{x}, \mathbf{p}', t) - f(\mathbf{x}, \mathbf{p}_1, t)\, f(\mathbf{x}, \mathbf{p}, t) \right] \mathrm{d}\Omega\, \mathrm{d}^3\mathrm{p}_1$$

1.2.3) Moments of the Boltzmann Equation

Consider the definitions: $\rho(\mathbf{x},t) \equiv \int m\, f(\mathbf{x},\mathbf{p},t)\, d^3p \;\; ; \;\; \rho(\mathbf{x},t)\,\mathbf{v}(\mathbf{x},t) \equiv \int \mathbf{p}\, f(\mathbf{x},\mathbf{p},t)\, d^3p \;\; ;$

$$\varepsilon(\mathbf{x},t) \equiv \int \frac{\mathbf{p}^2}{2m}\, f(\mathbf{x},\mathbf{p},t)\, d^3p$$

We see that they are just moments of the distribution function. Boltzmann equation gave us dynamics, but not quite in the variables we need.

Consequently, to obtain *time-evolutionary equations* of the above *fluid variables*, take the following moments of the Boltzmann equation:

$$\psi(\mathbf{p}) = m \quad ; \quad \psi(\mathbf{p}) = \mathbf{p} \quad ; \quad \psi(\mathbf{p}) = \frac{\mathbf{p}^2}{2m}$$

We get

$$\int \left(\psi(\mathbf{p})\, \frac{\partial f}{\partial t} + \psi(\mathbf{p})\, \frac{\mathbf{p}}{m} \bullet \frac{\partial f}{\partial \mathbf{x}} + \psi(\mathbf{p})\, \mathbf{F} \bullet \frac{\partial f}{\partial \mathbf{p}} \right) d^3p = \int \psi(\mathbf{p}) \left(\frac{\delta f}{\delta t} \right)_c d^3p$$

Derivatives and integrals can be interchanged for the first two terms. The third term requires a little bit of deft manipulation, as we will show.

Question: Why does the last term, i.e. RHS, integrate to zero?

We have the following supporting definition for the averaging process:

$$\langle \psi \rangle \equiv \frac{1}{n} \int \psi(\mathbf{p}) \; f(\mathbf{x},\mathbf{p},t)\, d^3 p \quad ; \quad n \equiv \int f(\mathbf{x},\mathbf{p},t)\, d^3 p$$

$$\int\left(\psi(\mathbf{p})\ \frac{\partial f}{\partial t} + \psi(\mathbf{p})\ \frac{\mathbf{p}}{m}\bullet\frac{\partial f}{\partial \mathbf{x}} + \psi(\mathbf{p})\ \mathbf{F}\bullet\frac{\partial f}{\partial \mathbf{p}}\right) d^3p = \int\psi(\mathbf{p})\ \left(\frac{\delta f}{\delta t}\right)_{\mathrm{c}}\ d^3p$$

$$\boxed{\frac{\partial}{\partial t}\left(n\langle\psi\rangle\right) + \nabla\bullet\left(n\left\langle\frac{\mathbf{p}}{m}\psi\right\rangle\right) - n\,\mathbf{F}\bullet\left\langle\nabla_{\mathbf{p}}\psi(\mathbf{p})\right\rangle = 0}$$

The net result is

$$\boxed{\frac{\partial}{\partial t}\left(n\langle\psi\rangle\right) + \nabla \bullet \left(n\left\langle \frac{\mathbf{p}}{m}\psi \right\rangle \right) - n\,\mathbf{F} \bullet \left\langle \nabla_{\mathbf{p}}\psi(\mathbf{p}) \right\rangle = 0}$$

The averages in the above equation require the supporting definitions :

$$\langle\psi\rangle \equiv \frac{1}{n}\int \psi(\mathbf{p})\; f(\mathbf{x},\mathbf{p},t)\, d^3p \quad ; \quad n \equiv \int f(\mathbf{x},\mathbf{p},t)\, d^3p$$

This is known as ***Chapman-Enskog theory*** for fluids or the ***BBGKY hierarchy***.

1.2.4) The Continuity Equation

Set $\psi(\mathbf{p}) = m$ in $\frac{\partial}{\partial t}\left(n\langle\psi\rangle\right) + \nabla \bullet \left(n\left\langle \frac{\mathbf{p}}{m}\psi \right\rangle\right) - n\,\mathbf{F} \bullet \left\langle \nabla_{\mathbf{p}}\psi(\mathbf{p})\right\rangle = 0$

to get :
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho\, \mathrm{v}_i\right) = 0 \quad \Leftrightarrow \quad \frac{\partial \rho}{\partial t} + \nabla \bullet \left(\rho\, \mathbf{v}\right) = 0$$

Notice, very importantly, that this is a *conservation form*. Question: Can you think of other equations in conservation form? What does an equation in conservation form tell us? What are the units of mass flux?

An equivalent non-conservative, i.e. *primitive*, form :

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \bullet \nabla\rho = -\rho\, \nabla \bullet \mathbf{v} \quad \Leftrightarrow \quad \frac{D\rho}{Dt} \equiv \frac{\partial \rho}{\partial t} + \mathbf{v} \bullet \nabla\rho = -\rho\, \nabla \bullet \mathbf{v}$$

Question: When is the *primitive form* useful? What is the import of the RHS?

$$\boxed{\frac{\partial}{\partial t}\left(n\langle\psi\rangle\right) + \nabla \bullet \left(n\left\langle\frac{\mathbf{p}}{m}\psi\right\rangle\right) - n\,\mathbf{F} \bullet \left\langle\nabla_{\mathbf{p}}\psi(\mathbf{p})\right\rangle = 0 \quad \text{with } \psi(\mathbf{p}) = m}$$

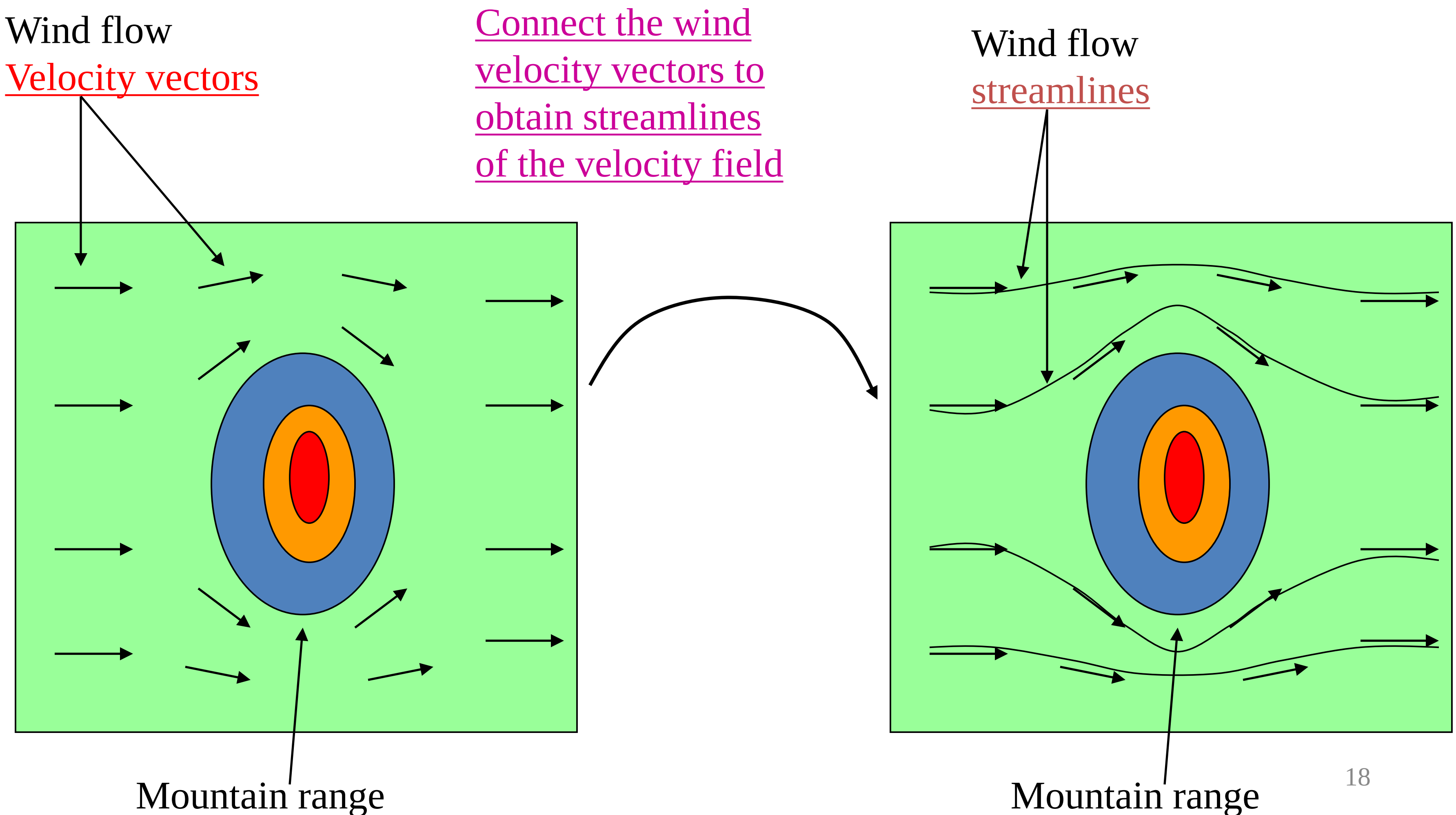
$$\frac{\partial \rho}{\partial t} + \nabla \bullet \left(\rho\, \mathbf{v}\right) = 0$$

The *Lagrangian derivative / material derivative*, shown above, has a form that occurs very often in fluid dynamics:

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \bullet \nabla$$

It is very important to intuitively understand the *Lagrangian derivative* .

<u>Question</u>: What is it tracing?



1.2.5) The Momentum Equation

Set $\psi(\mathbf{p}) = \mathbf{p} = m\,(\mathbf{v} + \mathbf{w})$ ($\mathbf{v}$ == mean velocity; $\mathbf{w}$ == fluctuation) to get:

$$\frac{\partial}{\partial t}(\rho \mathrm{v}_i) + \frac{\partial}{\partial x_j}\left(\rho\left\langle(\mathrm{v}_i + \mathrm{w}_i)(\mathrm{v}_j + \mathrm{w}_j)\right\rangle\right) = \rho\, \mathrm{a}_i$$

Notice now that : $\left\langle(\mathrm{v}_i + \mathrm{w}_i)(\mathrm{v}_j + \mathrm{w}_j)\right\rangle = \mathrm{v}_i \mathrm{v}_j + \left\langle \mathrm{w}_i \mathrm{w}_j \right\rangle$

Question: What can we say about the correlation between the fluctuating parts of the velocity? Describe situations where you might expect a correlation.

Using our definition of pressure : $\mathrm{P} \equiv \frac{1}{3}\rho(\mathbf{x},t)\int \frac{(\mathbf{p} - m\,\mathbf{v}(\mathbf{x},t))^2}{m^2} f(\mathbf{x},\mathbf{p},t)\, d^3p \equiv \frac{1}{3}\rho\left\langle \mathbf{w}^2 \right\rangle$
we get:

$$\rho\left\langle \mathrm{w}_i \mathrm{w}_j \right\rangle = \mathrm{P}\,\delta_{ij} - \pi_{ij} \quad ; \quad \pi_{ij} \equiv \rho\left\langle \frac{1}{3}\mathbf{w}^2\,\delta_{ij} - \mathrm{w}_i \mathrm{w}_j \right\rangle \quad \leftarrow \text{viscous stress}$$

$$\frac{\partial}{\partial t}\left(n\langle\psi\rangle\right) + \nabla \bullet \left(n\left\langle\frac{\mathbf{p}}{m}\psi\right\rangle\right) - n\,\mathbf{F} \bullet \left\langle\nabla_{\mathbf{p}}\psi\left(\mathbf{p}\right)\right\rangle = 0 \quad \text{with } \psi\left(\mathbf{p}\right) = \mathbf{p} = m\left(\mathbf{v} + \mathbf{w}\right)$$

$$\frac{\partial}{\partial t}\left(\rho \mathrm{v}_i\right) + \frac{\partial}{\partial x_j}\left(\rho\left\langle\left(\mathrm{v}_i + \mathrm{w}_i\right)\left(\mathrm{v}_j + \mathrm{w}_j\right)\right\rangle\right) = \rho\, \mathrm{a}_i$$

The final *momentum equation in conserved form* then becomes:

$$\frac{\partial}{\partial t}\left(\rho \mathrm{v}_i\right) + \frac{\partial}{\partial x_j}\left(\rho \mathrm{v}_i \mathrm{v}_j + \mathrm{P}\,\delta_{ij} - \pi_{ij}\right) = \rho\,\mathrm{a}_i$$

Here a_i is an acceleration, $\mathbf{a} = \mathbf{F}/m$.

It is an interesting trick to learn how to find the corresponding *primitive form of the momentum equation*:

$$\rho \frac{D\,\mathrm{v}_i}{D\,t} = -\frac{\partial\,\mathrm{P}}{\partial\,x_i} + \rho\,\mathrm{a}_i + \frac{\partial}{\partial\,x_j}\pi_{ij}$$

<u>Question</u>: Which one of Mr. Newton’s 3 laws are you reminded of?

With viscous terms, this is known as the ***Navier Stokes equations***; without the viscous terms, we call it the ***Euler equations***.

1.2.6) The Energy Equation

Set $\psi(\mathbf{p}) = m\,(\mathbf{v} + \mathbf{w})^2 / 2$ to get an equation for the total energy density $\varepsilon = \mathrm{e} + \rho\,\mathbf{v}^2/2$. We get:

$$\frac{\partial\,\varepsilon}{\partial\,t} + \frac{\partial}{\partial\,x_i}\left(\frac{\rho}{2}\left\langle\left(\mathrm{v}_i + \mathrm{w}_i\right)\left(\mathbf{v} + \mathbf{w}\right)^2\right\rangle\right) = \rho\,\mathrm{v}_i\;\mathrm{a}_i$$

<u>Question</u>: Can you interpret the RHS intuitively? On dimensional grounds, what does it tell you?

