Chp 4: Non-linear Conservation Laws; the Scalar Case

By

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4.1) Introduction

We have seen that *monotonicity preserving reconstruction* and *Riemann solvers* are essential building blocks for numerically solving a linear hyperbolic system.

While the same remains true for a ***non-linear*** *system of conservation laws*, the emphasis shifts.

For non-linear systems, the Riemann solver and the reconstruction problem become more complicated.

The presence of *non-linearity* introduces additional complications – the presence of *shocks* and *rarefactions*. We study them for the simplest *scalar* case:

$u_t + f(u)_x = 0$

Need to focus on df(u)/du , the *wave speed*, and $d^2f(u)/du^2$, *convexity*.

With $f(u) = u^2 / 2$ we get *Burgers equation*. Interesting because it can produce prototypes of many of the shocks and rarefactions we will study later.

Conceptual simplification if $f^{//}(u)$ does not change sign, i.e. eqn. is ***convex***. Then the wavespeed either monotonically increases or decreases with “u”. Burgers eqn. is convex. Euler system can also be shown to be convex.

For hyperbolic conservation laws we will see that:
Convexity + strict hyperbolicity ➔ *several advantages* in designing numerical solution methods.

If $f^{//}(u)$ does change sign, the eqn. is ***non-convex***. When the PDE is non-convex, we are not on very firm ground. Examples, multiphase flow, non-linear elasticity equations, MHD.

4.2) A Gentle Introduction to Rarefaction Waves and Shocks

4.2.1) A Mechanistic Model for Rarefaction Waves and Shocks