As before, we have to interpret the angled brackets as follows:

$$\frac{\rho}{2}\left\langle\left(\mathrm{v}_i + \mathrm{w}_i\right)\left(\mathbf{v} + \mathbf{w}\right)^2\right\rangle = \underbrace{\frac{\rho}{2}\,\mathbf{v}^2}_{\text{Kinetic Energy Density}}\,\mathrm{v}_i + \underbrace{\rho\left\langle\frac{1}{2}\,\mathbf{w}^2\right\rangle}_{\text{Thermal Energy Density}}\mathrm{v}_i + \underbrace{\rho\left\langle\mathrm{w}_i\,\mathrm{w}_j\right\rangle}_{\mathrm{P}\,\delta_{ij} - \pi_{ij}}\,\mathrm{v}_j + \underbrace{\rho\left\langle\mathrm{w}_i\,\frac{1}{2}\,\mathbf{w}^2\right\rangle}_{\text{Question: What is this term?}}$$

Kinetic Energy Density

Thermal Energy Density

$\mathrm{P}\,\delta_{ij} - \pi_{ij}$

pressure & viscous stresses

<u>Question</u>: What is this term?

$$\boxed{\frac{\partial}{\partial t}\left(n\langle\psi\rangle\right) + \nabla \bullet \left(n\left\langle\frac{\mathbf{p}}{m}\psi\right\rangle\right) - n\,\mathbf{F} \bullet \left\langle\nabla_{\mathbf{p}}\psi\left(\mathbf{p}\right)\right\rangle = 0}$$

$$\frac{\partial\,\varepsilon}{\partial t} + \frac{\partial}{\partial x_i}\left(\frac{\rho}{2}\left\langle\left(\mathrm{v}_i + \mathrm{w}_i\right)\left(\mathbf{v} + \mathbf{w}\right)^2\right\rangle\right) = \rho\,\mathrm{v}_i\,\mathrm{a}_i$$

The *energy equation in conservation form* is then given by:

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial x_i}\Big(\big(\varepsilon + \mathrm{P}\big)\, \mathrm{v}_i \;-\; \mathrm{v}_j\, \pi_{ji} + \mathrm{F}_i^{cond}\Big) = \rho\, \mathrm{v}_i\, \mathrm{a}_i$$

with $\mathrm{F}_i^{cond} \equiv \rho \left\langle \mathrm{w}_i \, \frac{1}{2}\, \mathbf{w}^2 \right\rangle$. Also recall, $\varepsilon = e + \frac{1}{2}\rho \mathbf{v}^2$.

A little bit of work (i.e. subtracting off the kinetic energy density) gives us an equation for the *internal energy density in primitive form* as:

$$\frac{\partial \mathrm{e}}{\partial t} + \frac{\partial}{\partial x_i}\left(\mathrm{e}\, \mathrm{v}_i\right) = -\,\mathrm{P}\frac{\partial \mathrm{v}_i}{\partial x_i} - \frac{\partial \mathrm{F}_i^{cond}}{\partial x_i} + \pi_{ij}\frac{\partial \mathrm{v}_i}{\partial x_j}$$

<u>Questions</u>: Interpret each of the terms on the RHS in the equation above. What is the connection with the *first law of thermodynamics*?

One can make connection with the *second law of thermodynamics* as:

$$\rho\, \mathrm{T}\, \frac{Ds}{Dt} = -\,\nabla \bullet \mathbf{F}^{cond} + \pi_{ij}\frac{\partial \mathrm{v}_i}{\partial x_j}$$ $\leftarrow$ What does this equation do at shocks?

1.3) The Euler Equations

Conservation form:		Primitive form:
$\dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial x_i}\left(\rho \, \mathrm{v}_i\right) = 0$		$\dfrac{D\rho}{Dt} = -\rho \, \nabla \bullet \mathbf{v}$
$\dfrac{\partial}{\partial t}\left(\rho \mathrm{v}_i\right) + \dfrac{\partial}{\partial x_j}\left(\rho \mathrm{v}_i \mathrm{v}_j + \mathrm{P}\,\delta_{ij}\right) = 0$	$\Leftrightarrow$	$\rho \dfrac{D \, \mathrm{v}_i}{D\,t} + \dfrac{\partial \, \mathrm{P}}{\partial x_i} = 0$
$\dfrac{\partial \varepsilon}{\partial t} + \dfrac{\partial}{\partial x_i}\left(\left(\varepsilon + \mathrm{P}\right) \mathrm{v}_i\right) = 0$		$\dfrac{\mathrm{D\,e}}{\mathrm{D}\,t} = -\left(\mathrm{e} + \mathrm{P}\right) \nabla \bullet \mathbf{v}$

$$\varepsilon = \mathrm{e} + \rho \, \mathbf{v}^2 / 2$$

Notice that we have 6 variables (density, 3 velocities, internal energy and pressure) but only 5 equations. This is called the *closure problem*. The *equation of state* helps close the system, i.e. provides the one extra equation needed to get the number of equations == number of unknowns.

Question: How do the dimensions of the fluxes relate to the dimensions of the conserved variables? Interpret your answer physically?

Primitive form is v. useful for analytic work; conserved form for computation (especially when discontinuities present). Question : Why?

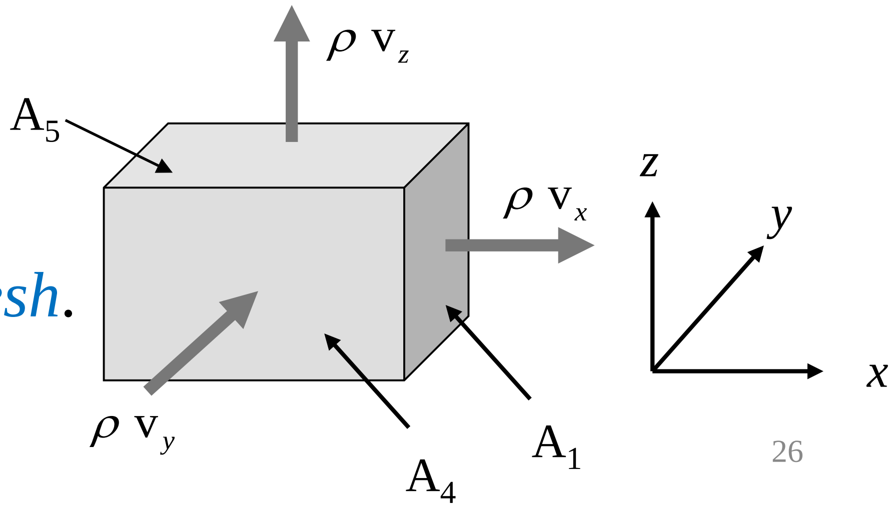
Notice that the fluid variables evolve in time in response to their own spatial gradients. This is often the case with most PDEs.

<u>Question</u>: So what makes the ***conservation form*** so special?
<u>Answer</u>: Gauss' Law.
Let's focus on the continuity equation and the figure below.

$$\iiint_V \left(\frac{\partial \rho}{\partial t} + \frac{\partial (\rho \mathrm{v}_x)}{\partial x} + \frac{\partial (\rho \mathrm{v}_y)}{\partial y} + \frac{\partial (\rho \mathrm{v}_z)}{\partial z} \right) dx\, dy\, dz = 0 \quad \Rightarrow$$

$$\frac{\partial}{\partial t} \iiint_V \rho \, dx\, dy\, dz + \iint_{A_1} \rho \mathrm{v}_x \, dy\, dz - \iint_{A_2} \rho \mathrm{v}_x \, dy\, dz + \iint_{A_3} \rho \mathrm{v}_y \, dx\, dz - \iint_{A_4} \rho \mathrm{v}_y \, dx\, dz + \iint_{A_5} \rho \mathrm{v}_z \, dx\, dy - \iint_{A_6} \rho \mathrm{v}_z \, dx\, dy = 0$$

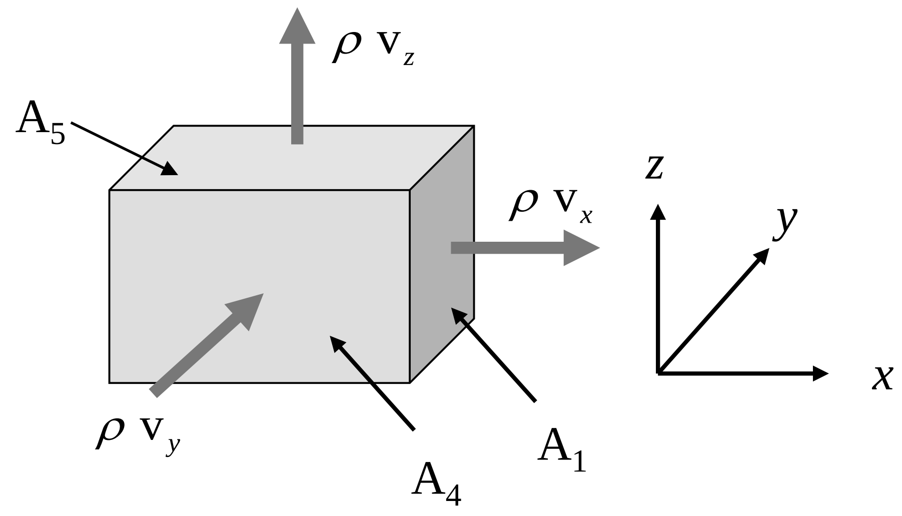
When ***discontinuities/shocks*** are present, we have no hope of predicting the flow structure inside a *zone* in our computational *mesh*. However, the ***conservation form remains valid***!



$$\iiint_V \left(\frac{\partial \rho}{\partial t} + \frac{\partial(\rho \, v_x)}{\partial x} + \frac{\partial(\rho \, v_y)}{\partial y} + \frac{\partial(\rho \, v_z)}{\partial z} \right) dx\, dy\, dz = 0$$

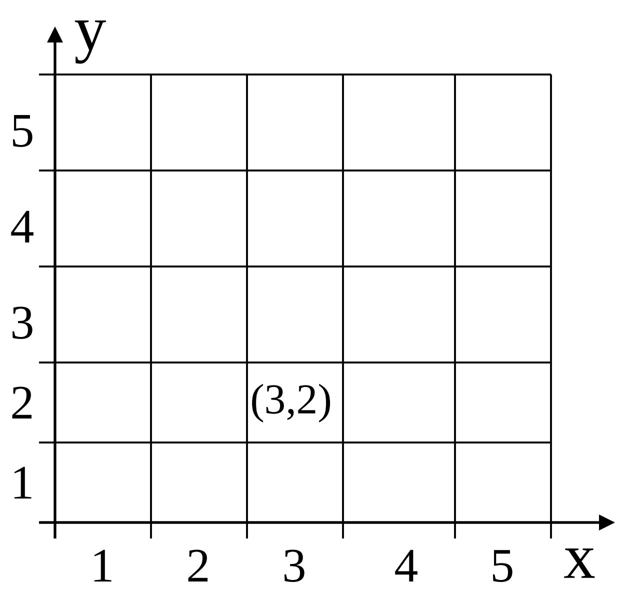
$$\iiint_V \frac{\partial \rho}{\partial t} \, dx\, dy\, dz =$$

$$\iiint_V \frac{\partial(\rho \, v_x)}{\partial x} \, dx\, dy\, dz =$$



$$\frac{\partial}{\partial t} \iiint_V \rho \, dx\, dy\, dz + \iint_{A_1} \rho \, v_x \, dy\, dz - \iint_{A_2} \rho \, v_x \, dy\, dz + \iint_{A_3} \rho \, v_y \, dx\, dz - \iint_{A_4} \rho \, v_y \, dx\, dz$$

$$+ \iint_{A_5} \rho \, v_z \, dx\, dy - \iint_{A_6} \rho \, v_z \, dx\, dy = 0$$



Introducing concepts of a mesh, zones and timestep.

Role of timestep in conveying information.

Value of conservation form on a mesh.

1.4) The Navier Stokes Equations

Conservation form:		Primitive form:
$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho \mathrm{v}_i\right) = 0$		$\frac{D\rho}{Dt} = -\rho \nabla \bullet \mathbf{v}$
$\frac{\partial}{\partial t}\left(\rho \mathrm{v}_i\right) + \frac{\partial}{\partial x_j}\left(\rho \mathrm{v}_i \mathrm{v}_j + \mathrm{P}\,\delta_{ij} - \pi_{ij}\right) = 0$	$\Leftrightarrow$	$\rho \frac{D \mathrm{v}_i}{D t} + \frac{\partial \mathrm{P}}{\partial x_i} - \frac{\partial}{\partial x_j}\pi_{ij} = 0$
$\frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial x_i}\left(\left(\varepsilon + \mathrm{P}\right)\mathrm{v}_i - \mathrm{v}_j\,\pi_{ji} + \mathrm{F}_i^{cond}\right) = 0$		$\frac{D\,\mathrm{e}}{D\,t} = -\left(\mathrm{e} + \mathrm{P}\right)\nabla \bullet \mathbf{v} - \nabla \bullet \mathbf{F}^{cond} + \pi_{ij}\frac{\partial \mathrm{v}_i}{\partial x_j}$

As before, the equation of state helps close the system.

Notice though, that we now have extra terms for the viscosity and heat flux. Theory alone cannot specify these terms; experiments are needed. We have the following *stress-strain relation* for the *viscous stresses*:

$$\pi_{ij} \equiv \mu\, D_{ij} \quad ; \quad D_{ij} \equiv \frac{\partial \mathrm{v}_i}{\partial x_j} + \frac{\partial \mathrm{v}_j}{\partial x_i} - \frac{2}{3}\left(\nabla \bullet \mathbf{v}\right)\delta_{ij}$$

For the *thermal conduction* we have:

$$\mathbf{F}^{cond} \equiv -\,\kappa\,\nabla \mathrm{T}$$

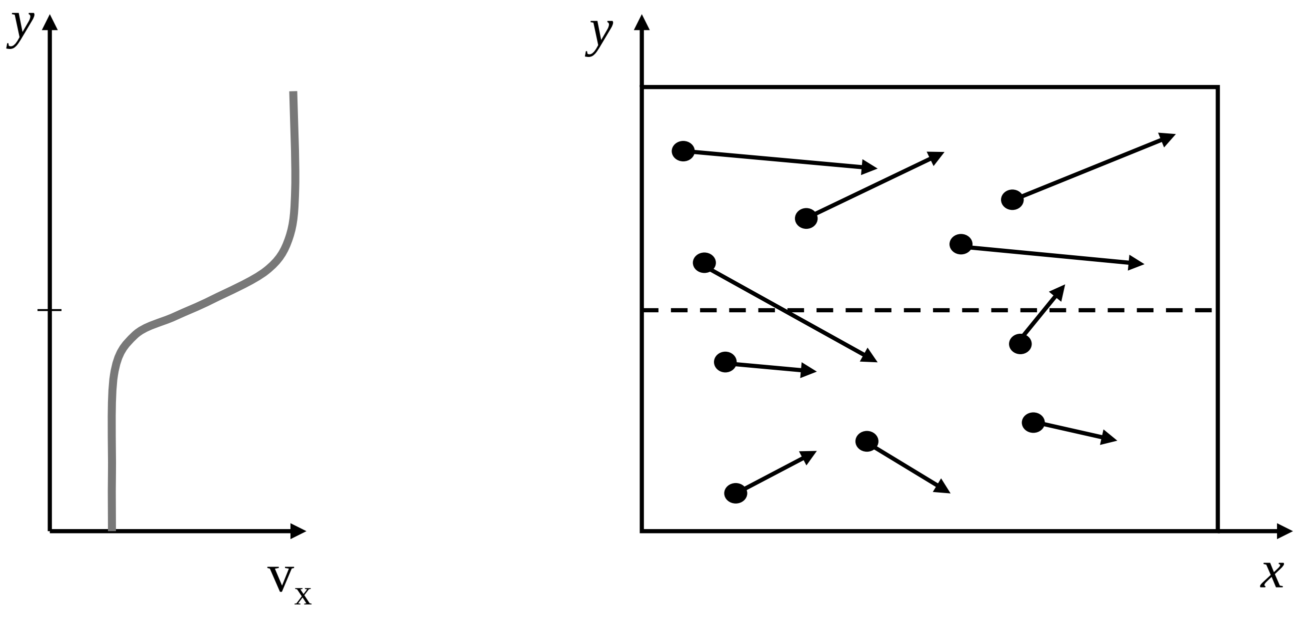
parabolic terms

It can be shown that $\pi_{ij} \dfrac{\partial \mathrm{v}_i}{\partial x_j}$ is always positive.

Question: What is the physical import of the above statement?

Intuitive introduction to viscosity for ***shear flow***: $\partial^2 \mathrm{v}_\mathrm{x} / \partial y^2$ is large.

x-momentum equation shows us that μ must scale as $\rho\, l^2 / \tau$.



Intuitive introduction to ***shocks***; a fast-moving stream of fluid collides with a stationary/slow-moving stream:

$\partial v_x / \partial x \ll 0$ and $\partial^2 v_x / \partial x^2$ is large at the shock.

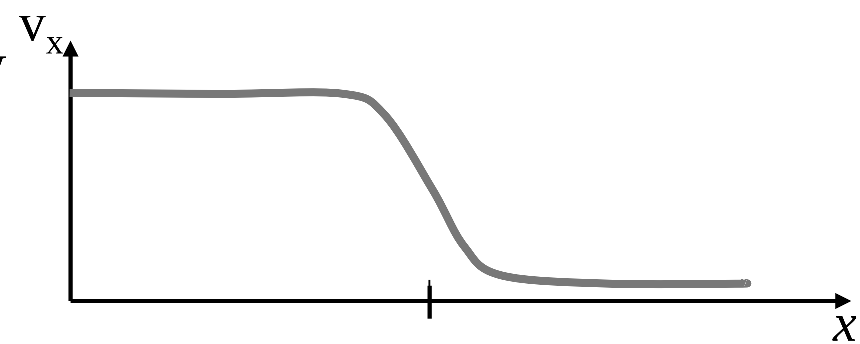
<u>Questions</u>: Can you make a traffic-flow analogy to this situation?

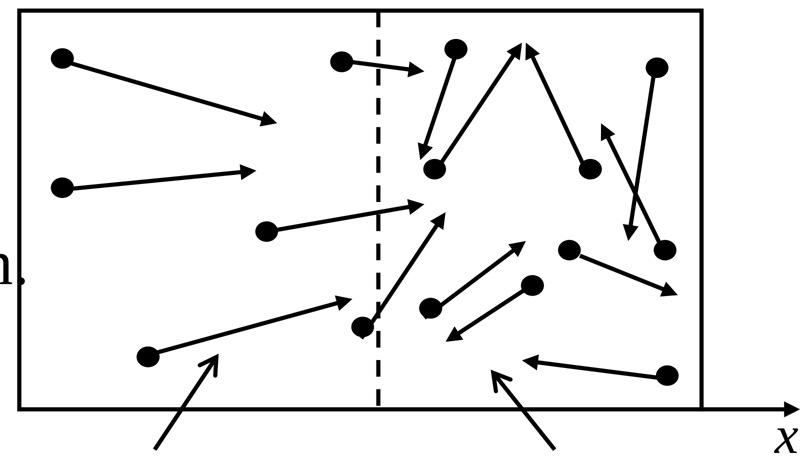
Which has higher density, shocked or unshocked gas? Explain using the continuity equation.

What about entropy?
Explain your answer via entropy equation.

What is the role of the viscous terms?

Can one have shocks form without having the viscous terms dominate at small scales? (Multiple answers possible.)





Unshocked Gas:	Shocked Gas:
Density low	Density high
Mean velocity high	Mean velocity low
RMS velocity low	RMS velocity high

1.5) Classifying and Understanding PDEs

(Or How to read a PDE like a book)

1.5.1) Motivation

Let's start simple with scalar PDEs. Consider our first example:

$$\frac{\partial \rho}{\partial t} + a \frac{\partial \rho}{\partial x} + b \frac{\partial \rho}{\partial y} = 0 \quad \leftarrow$$ What does it remind you of? How does ρ move in 2d?

Analyzing a PDE is same as bringing out its character. Since the above PDE has a wave character, i.e. it is *hyperbolic*, we try harmonic modes:

$$\rho(x, y, t) = \rho_0 + \rho_1 \, e^{i\left(k_x x + k_y y - \omega t\right)}$$ Substitution in the PDE gives : $\omega = k_x \, a + k_y \, b$

$$\Rightarrow \rho(x, y, t) = \rho_0 + \rho_1 \, e^{i\left[k_x(x - a\,t) + k_y(y - b\,t)\right]} \quad \leftarrow$$ Propagating wave-like modes.

The harmonic modes take on a time-evolution that is purely multiplicative with no change in the amplitude. Such modes are called the ***eigenmodes*** of the PDE. "ω" is the ***eigenvalue*** associated with that eigenmode.

So here we started with a PDE that we knew to be hyperbolic and divined its mathematical character.

Substitute $\rho(x,y,t) = \rho_0 + \rho_1 \, \mathrm{e}^{i\left(k_x x + k_y y - \omega t\right)}$ in $\frac{\partial \rho}{\partial t} + \mathrm{a} \frac{\partial \rho}{\partial x} + \mathrm{b} \frac{\partial \rho}{\partial y} = 0$

$\rho(x,y,t) = \rho_0 + \rho_1 \, \mathrm{e}^{i\left[k_x(x - \mathrm{a}\,t) + k_y(y - \mathrm{b}\,t)\right]}$ with $\omega = k_x \, \mathrm{a} + k_y \, \mathrm{b}$

Consider the heat equation in 2d with constant coefficients:

$$\frac{\partial T}{\partial t} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

Again, let us put in harmonic modes to bring out the character of this well-known *parabolic* PDE:

$$T(x, y, t) = T_0 + T_1 e^{i\left(k_x x + k_y y - \omega t\right)} \quad \text{Substitution in the PDE gives : } \omega = -i \kappa \left(k_x^2 + k_y^2\right)$$

$$\Rightarrow T(x, y, t) = T_0 + T_1 e^{i\left(k_x x + k_y y\right) - \kappa \left(k_x^2 + k_y^2\right) t} \quad \leftarrow \text{ A } \textit{time-decaying} \text{ solution}$$

Such modes are called the *eigenmodes* of the PDE. "ω" is the *eigenvalue* associated with that eigenmode.

So here we started with a PDE that we knew to be parabolic and divined its mathematical character.

Now the ideas can be combined. -- A super simple chemo-taxis example:

$$\frac{\partial \rho}{\partial t} + a \frac{\partial \rho}{\partial x} + b \frac{\partial \rho}{\partial y} - \kappa \left(\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} \right) = s$$

Substitute $\mathrm{T}(x,y,t) = \mathrm{T}_0 + \mathrm{T}_1\, \mathrm{e}^{i\left(k_x x + k_y y - \omega t\right)}$ in $\dfrac{\partial\, \mathrm{T}}{\partial\, t} = \kappa \left(\dfrac{\partial^2\, \mathrm{T}}{\partial\, x^2} + \dfrac{\partial^2\, \mathrm{T}}{\partial\, y^2} \right)$

$\mathrm{T}(x,y,t) = \mathrm{T}_0 + \mathrm{T}_1\, \mathrm{e}^{i\left(k_x x + k_y y\right) - \kappa\left(k_x^2 + k_y^2\right)t}$ with $\omega = -i\,\kappa\,\left(k_x^2 + k_y^2\right)$

1.5.2) Characteristic Analysis of the Euler Equations

Let us write the 1d Euler equations in a matrix form:

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ \mathrm{v_x} \\ \mathrm{P}\end{pmatrix} + \begin{pmatrix}\mathrm{v_x} & \rho & 0 \\ 0 & \mathrm{v_x} & \frac{1}{\rho} \\ 0 & \Gamma\,\mathrm{P} & \mathrm{v_x}\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\rho \\ \mathrm{v_x} \\ \mathrm{P}\end{pmatrix} = 0\,. \quad \text{Try the wave-like solution :}\ \begin{pmatrix}\rho(x,t) \\ \mathrm{v_x}(x,t) \\ \mathrm{P}(x,t)\end{pmatrix} = \begin{pmatrix}\rho_0 \\ \mathrm{v_{x0}} \\ \mathrm{P_0}\end{pmatrix} + \begin{pmatrix}\rho_1 \\ \mathrm{v_{x1}} \\ \mathrm{P_1}\end{pmatrix} e^{\,i\,(k\,x - \omega\,t)}$$

We think of the variables with subscript “0” as the constant values around which we introduce small fluctuations, i.e. the variables with subscript “1”. This has the advantage that it freezes the matrix. We get:

$$-\,i\,\omega\begin{pmatrix}\rho_1 \\ \mathrm{v_{x1}} \\ \mathrm{P_1}\end{pmatrix} + i\,k\begin{pmatrix}\mathrm{v_{x0}} & \rho_0 & 0 \\ 0 & \mathrm{v_{x0}} & \frac{1}{\rho_0} \\ 0 & \Gamma\,\mathrm{P_0} & \mathrm{v_{x0}}\end{pmatrix}\begin{pmatrix}\rho_1 \\ \mathrm{v_{x1}} \\ \mathrm{P_1}\end{pmatrix} = 0\,.\ \text{Use}\ \lambda \equiv \omega/k\ \text{to get :}\ \begin{pmatrix}\mathrm{v_{x0}}-\lambda & \rho_0 & 0 \\ 0 & \mathrm{v_{x0}}-\lambda & \frac{1}{\rho_0} \\ 0 & \Gamma\,\mathrm{P_0} & \mathrm{v_{x0}}-\lambda\end{pmatrix}\begin{pmatrix}\rho_1 \\ \mathrm{v_{x1}} \\ \mathrm{P_1}\end{pmatrix} = 0$$

Notice that for systems we have to analyze the above *characteristic matrix*. Its determinant yields the *characteristic equation* with solutions:

$$\lambda^1 = \mathrm{v_{x0}} - \mathrm{c_0} \quad ; \quad \lambda^2 = \mathrm{v_{x0}} \quad ; \quad \lambda^3 = \mathrm{v_{x0}} + \mathrm{c_0} \quad ; \quad \mathrm{c_0} \equiv \sqrt{\frac{\Gamma\,\mathrm{P_0}}{\rho_0}} \leftarrow \text{speed of sound}$$

$$\frac{\partial \rho}{\partial t} + \mathrm{v}_x \frac{\partial \rho}{\partial x} + \rho \frac{\partial \mathrm{v}_x}{\partial x} = 0 \ ;$$

$$\rho \left(\frac{\partial \mathrm{v}_x}{\partial t} + \mathrm{v}_x \frac{\partial \mathrm{v}_x}{\partial x} \right) + \frac{\partial \mathrm{P}}{\partial x} = 0 \ ;$$

$$\frac{\partial \mathrm{e}}{\partial t} + \mathrm{v}_x \frac{\partial \mathrm{e}}{\partial x} + \mathrm{e} \frac{\partial \mathrm{v}_x}{\partial x} + \mathrm{P} \frac{\partial \mathrm{v}_x}{\partial x} = 0 \quad \text{with } \mathrm{e} = \frac{\mathrm{P}}{\Gamma - 1}$$

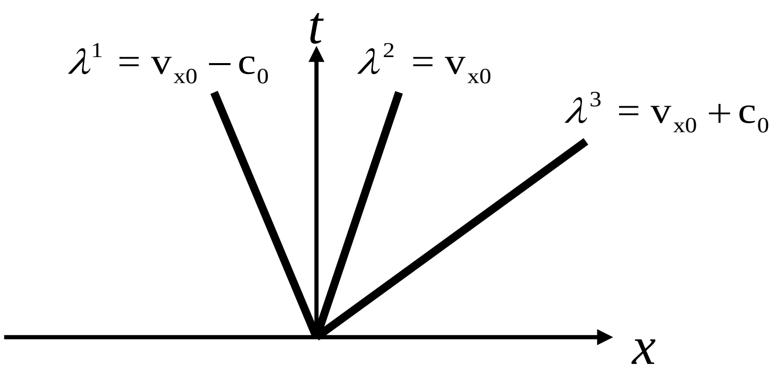
$$\longrightarrow \quad \frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \mathrm{v_x} \\ \mathrm{P} \end{pmatrix} + \begin{pmatrix} \mathrm{v_x} & \rho & 0 \\ 0 & \mathrm{v_x} & \frac{1}{\rho} \\ 0 & \Gamma\,\mathrm{P} & \mathrm{v_x} \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} \rho \\ \mathrm{v_x} \\ \mathrm{P} \end{pmatrix} = 0$$

Equations *linearized* about a constant state: $\left(\rho_0,\ \mathrm{v}_{x0},\ \mathrm{P}_0\right)$

$$-i\ \omega \begin{pmatrix} \rho_1 \\ \mathrm{v}_{x1} \\ \mathrm{P}_1 \end{pmatrix} + i\ \mathrm{k} \begin{pmatrix} \mathrm{v}_{x0} & \rho_0 & 0 \\ 0 & \mathrm{v}_{x0} & \dfrac{1}{\rho_0} \\ 0 & \Gamma\ \mathrm{P}_0 & \mathrm{v}_{x0} \end{pmatrix} \begin{pmatrix} \rho_1 \\ \mathrm{v}_{x1} \\ \mathrm{P}_1 \end{pmatrix} = 0 \quad \text{with } \lambda \equiv \omega / k$$

$$\lambda^1 = \mathrm{v}_{x0} - \mathrm{c}_0 \quad ; \quad \lambda^2 = \mathrm{v}_{x0} \quad ; \quad \lambda^3 = \mathrm{v}_{x0} + \mathrm{c}_0 \quad ; \quad \mathrm{c}_0 \equiv \sqrt{\frac{\Gamma\ \mathrm{P}_0}{\rho_0}}$$

We can use a *space-time diagram* → to trace the waves. The lines are called ***characteristic curves***.



Question: Identify sub-sonic and supersonic flow situations by drawing space-time diagrams for them. Are the characteristic curves always straight lines?

Analysis of the ***right eigenvectors*** yields a lot of further insight:

$$r^1 = \begin{pmatrix} \rho_0 \\ -c_0 \\ \rho_0 \, c_0^2 \end{pmatrix} \quad ; \quad r^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad ; \quad r^3 = \begin{pmatrix} \rho_0 \\ c_0 \\ \rho_0 \, c_0^2 \end{pmatrix}$$

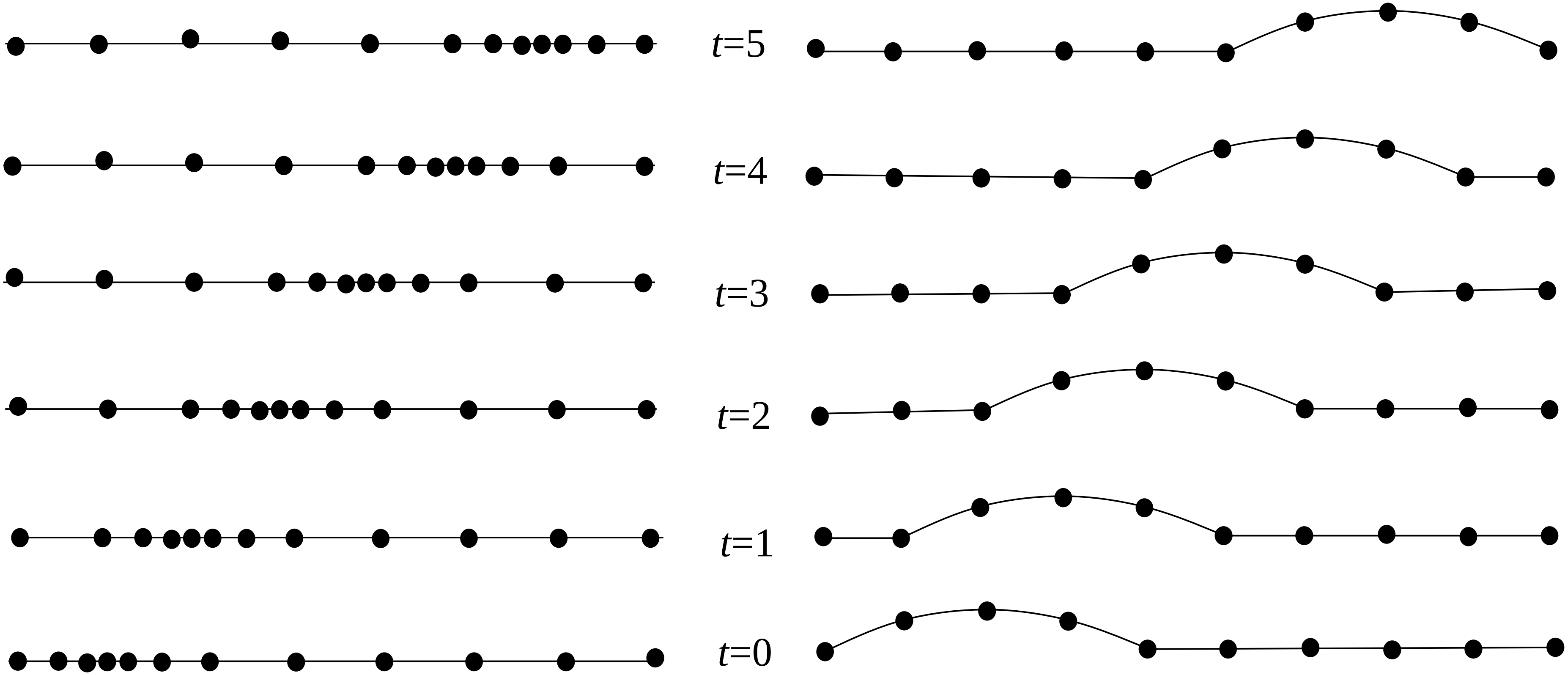
Observe from r^3 that the fluctuations have to have the ratios:- $\rho_1 : v_{x1} : P_1 = \rho_0 : c_0 : \rho_0 \, c_0^2$

I.e. the ratios are preset and all these fluctuations produce *compressions* in the velocity and pressure fluctuations $\Rightarrow$ these are *right-going sound waves* (see the eigenvalue λ^3).

Observe from r^2 that the fluctuations have to have the ratios:- $\rho_1 : v_{x1} : P_1 = 1:0:0$

I.e. no pressure or velocity fluctuations; only changes in the density $\Rightarrow$ these are *entropy waves* (see the eigenvalue λ^2). They are *advected* with the fluid velocity.

Wave propagation can be via *longitudinal* or *transverse* fluctuations.
Question: Which one is which in the figure below? Which one represents sound waves?



The right eigenvectors form a *complete* basis set in a 3-d vector space.➔
There exists an *orthonormal* set of ***left eigenvectors***. They are:

$$l^1 = \begin{pmatrix} 0 & \frac{-1}{2\,c_0} & \frac{1}{2\,\rho\,c_0^2} \end{pmatrix} \quad ; \quad l^2 = \begin{pmatrix} 1 & 0 & -\frac{1}{c_0^2} \end{pmatrix} \quad ; \quad l^3 = \begin{pmatrix} 0 & \frac{1}{2\,c_0} & \frac{1}{2\,\rho\,c_0^2} \end{pmatrix}$$

$$\begin{pmatrix}\rho_1 & v_{x1} & P_1\end{pmatrix}\begin{pmatrix} v_{x0}-\lambda & \rho_0 & 0 \\ 0 & v_{x0}-\lambda & \dfrac{1}{\rho_0} \\ 0 & \Gamma\ P_0 & v_{x0}-\lambda \end{pmatrix} = 0$$

$$\lambda^1 = v_{x0} - c_0 \quad ; \quad \lambda^2 = v_{x0} \quad ; \quad \lambda^3 = v_{x0} + c_0 \quad ;$$

$$l^1 = \begin{pmatrix} 0 & \dfrac{-1}{2\ c_0} & \dfrac{1}{2\ \rho\ c_0^2} \end{pmatrix} \quad ; \quad l^2 = \begin{pmatrix} 1 & 0 & -\dfrac{1}{c_0^2} \end{pmatrix} \quad ; \quad l^3 = \begin{pmatrix} 0 & \dfrac{1}{2\ c_0} & \dfrac{1}{2\ \rho\ c_0^2} \end{pmatrix}$$

How is all this machinery of eigenvalues and eigenvectors useful?

Imagine you have a mean fluid state and you impose a small Gaussian pulse, i.e. a perturbation. How do you predict its time-evolution?

$$\begin{pmatrix} \rho(x,t=0) \\ \mathrm{v_x}(x,t=0) \\ \mathrm{P}(x,t=0) \end{pmatrix} = \begin{pmatrix} \rho_0 \\ \mathrm{v_{x0}} \\ \mathrm{P_0} \end{pmatrix} + \begin{pmatrix} \rho_1 \\ \mathrm{v_{x1}} \\ \mathrm{P_1} \end{pmatrix} \mathrm{e}^{-x^2} = \mathrm{V_0} + \mathrm{V_1}\,\mathrm{e}^{-x^2}$$

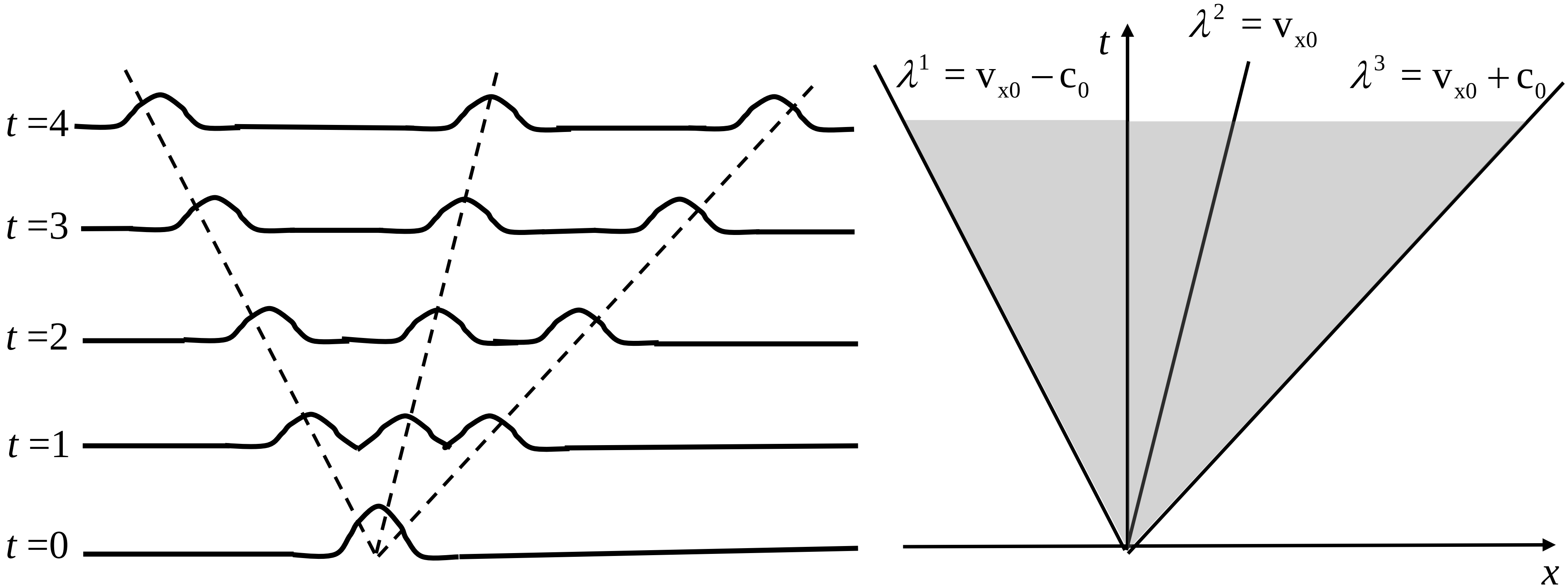
We know that each right eigenvector is a pure wave that travels with a preset speed. So use the left eigenvectors to find out what fraction of the initial perturbation contributes to each of the waves. This is given to us by the ***eigenweights***:

i.e. in order to write $\mathrm{V_1} = \alpha^1 r^1 + \alpha^2 r^2 + \alpha^3 r^3$ we need to find α^1, α^2, and α^3.

$$\alpha^1 = \left(l^1 \bullet \mathrm{V_1}\right) \quad ; \quad \alpha^2 = \left(l^2 \bullet \mathrm{V_1}\right) \quad ; \quad \alpha^3 = \left(l^3 \bullet \mathrm{V_1}\right)$$

The time-evolution of the Gaussian perturbation is then given by:

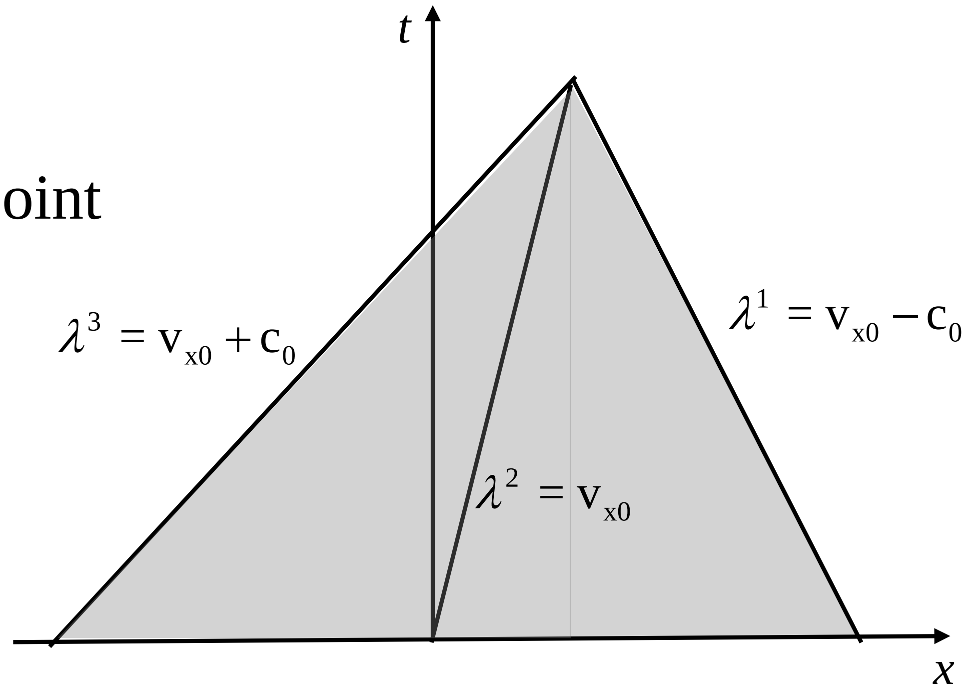
$$\begin{pmatrix} \rho(x,t) \\ \mathrm{v_x}(x,t) \\ \mathrm{P}(x,t) \end{pmatrix} = \begin{pmatrix} \rho_0 \\ \mathrm{v_{x0}} \\ \mathrm{P_0} \end{pmatrix} + \alpha^1 \begin{pmatrix} \rho_0 \\ -\mathrm{c_0} \\ \rho_0\,\mathrm{c_0^2} \end{pmatrix} \mathrm{e}^{-(x-\lambda_1 t)^2} + \alpha^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathrm{e}^{-(x-\lambda_2 t)^2} + \alpha^3 \begin{pmatrix} \rho_0 \\ \mathrm{c_0} \\ \rho_0\,\mathrm{c_0^2} \end{pmatrix} \mathrm{e}^{-(x-\lambda_3 t)^2}$$



The initial Gaussian pulse propagates away as three Gaussian pulses with amplitudes given by the eigenweights and speeds given by the eigenvalues.

The shaded region in the space-time diagram shows the ***range of influence***. I.e. it gives us the portions of space-time that get influenced by the initial perturbation.
Notice that the extremal wave speeds, λ^1 and λ^3 , determine the range of influence.

Say that the x-axis is seeded with small fluctuations at t=0.
Question: If we pick a space-time point (x,t) with t>0, which points on the original x-axis will influence its evolution?



Realize that information travels at a finite speed in a hyperbolic system.

Since the characteristics curves are straight lines, in our linearized system, it is easy to find that domain by propagating the characteristics backward. Again, the extremal wave speeds, λ^1 and λ^3 , are very useful in identifying this domain.

The shaded region shows the ***domain of dependence*** in space-time.

1.5.3) Generalized Definition of a Hyperbolic PDE

Many, though not all, hyperbolic systems can be written in conservation form:

$$U_t + F(U)_x + G(U)_y + H(U)_z = S(U)$$

"U" is the vector of "*M*" conserved variables and "F", "G" and "H" are the flux vectors. "S" is the vector of source terms. One can then obtain the characteristic matrices "A", "B" and "C", which are all $M \times M$:

$$U_t + A(U)\, U_x + B(U)\, U_y + C(U)\, U_z = S(U) \quad \left(\text{In component form } A(U)_{i,j} \equiv \partial F_i(U) / \partial U_j\right)$$

$$\text{with } A(U) \equiv \frac{\partial F(U)}{\partial U} \; ; \; B(U) \equiv \frac{\partial G(U)}{\partial U} \; ; \; C(U) \equiv \frac{\partial H(U)}{\partial U}$$

The *hyperbolic property* then depends on the *eigenstructure of the characteristic matrices*.

Often times, as with Euler eqns., it is easier to analyze the hyperbolic system in terms of primitive variables V : $\delta U = \left(\frac{\partial U}{\partial V}\right) \delta V \; ; \; \delta V = \left(\frac{\partial V}{\partial U}\right) \delta U$

Any general hyperbolic system can then be written as:

$$V_t + A(V)\,V_x + B(V)\,V_y + C(V)\,V_z = S(V)$$

with $A(V) \equiv \left(\frac{\partial V}{\partial U}\right) \frac{\partial F(U)}{\partial U} \left(\frac{\partial U}{\partial V}\right)$; $S(V) \equiv \left(\frac{\partial V}{\partial U}\right)$ S(U) if it is in conservation form.

The system is then said to be ***hyperbolic*** if each of the matrices “A”, “B” and “C” admit “M” *real eigenvalues* and a *complete* set of “M” *right eigenvectors*. This also ensures the existence of orthonormal left eigenvectors. I.e. solutions are wave-like when propagating in all directions.

Above definition gives us the useful properties that :

1) Waves can propagate in any direction.

2) Any small initial fluctuation can be evolved forward in time for at least a short amount of time. It enables us to do dynamics.

$$\mathrm{U}_t + \mathrm{F}(\mathrm{U})_x + \mathrm{G}(\mathrm{U})_y + \mathrm{H}(\mathrm{U})_z = \mathrm{S}(\mathrm{U}) \;\; \rightarrow \;\; \mathrm{U}_t + \mathrm{A}(\mathrm{U})\,\mathrm{U}_x + \mathrm{B}(\mathrm{U})\,\mathrm{U}_y + \mathrm{C}(\mathrm{U})\,\mathrm{U}_z = \mathrm{S}(\mathrm{U})$$

$$\delta\mathrm{U} = \left(\frac{\partial \mathrm{U}}{\partial \mathrm{V}}\right) \delta\mathrm{V} \; ; \; \delta\mathrm{V} = \left(\frac{\partial \mathrm{V}}{\partial \mathrm{U}}\right) \delta\mathrm{U} \;\; \rightarrow \;\; \mathrm{V}_t + \mathrm{A}(\mathrm{V})\,\mathrm{V}_x + \mathrm{B}(\mathrm{V})\,\mathrm{V}_y + \mathrm{C}(\mathrm{V})\,\mathrm{V}_z = \mathrm{S}(\mathrm{V})$$

Let us now focus on the 1d case. We assume that the characteristic matrix "A" is frozen about some constant state V_0 . For small fluctuations V_1 about that constant state, we have

$$\frac{\partial V_1}{\partial t} + A \, \frac{\partial V_1}{\partial x} = 0$$

where A admits an ordered set of M real eigenvalues: $\lambda^1 \le \lambda^2 \le ... \le \lambda^M$

We have M left and right eigenvectors of A

so that: $A \, r^m = \lambda^m \, r^m \; ; \; l^m \, A = l^m \lambda^m \quad \forall \; m=1,..,M$

Let the left and right eigenvectors be orthonormalized w.r.t. each other.

When the eigenvalues are degenerate we use Gram-Schmidt orthonormalization to obtain linearly independent eigenvectors.

The next three steps are purely formal but yields a very compact and useful notation that is used over and over in this field:

1) Let "*R*" be a matrix of right eigenvectors whose *m*th column is given by r^m .

2) Let "*L*" be the matrix of left eigenvectors whose *m*th row is given by l^m . We want $L\,R = I$, *i.e. L* is left inverse of *R*.

3) Define $\Lambda \equiv \mathrm{diag}\left\{\lambda^1, \lambda^2, \ldots, \lambda^M\right\}$.

We then have:

$$\boxed{A\,R = R\,\Lambda \quad ; \quad L\,A = \Lambda\,L \quad ; \quad L\,A\,R = \Lambda \quad ; \quad A = R\,\Lambda\,L}$$

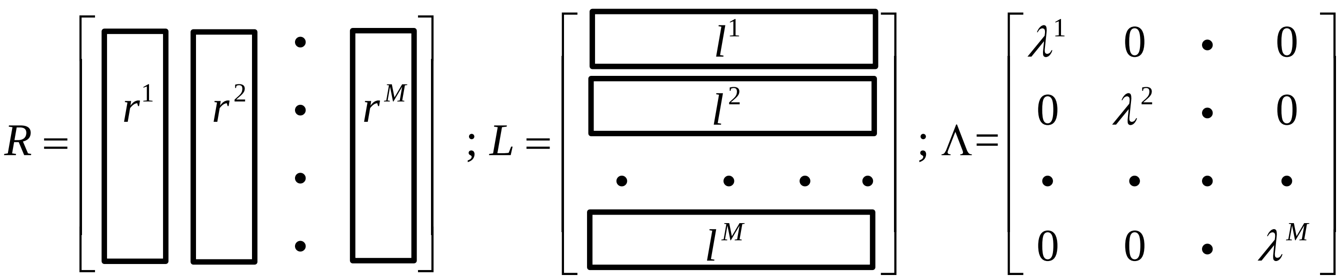


Fig. schematically shows the structure of the matrices R, L and Λ .

The whole purpose of the formal build-up so far is so that we can ***do dynamics***. I.e. , given a constant state V_0 and a small initial fluctuation $V_1(x)$ about it, we wish to predict the time-evolution of $V_1(x)$.

Left-multiply the evolution equation to get:

$$L \frac{\partial V_1}{\partial t} + \left(L\, A\, R\right) L \frac{\partial V_1}{\partial x} = 0 \qquad \Rightarrow \qquad W_t + \Lambda\, W_x = 0 \quad \text{with} \quad W \equiv L V_1$$

For the m^{th} component of W we have: $w_t^m + \lambda^m w_x^m = 0 \quad \text{for } m = 1, \ldots, M$

Now say that we start with initial conditions $V_0 + V_1(x)$ at $t = 0$.

The fluctuation in the m^{th} eigenweight at t=0 is given by: $w^m(x) \equiv l^m \bullet V_1(x)$

At a later time, $t > 0$, in light of its evolution equation, the eigenweight is given by: $w^m\left(x - \lambda^m t\right)$

The time-dependent solution for $t > 0$ is given by:

$$V(x,t) = V_0 + \sum_{m=1}^{M} w^m\left(x - \lambda^m t\right) r^m$$

<u>V. Imp. Question</u>: Why is the material in this Sub-section so very useful?

$$\frac{\partial \mathrm{V}_1}{\partial t} + \mathrm{A}\ \frac{\partial \mathrm{V}_1}{\partial x} = 0 \qquad \rightarrow \qquad W_t + \Lambda\, W_x = 0 \quad \text{with} \quad W \equiv L\mathrm{V}_1$$

$$w_t^m + \lambda^m w_x^m = 0 \quad \text{for } m = 1, \ldots, M \qquad \text{with} \quad w^m\left(x\right) \equiv l^m \bullet \mathrm{V}_1\left(x\right) \text{ at } t = 0$$

1.5.4) Analysis of the Navier Stokes Equation

Having studied parabolic scalar equations, let's study the Navier Stokes equations as a system of parabolic equations.

Viscous and conductive terms involve *higher spatial derivatives* – make equation set *parabolic*. Question: For the Navier Stokes equations, can you identify a good set of primitive variables?

Equations in 1D:-

$$\frac{\partial \rho}{\partial t} + \mathrm{v_x} \frac{\partial \rho}{\partial x} + \rho \frac{\partial \, \mathrm{v_x}}{\partial x} = 0$$

$$\frac{\partial \, \mathrm{v_x}}{\partial t} + \mathrm{v_x} \frac{\partial \, \mathrm{v_x}}{\partial x} + \frac{R \, \mathrm{T}}{\bar{\mu} \, \rho} \frac{\partial \rho}{\partial x} + \frac{R}{\bar{\mu}} \frac{\partial \mathrm{T}}{\partial x} - \frac{4 \mu}{3 \rho} \frac{\partial^2 \, \mathrm{v_x}}{\partial x^2} = 0$$

$$\frac{\partial \mathrm{T}}{\partial t} + \mathrm{v_x} \frac{\partial \mathrm{T}}{\partial x} + (\Gamma - 1) \, \mathrm{T} \frac{\partial \, \mathrm{v_x}}{\partial x} - \frac{4(\Gamma - 1) \bar{\mu} \, \mu}{3 \, R} \left(\frac{\partial \, \mathrm{v_x}}{\partial x} \right)^2 - \frac{(\Gamma - 1) \kappa \, \bar{\mu}}{R} \frac{\partial^2 \mathrm{T}}{\partial x^2} = 0$$

Linearize about a constant state:-

$$\begin{pmatrix} \rho(x,t) \\ \mathrm{v_x}(x,t) \\ \mathrm{T}(x,t) \end{pmatrix} = \begin{pmatrix} \rho_0 \\ \mathrm{v_{x0}} \\ \mathrm{T_0} \end{pmatrix} + \begin{pmatrix} \rho_1 \\ \mathrm{v_{x1}} \\ \mathrm{T_1} \end{pmatrix} \mathrm{e}^{\, i \, (\mathrm{k} \, \mathrm{x} - \omega \, \mathrm{t})}$$

$$\frac{\partial\rho}{\partial t} + \mathrm{v}_\mathrm{x}\frac{\partial\rho}{\partial x} + \rho\frac{\partial\,\mathrm{v}_\mathrm{x}}{\partial x} = 0$$

$$\frac{\partial\,\mathrm{v}_\mathrm{x}}{\partial t} + \mathrm{v}_\mathrm{x}\frac{\partial\,\mathrm{v}_\mathrm{x}}{\partial x} + \frac{R\,\mathrm{T}}{\bar{\mu}\,\rho}\frac{\partial\rho}{\partial x} + \frac{R}{\bar{\mu}}\frac{\partial\mathrm{T}}{\partial x} - \frac{4\mu}{3\rho}\frac{\partial^2\,\mathrm{v}_\mathrm{x}}{\partial x^2} = 0$$

$$\frac{\partial\mathrm{T}}{\partial t} + \mathrm{v}_\mathrm{x}\frac{\partial\mathrm{T}}{\partial x} + (\Gamma-1)\,\mathrm{T}\,\frac{\partial\,\mathrm{v}_\mathrm{x}}{\partial x} - \frac{4(\Gamma-1)\bar{\mu}\,\mu}{3\,R}\left(\frac{\partial\,\mathrm{v}_\mathrm{x}}{\partial x}\right)^2 = 0 \quad \text{Setting } \kappa=0$$

$$\begin{pmatrix} \mathrm{v}_{\mathrm{x}0}-\lambda & \rho_0 & 0 \\ \frac{R\mathrm{T}_0}{\bar{\mu}\rho_0} & \mathrm{v}_{\mathrm{x}0}-\lambda-i\,\mathrm{k}\frac{4\mu}{3\rho_0} & \frac{R}{\bar{\mu}} \\ 0 & (\Gamma-1)\mathrm{T}_0 & \mathrm{v}_{\mathrm{x}0}-\lambda \end{pmatrix}\begin{pmatrix} \rho_1 \\ \mathrm{v}_{\mathrm{x}1} \\ \mathrm{T}_1 \end{pmatrix} = 0$$

$$\begin{pmatrix} \mathrm{v}_{x0}-\lambda & \rho_0 & 0 \\ \dfrac{R\mathrm{T}_0}{\bar{\mu}\rho_0} & \mathrm{v}_{x0}-\lambda-i\ \mathrm{k}\dfrac{4\mu}{3\rho_0} & \dfrac{R}{\bar{\mu}} \\ 0 & (\Gamma-1)\mathrm{T}_0 & \mathrm{v}_{x0}-\lambda \end{pmatrix} \begin{pmatrix} \rho_1 \\ \mathrm{v}_{x1} \\ \mathrm{T}_1 \end{pmatrix} = 0$$

; no thermal conduction -- κ=0

Let us simplify by setting κ = 0. The resulting eigenvalues are:

$$\lambda_1 = \mathrm{v}_{x0} - \sqrt{\mathrm{c}_0^2 - \mathrm{k}^2 \frac{4\mu^2}{9\rho_0^2}} - i\,\mathrm{k}\,\frac{2\mu}{3\rho_0} \quad ; \quad \lambda_2 = \mathrm{v}_{x0} \quad ; \quad \lambda_3 = \mathrm{v}_{x0} + \sqrt{\mathrm{c}_0^2 - \mathrm{k}^2 \frac{4\mu^2}{9\rho_0^2}} - i\,\mathrm{k}\,\frac{2\mu}{3\rho_0}$$

<u>Questions</u>: What happens to the eigenvalues for the sound waves? Interpret real and imaginary parts physically.

Generalizing: We see that the equations become parabolic because of the extra presence of a gradient of the solution in the fluxes. We can write the general form as:

$$\mathrm{U}_t + \mathrm{F}(\mathrm{U})_x + \mathrm{G}(\mathrm{U})_y + \mathrm{H}(\mathrm{U})_z + \mathrm{F}_{ni}(\mathrm{U}, \nabla \mathrm{U})_x + \mathrm{G}_{ni}(\mathrm{U}, \nabla \mathrm{U})_y + \mathrm{H}_{ni}(\mathrm{U}, \nabla \mathrm{U})_z = \mathrm{S}(\mathrm{U})$$

Note though that we do need to have solutions that are decaying in time.

When spatial derivatives with third and higher order are present, analyze on a case-by-case basis. General-purpose numerical methods not available yet.

1.6) Incompressible Flow Equations

Sound speed in water ~ 2000 m/s. A v. fast speedboat goes ~ 20 m/s. If the sound waves are to be captured numerically in a code, the code's time step would drop by a factor of 100! Thus we really desire an approximation that can make the sound waves drop out of the system.

Liquids, like water, are almost incompressible – ρ remains constant.

The sound waves drop out of the system when that approximation is made. Recall that to sustain a sound wave, we do need density fluctuations.

The continuity equation then gives $\nabla \bullet \mathbf{v} = 0 \;\Rightarrow$ there is no eqn for density evolution.

The momentum equation gives $\dfrac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \bullet \nabla \mathbf{v} = -\dfrac{1}{\rho} \nabla P$

Taking its divergence gives an elliptic equation for the pressure :

$\nabla^2 P = -\rho \nabla \bullet \left(\mathbf{v} \bullet \nabla \mathbf{v} \right) \;\Rightarrow$ no need for an evolutionary eqn for thermal energy.

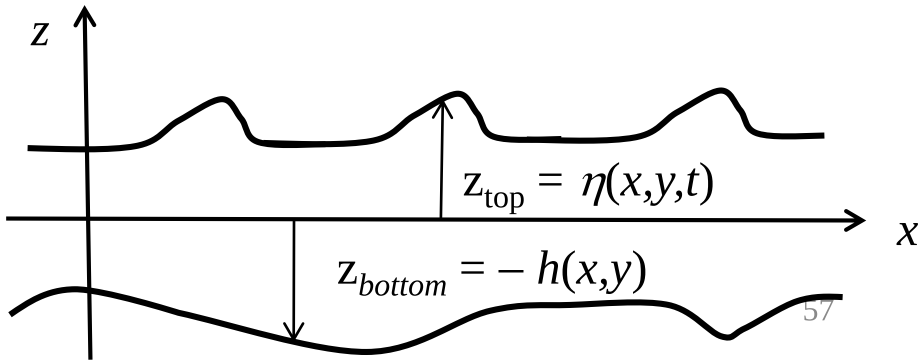
1.7) Shallow Water Equations

Water in lakes and oceans forms a *very thin layer of incompressible fluid* on the earth's surface. We want to treat it like a 2d problem.

We simplify the problem by considering a flat patch in the (x,y) plane. Details of the z-velocity are not important to us.

However, $z_{bottom} = -h(x,y)$ changes with position. We care about $z_{top} = \eta(x,y,t)$ because it determines waves/tsunamis etc.

The gravitational potential drives the evolution of such waves. Thus, even though we neglect z-velocity, we have to keep track of the *geopotential* : $\phi(x,y,t) = g\,(\eta(x,y,t) + h(x,y))$.



$$\frac{\partial}{\partial t}\begin{bmatrix}\phi \\ \phi\,\mathrm{v_x} \\ \phi\,\mathrm{v_y}\end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix}\phi\,\mathrm{v_x} \\ \phi\,\mathrm{v_x^2}+\phi^2/2 \\ \phi\,\mathrm{v_x}\;\mathrm{v_y}\end{bmatrix} + \frac{\partial}{\partial y}\begin{bmatrix}\phi\,\mathrm{v_y} \\ \phi\,\mathrm{v_x}\;\mathrm{v_y} \\ \phi\,\mathrm{v_y^2}+\phi^2/2\end{bmatrix} = \begin{bmatrix}0 \\ g\;\phi\,\partial_x h \\ g\;\phi\,\partial_y h\end{bmatrix}$$

$\Rightarrow$ Source terms unavoidable when we have varying bathymetry.

In primitive form, we have:

$$\frac{\partial}{\partial t}\begin{bmatrix}\phi \\ \mathrm{v_x} \\ \mathrm{v_y}\end{bmatrix} + \begin{bmatrix}\mathrm{v_x} & \phi & 0 \\ 1 & \mathrm{v_x} & 0 \\ 0 & 0 & \mathrm{v_x}\end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix}\phi \\ \mathrm{v_x} \\ \mathrm{v_y}\end{bmatrix} + \begin{bmatrix}\mathrm{v_y} & 0 & \phi \\ 0 & \mathrm{v_y} & 0 \\ 1 & 0 & \mathrm{v_y}\end{bmatrix}\frac{\partial}{\partial y}\begin{bmatrix}\phi \\ \mathrm{v_x} \\ \mathrm{v_y}\end{bmatrix} = \begin{bmatrix}0 \\ g\,\partial_x h \\ g\,\partial_y h\end{bmatrix}$$

The characterisitic analysis of these equations is easy enough and yields:

$\lambda^1 = \mathrm{v_x} - \sqrt{\phi}\;\;;\;\;\lambda^2 = \mathrm{v_x}\;\;;\;\;\lambda^3 = \mathrm{v_x} + \sqrt{\phi}$ $\leftarrow$ Question: Interpret these waves.

1.8) Maxwell's Equations

Useful in non-linear optics, designing fiber optic cables. Also needed for designing stealth technology.

Written as:

$$\frac{\partial \mathbf{B}}{\partial t} + c \, \nabla \times \mathbf{E} = 0 \qquad \leftarrow \text{ Faraday's Law}$$

$$\frac{\partial \mathbf{D}}{\partial t} - c \, \nabla \times \mathbf{H} = - \, 4 \, \pi \, \mathbf{J} \qquad \leftarrow \text{ Generalized Ampere's Law}$$

$$\nabla \bullet \mathbf{D} = 4 \, \pi \, \rho \qquad \leftarrow \text{ Gauss's Law}$$

$$\nabla \bullet \mathbf{B} = 0 \qquad \leftarrow \text{ Divergence-Free Constraint}$$

Closure only obtained by constitutive relationships:-

Relate magnetic induction to magnetic field : $\mathbf{B} = \mu \, \mathbf{H}$

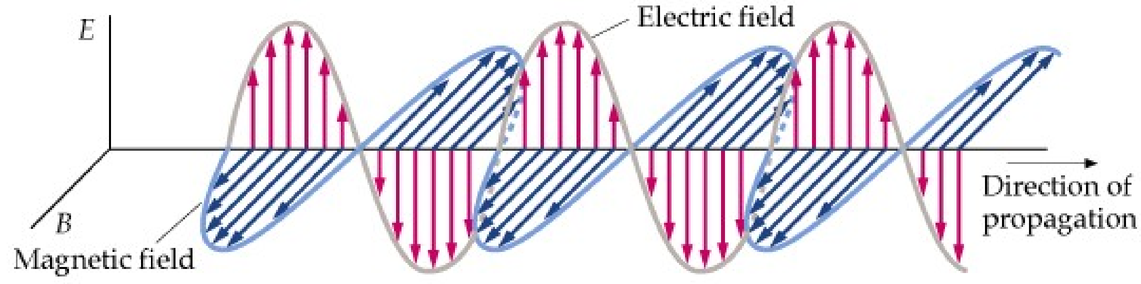
Relate displacement vector to the electric field : $\mathbf{D} = \varepsilon \, \mathbf{E}$

Assume simple, linear scalar relations (material media require tensorial relationships).

$$\frac{\partial \mathbf{H}}{\partial t} + \frac{c}{\mu} \nabla \times \mathbf{E} = 0$$

$$\frac{\partial \mathbf{E}}{\partial t} - \frac{c}{\varepsilon} \nabla \times \mathbf{H} = -\frac{4\pi}{\varepsilon} \mathbf{J}$$

$$\nabla \bullet \mathbf{E} = \frac{4\pi}{\varepsilon} \rho$$



$\nabla \bullet \mathbf{B} = 0 \quad \leftarrow$ Constraint can be important to the fidelity of the solution process

Methods have also been developed to sweep any magnetic divergence off the mesh.

For now, we write the linear equations in a form that it designed to reveal the characteristic matrix:

$$\frac{\partial}{\partial t}\begin{pmatrix} E_x \\ E_y \\ E_z \\ B_x \\ B_y \\ B_z \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c/\varepsilon \\ 0 & 0 & 0 & 0 & -c/\varepsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -c/\mu & 0 & 0 & 0 \\ 0 & c/\mu & 0 & 0 & 0 & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} E_x \\ E_y \\ E_z \\ B_x \\ B_y \\ B_z \end{pmatrix} = 0$$

Question: What are the eigenvectors of this system telling us?

1.9) The Magnetohydrodynamic Equations

Very useful in terrestrial fusion expts., interiors of stars, solar wind, magnetospheres of stars and planets. Needed: large system + lo resistivity

When gas it hot enough, it becomes partially/fully ionized. Charged particles gyrate around magnetic fields ➔ *matter and field are tightly coupled*. Magnetohydrodynamics (***MHD***) is the simplest approximation.

Viewed over large enough length scales, plasma is neutral ➔ local charge imbalances are v. rapidly neutralized in plasma's rest frame.

Viewed over large enough time scales, i.e. longer than plasma waves ➔

$$\partial \mathbf{D} / \partial t = 0$$

The B-field is locked in the plasma ➔ Lorenz-force acts in momentum eqn:

$$\rho \frac{D\,\mathbf{v}}{D\,t} = -\nabla P + \frac{1}{c}\left(\mathbf{J} \times \mathbf{B}\right) + \overleftrightarrow{\nabla}\pi \quad \leftarrow \quad \text{Recall: current needed to sustain B-field!}$$

We also need an evolutionary equation for B – field :

$$\frac{\partial \mathbf{B}}{\partial t} + c\,\nabla \times \mathbf{E} = 0 \quad \leftarrow \quad \text{Faraday's Law;} \qquad \mathbf{J} = \frac{c}{4\,\pi}\,\nabla \times \mathbf{B} \quad \leftarrow \text{ Ampere's Law}$$

In the fluid's (primed) rest frame we have Ohm's law: $\mathbf{J}' = \sigma\,\mathbf{E}'$

Lorenz Transform to the fluid's rest frame to get : $\mathbf{J}' = \mathbf{J} \quad ; \quad \mathbf{E}' = \mathbf{E} + \frac{1}{c}\left(\mathbf{v} \times \mathbf{B}\right)$

Ohm's Law in the Eulerian frame of reference then gives:

$$\mathbf{J}' = \sigma\,\mathbf{E}' \quad \Rightarrow \quad \mathbf{J} = \sigma\left(\mathbf{E} + \frac{1}{c}\,\mathbf{v} \times \mathbf{B}\right) \quad \Rightarrow \quad \mathbf{E} = -\frac{1}{c}\,\mathbf{v} \times \mathbf{B} + \frac{c}{4\pi\sigma}\,\nabla \times \mathbf{B}$$

This gives us our *evolutionary equation for the B - field* :

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{v} \times \mathbf{B}\right) + \frac{c^2}{4\pi\sigma}\,\nabla^2 \mathbf{B}}$$

Question: Interpret these 2 terms

Write the Lorenz force as : $\frac{1}{c}\left(\mathbf{J} \times \mathbf{B}\right) = -\frac{1}{4\pi}\,\mathbf{B} \times \left(\nabla \times \mathbf{B}\right) = -\nabla\left(\frac{\mathbf{B}^2}{8\pi}\right) + \frac{1}{4\pi}\left(\mathbf{B} \bullet \nabla\right)\mathbf{B}$

The momentum equation then gives : $$\rho \frac{D\,\mathbf{v}}{D\,t} = -\nabla\left(\mathrm{P} + \frac{\mathbf{B}^2}{8\pi}\right) + \frac{1}{4\pi}\left(\mathbf{B}\bullet\nabla\right)\mathbf{B} + \overleftrightarrow{\nabla}\pi$$

The full set of ideal MHD equations in Conservation form can then be written as:

$$\frac{\partial\,\rho}{\partial\,t} + \frac{\partial}{\partial\,x_i}\left(\rho\,\mathrm{v}_i\right) = 0$$

$$\frac{\partial}{\partial\,t}\left(\rho\mathrm{v}_i\right) + \frac{\partial}{\partial\,x_j}\left(\rho\mathrm{v}_i\mathrm{v}_j + \left(\mathrm{P} + \mathbf{B}^2/8\pi\right)\delta_{ij} - \mathrm{B}_i\,\mathrm{B}_j/4\pi\right) = 0$$

$$\frac{\partial\,\varepsilon}{\partial\,t} + \frac{\partial}{\partial\,x_i}\left(\left(\varepsilon + \mathrm{P} + \mathbf{B}^2/8\pi\right)\mathrm{v}_i - \mathrm{B}_i\left(\mathbf{v}\cdot\mathbf{B}\right)/4\pi\right) = 0$$

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times\left(\mathbf{v}\times\mathbf{B}\right) \qquad \text{with } \nabla\bullet\mathbf{B} = 0$$

with $\varepsilon = \mathrm{e} + \frac{1}{2}\rho\,\mathbf{v}^2 + \frac{\mathbf{B}^2}{8\pi}$ and $\mathrm{e} \equiv \frac{\mathrm{P}}{\left(\Gamma-1\right)}$

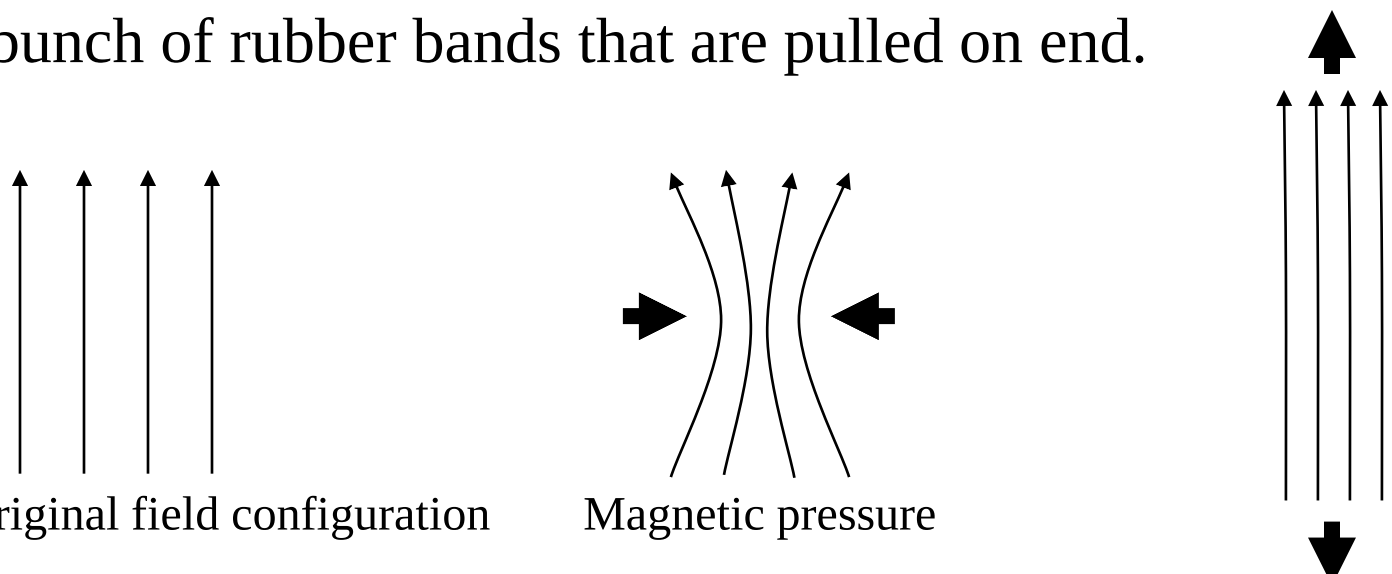
Question: Can you spot the magnetic energy density and Poynting flux on this page?

Flux Freezing Approximation for Ideal MHD

The magnetic fields are frozen into the fluid and move with the fluid in the ideal MHD limit.

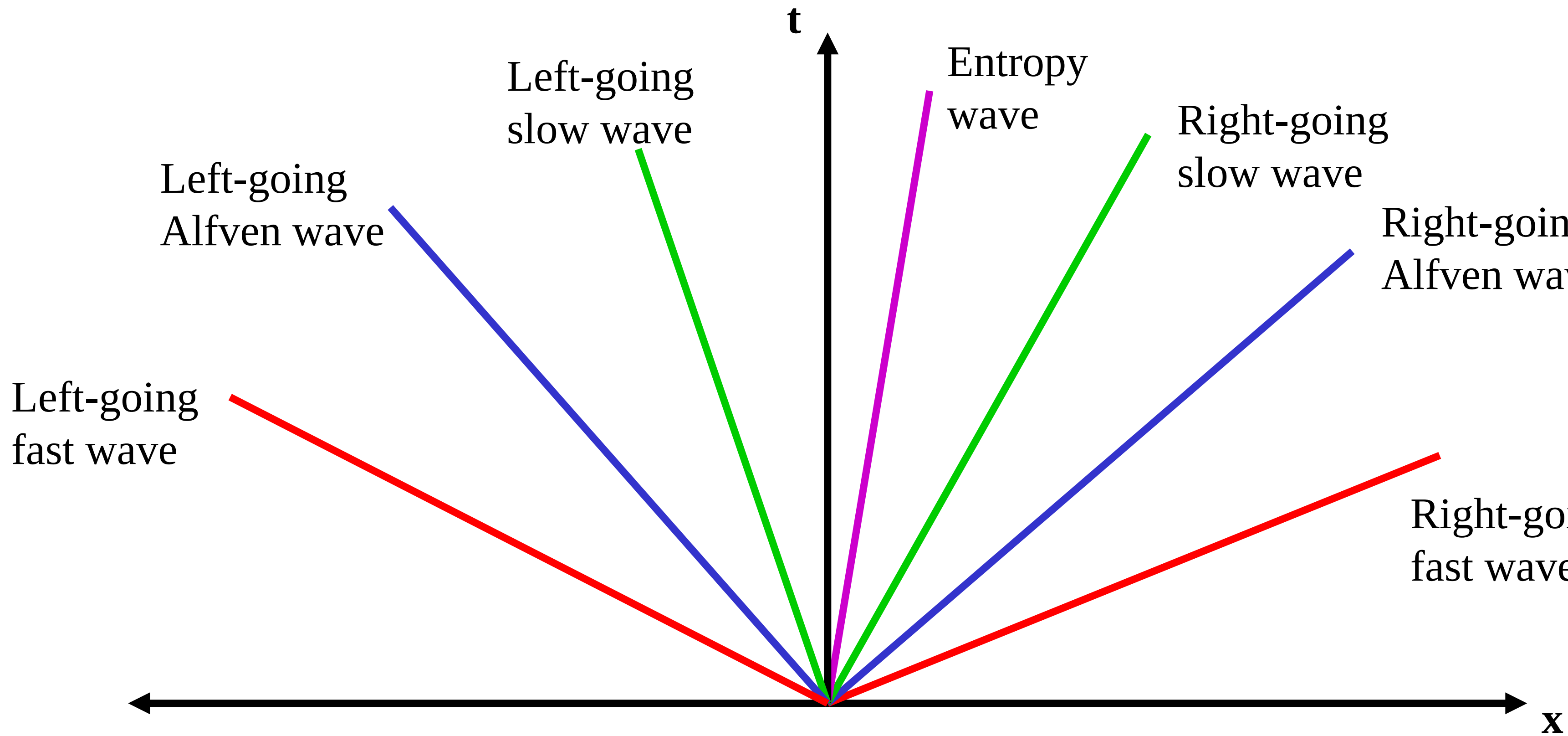
Thus if the fluid is compressed transversely to the magnetic field direction, the magnetic fields get squished too. This gives us an extra magnetic pressure term. Hint: Think of $B^2 / 8\pi$ to see how it works.

If the fluid is pulled longitudinal to the magnetic field direction, the magnetic fields also produce extra magnetic tensional forces, just like a bunch of rubber bands that are pulled on end.



The equations of MHD have seven propagating wave families.

The Riemann problem (I.e. the cell-break problem) tells us how these families propagate at discontinuities. This eminently physical procedure is used to build some of the most robust and accurate numerical schemes for MHD.



Space-Time Diagram for MHD Waves

1.10) Flux Limited Diffusion (FLD) Radiation Hydrodynamics

It is tempting to build a "hydrodynamic" approximation for photons interacting with atoms. Question: What are the deficiencies in that? (Hint: Compare $\sigma_{Couloumb}$ to $\sigma_{Thompson}$.)

The Flux Limited Diffusion approximation does *not* solve this problem. It does, however, make it possible to arrive at a more tractable set of equations that can be solved.

Works best in the *optically thick regime*. Provides gracious breakdown in the *optically thin regime*. Question: What do these two regimes mean?

For some problems, we only care for the optically thick regime. FLD is ok in such situations because it assumes that photons *diffuse* through matter.

Mathematically: Assert that the *radiation energy density* E is the only variable of interest. Claim that *radiation flux* **F** and *radiation pressure tensor* **P** obtained from it. We will not derive, but give feel for equations.

$$\mathbf{F} = -\frac{c\lambda}{\kappa_{0R}} \nabla \mathrm{E} + \mathbf{v}\,\mathrm{E} + \mathbf{v} \bullet \mathbf{P}$$ ← Interpret these terms ; λ is flux limiter.

Question: With κ_{0R} being the reciprocal of a mean free path, can you interpret c/κ_{0R} ?

$$\mathbf{P} = \frac{\mathrm{E}}{2}\left[(1-\mathrm{R}_2)\,\mathbf{I} + (3\,\mathrm{R}_2 - 1)\,\mathbf{n}\otimes\mathbf{n}\right] \;;\; \mathrm{R}_2 = \lambda + \lambda^2\,\mathrm{R}^2 \;;\; \mathbf{n} = -\frac{\nabla \mathrm{E}}{|\nabla \mathrm{E}|} \;;\; \mathrm{R} = \frac{|\nabla\,\mathrm{E}|}{\kappa_{0R}\,\mathrm{E}}$$

The final equations are:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho\,\mathrm{v}_i) = 0$$

$$\frac{\partial}{\partial t}(\rho \mathrm{v}_i) + \frac{\partial}{\partial x_j}\left(\rho \mathrm{v}_i \mathrm{v}_j + \mathrm{P}\,\delta_{ij}\right) = -\lambda\,\frac{\partial\,\mathrm{E}}{\partial x_i}$$

$$\frac{\partial\,\varepsilon}{\partial t} + \frac{\partial}{\partial x_i}\left((\varepsilon + \mathrm{P})\,\mathrm{v}_i\right) = -\kappa_{0P}\left(4\pi\,\mathrm{B} - c\,\mathrm{E}\right) + \lambda\left(2\frac{\kappa_{0P}}{\kappa_{0R}} - 1\right)\mathbf{v}\bullet\nabla\mathrm{E} - \frac{3-\mathrm{R}_2}{2}\,\kappa_{0P}\,\frac{\mathbf{v}^2}{c}\,\mathrm{E}$$

$$\frac{\partial\,\mathrm{E}}{\partial t} + \frac{\partial}{\partial x_i}\left(\frac{3-\mathrm{R}_2}{2}\,\mathrm{E}\,\mathrm{v}_i\right) = \nabla\bullet\left(\frac{c\lambda}{\kappa_{0R}}\,\nabla\,\mathrm{E}\right)$$
$$+\,\kappa_{0P}\left(4\pi\,\mathrm{B} - c\,\mathrm{E}\right) - \lambda\left(2\frac{\kappa_{0P}}{\kappa_{0R}} - 1\right)\mathbf{v}\bullet\nabla\mathrm{E} + \frac{3-\mathrm{R}_2}{2}\,\kappa_{0P}\,\frac{\mathbf{v}^2}{c}\,\mathrm{E}$$

Question: Interpret the RHS terms. Interpret all the terms in the radiation energy equation.

1.11) Radiative Transfer

When the medium is not optically thick, photons do not propagate diffusively. We then have to treat the propagation of photons more carefully.

Photons can be absorbed or emitted by matter. (Give examples where this happens) They also scatter off the matter. (Question: Give terrestrial and stellar examples of scattering.)

In such situations, at each location "**x**", we study the propagation of photons in each direction "**Ω**".

The amount of radiant energy (in a frequency range ν to ν + dν) propagating per unit time through an infinitesimal area *dA* that is orthogonal to **Ω** is given by the *radiation intensity*: $I\left(\mathbf{x},\mathbf{\Omega},\nu,t\right)\ d\mathbf{\Omega}\ dA\ d\nu$

Question: How is this analogous to a distribution function for gas particles? Hint: for a photon, E = h ν = p c.

The *radiative transfer equation* for photons is the analogue of the collisional Boltzmann equation:

$$\frac{1}{c}\frac{\partial}{\partial t} I(\mathbf{x},\mathbf{\Omega},\nu,t) + \mathbf{\Omega} \bullet \nabla I(\mathbf{x},\mathbf{\Omega},\nu,t) = \kappa(\mathbf{x},\nu,t)\, I_b(\mathrm{T}(\mathbf{x},t),\nu)$$
$$-\left(\kappa(\mathbf{x},\nu,t)+\sigma(\mathbf{x},\nu,t)\right) I(\mathbf{x},\mathbf{\Omega},\nu,t)$$
$$+\ \frac{\sigma(\mathbf{x},\nu,t)}{4\pi}\int \Phi(\mathbf{\Omega},\mathbf{\Omega}')\, I(\mathbf{x},\mathbf{\Omega}',\nu,t)\, d\mathbf{\Omega}'$$

Note: This is not just one equation for a single $\mathbf{\Omega}$, but rather a set of equations for the *ordinates*, $\mathbf{\Omega}$, spanning *all* directions.

The left hand side just says that photons stream freely in the absence of matter (and, therefore, in the absence of collisions with matter).

The right hand side contains the effect of photon-matter interaction. Question: Interpret each of the terms. Why does I_b depend on the temperature of matter? κ is the absorption opacity and σ is the scattering opacity. How do those terms differ in the right hand side?

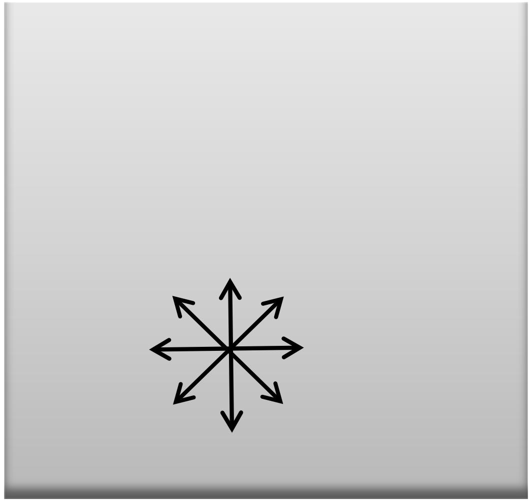
First simplification: Speed of light >> all other speeds. ➔ time-dependence can be dropped.

Second simplification: Solve for a small number of frequency bins – *picket fence approximation*. Alternatively, integrate over all frequencies – *gray approximation*.

Third simplification: Solve only for a small set of ordinate directions. For each integer "*N*" there are only *N (N + 2)* ordinates. Gives rise to an S_N method.

Fourth simplification: Oftentimes, the matter is assumed stationary.

$$\begin{aligned}\boldsymbol{\Omega}_i \bullet \nabla I\left(\mathbf{x},\boldsymbol{\Omega}_i,\nu\right) &= \kappa\left(\mathbf{x},\nu\right)\, I_b\left(\mathrm{T}\left(\mathbf{x}\right),\nu\right) \\ &\quad -\left(\kappa\left(\mathbf{x},\nu\right)+\sigma\left(\mathbf{x},\nu\right)\right) I\left(\mathbf{x},\boldsymbol{\Omega}_i,\nu\right) \\ &\quad + \frac{\sigma\left(\mathbf{x},\nu\right)}{4\pi} \sum_{j=1}^{N(N+2)} w_j\, \Phi\left(\boldsymbol{\Omega}_i,\boldsymbol{\Omega}_j\right) I\left(\mathbf{x},\boldsymbol{\Omega}_j,\nu\right)\end{aligned}$$



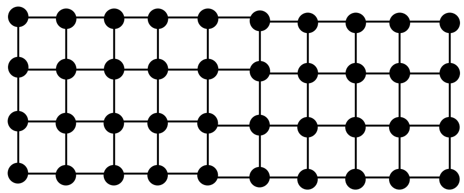
1.12) Equations of Linear Elasticity

We can stretch a spring and verify the linear relation in Hooke's law. Restoring force (*stress*) *proportional to* deformation (*strain*).

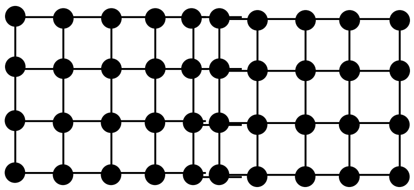
Imagine plucking the spring. Set up *compressible oscillations*.

We can also set up *transverse oscillations*.

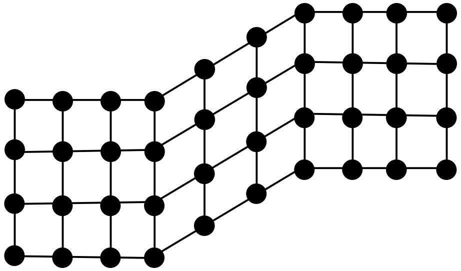
At an atomic level, the force comes from the deformation of atomic bonds



Undeformed

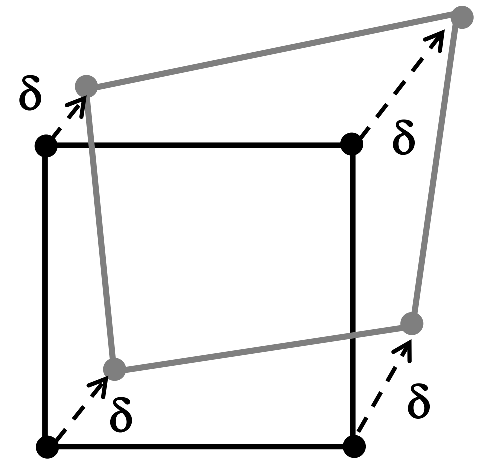


Extensional strain



Shear strain

Let the original positions of the "atoms"/mass points be : $\mathbf{x} = (x, y, z)$



After application of strain, the shifted positons are :

$$\mathbf{X} = \left(\mathrm{X}(x, y, z, t), \mathrm{Y}(x, y, z, t), \mathrm{Z}(x, y, z, t)\right)$$

Displacement vector : $\boldsymbol{\delta} \equiv \mathbf{X} - \mathbf{x}$

We seek a definition for strain that is independent of translations or solid body rotations.

Consider 3×3 tensor : $\partial \boldsymbol{\delta} / \partial \mathbf{x}$. It does eliminate translations.

Solid body rotations are given by anti-symmetric tensor : $\left[(\partial \boldsymbol{\delta}/\partial \mathbf{x}) - (\partial \boldsymbol{\delta}/\partial \mathbf{x})^T \right] / 2$

The *strain* is , therefore, given by the *symmetric* tensor : $\boldsymbol{\varepsilon} \equiv \left[(\partial \boldsymbol{\delta}/\partial \mathbf{x}) + (\partial \boldsymbol{\delta}/\partial \mathbf{x})^T \right] / 2$

We will shortly see how the stress tensor $\boldsymbol{\sigma}$ *linearly* relates to the strain tensor $\boldsymbol{\varepsilon}$.

The velocity is defined by : $\mathbf{v} \equiv (\mathrm{v_x}, \mathrm{v_y}, \mathrm{v_z}) = (\partial \mathrm{X}/\partial t, \partial \mathrm{Y}/\partial t, \partial \mathrm{Z}/\partial t)$

and satisfies consistency conditions of the form :

$$\partial_t \varepsilon_{11} = \frac{\partial}{\partial t}\left(\frac{\partial \mathrm{X}(x, y, z, t)}{\partial x} - 1 \right) = \frac{\partial}{\partial x}\left(\frac{\partial \mathrm{X}(x, y, z, t)}{\partial t} \right) = \partial_x \mathrm{v_x}$$

Dynamical equations given by:

$$\partial_t \varepsilon_{11} - \partial_x \mathrm{v}_\mathrm{x} = 0 \quad ; \quad \partial_t \varepsilon_{22} - \partial_y \mathrm{v}_\mathrm{y} = 0 \quad ; \quad \partial_t \varepsilon_{33} - \partial_z \mathrm{v}_\mathrm{z} = 0 \quad ;$$

$$\partial_t \varepsilon_{12} - \frac{1}{2}\left(\partial_y \mathrm{v}_\mathrm{x} + \partial_x \mathrm{v}_\mathrm{y}\right) = 0 \;;\; \partial_t \varepsilon_{23} - \frac{1}{2}\left(\partial_z \mathrm{v}_\mathrm{y} + \partial_x \mathrm{v}_\mathrm{y}\right) = 0 \;;\; \partial_t \varepsilon_{13} - \frac{1}{2}\left(\partial_z \mathrm{v}_\mathrm{x} + \partial_x \mathrm{v}_\mathrm{z}\right) = 0 \;;$$

$$\rho\, \partial_t \mathrm{v}_\mathrm{x} - \partial_x \sigma_{11} - \partial_y \sigma_{12} - \partial_z \sigma_{13} = 0 \quad ;$$

$$\rho\, \partial_t \mathrm{v}_\mathrm{y} - \partial_x \sigma_{12} - \partial_y \sigma_{22} - \partial_z \sigma_{23} = 0 \quad ;$$

$$\rho\, \partial_t \mathrm{v}_\mathrm{z} - \partial_x \sigma_{13} - \partial_y \sigma_{23} - \partial_z \sigma_{33} = 0 \quad ;$$

i.e. 6 consistency conditions and 3 components of Newton's laws.

The linear stress-strain relation is a constitutive relationship given by:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} \Lambda + 2\mu & \Lambda & \Lambda & 0 & 0 & 0 \\ \Lambda & \Lambda + 2\mu & \Lambda & 0 & 0 & 0 \\ \Lambda & \Lambda & \Lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{13} \end{bmatrix}$$

Λ and μ are Lame$'$ parameters and are related to the Young's modulus "E" and Poisson ration"μ".

The final form of the equations for *linear elasticity* is:

$$\partial_t \sigma_{11} - (\Lambda + 2\mu)\partial_x \mathrm{v_x} - \Lambda\, \partial_y \mathrm{v_y} - \Lambda\, \partial_z \mathrm{v_z} = 0$$

$$\partial_t \sigma_{22} - \Lambda\, \partial_x \mathrm{v_x} - (\Lambda + 2\mu)\partial_y \mathrm{v_y} - \Lambda\, \partial_z \mathrm{v_z} = 0$$

$$\partial_t \sigma_{33} - \Lambda\, \partial_x \mathrm{v_x} - \Lambda\, \partial_y \mathrm{v_y} - (\Lambda + 2\mu)\partial_z \mathrm{v_z} = 0$$

$$\partial_t \sigma_{12} - \mu\left(\partial_y \mathrm{v_x} + \partial_x \mathrm{v_y}\right) = 0$$

$$\partial_t \sigma_{23} - \mu\left(\partial_z \mathrm{v_y} + \partial_x \mathrm{v_y}\right) = 0$$

$$\partial_t \sigma_{13} - \mu\left(\partial_z \mathrm{v_x} + \partial_x \mathrm{v_z}\right) = 0$$

$$\rho\, \partial_t \mathrm{v_x} - \partial_x \sigma_{11} - \partial_y \sigma_{12} - \partial_z \sigma_{13} = 0$$

$$\rho\, \partial_t \mathrm{v_y} - \partial_x \sigma_{12} - \partial_y \sigma_{22} - \partial_z \sigma_{23} = 0$$

$$\rho\, \partial_t \mathrm{v_z} - \partial_x \sigma_{13} - \partial_y \sigma_{23} - \partial_z \sigma_{33} = 0$$

For x-directional variations we can again write a characteristic matrix and obtain:

$$\lambda^1 = -c_P \ ; \ \lambda^2 = \lambda^3 = -c_S \ ; \ \lambda^4 = \lambda^5 = \lambda^6 = 0 \ ; \ \lambda^7 = \lambda^8 = c_S \ ; \ \lambda^9 = c_P$$

where $c_P = \sqrt{\dfrac{\Lambda + 2\mu}{\rho}} \ ; \ c_S = \sqrt{\dfrac{\mu}{\rho}}$

Question: Interpret these modes. What do they tell you? Explain why you get 6 propagating modes.

1.13) Relativistic Magnetohydrodynamics

(Rel-MHD) in Conservation form.(c=1)

1) Compare with non-relativistic case. Lab frame v/s rest frame.
2) Notice that the thermal energy and magnetic energy contribute to the inertia in the momentum equations. Similarly for Lorentz factor. Why?
3)The rest mass energy & magnetic energy contribute to the energy density.
4) Notice the promotion of the magnetic field to a 4-vector.
5) Notice that the induction equation remains unchanged owing to the fact that Maxwell’s Equations are already Lorentz-invariant!

$$\frac{\partial}{\partial t}(\rho\,\gamma)+\frac{\partial}{\partial x_i}(\rho\,\gamma\,\mathrm{v}_i)=0$$

$$\frac{\partial}{\partial t}\left(\rho\,h\,\gamma^2\,\mathrm{v}_i\,+\,(\mathbf{E}\times\mathbf{B})_i\right)\,+\,\frac{\partial}{\partial x_j}\left(\rho\,h\,\gamma^2\,\mathrm{v}_i\,\mathrm{v}_j\,-\,\mathrm{E}_i\,\mathrm{E}_j\,-\,\mathrm{B}_i\,\mathrm{B}_j\,+\,\left(\mathrm{P}+\frac{1}{2}\left(\mathbf{E}^2+\mathbf{B}^2\right)\right)\gamma\,\delta_{ij}\right)=0$$

$$\frac{\partial}{\partial t}\left(\rho\,h\,\gamma^2-\,\mathrm{P}+\frac{1}{2}\left(\mathbf{E}^2+\mathbf{B}^2\right)\right)\,+\,\frac{\partial}{\partial x_i}\left(\rho\,h\,\gamma^2\,\mathrm{v}_i\,+\,(\mathbf{E}\times\mathbf{B})_i\right)=0$$

$$\frac{\partial\mathbf{B}}{\partial t}\;=\;\nabla\times(\mathbf{v}\times\mathbf{B})\qquad;\qquad\nabla\bullet\mathbf{B}=0\qquad;\qquad\mathbf{E}=-\mathbf{v}\times\mathbf{B}$$

$$h=1+\,\mathrm{P}\,\Gamma\big/\left[\rho\,\left(\Gamma-1\right)\right]$$

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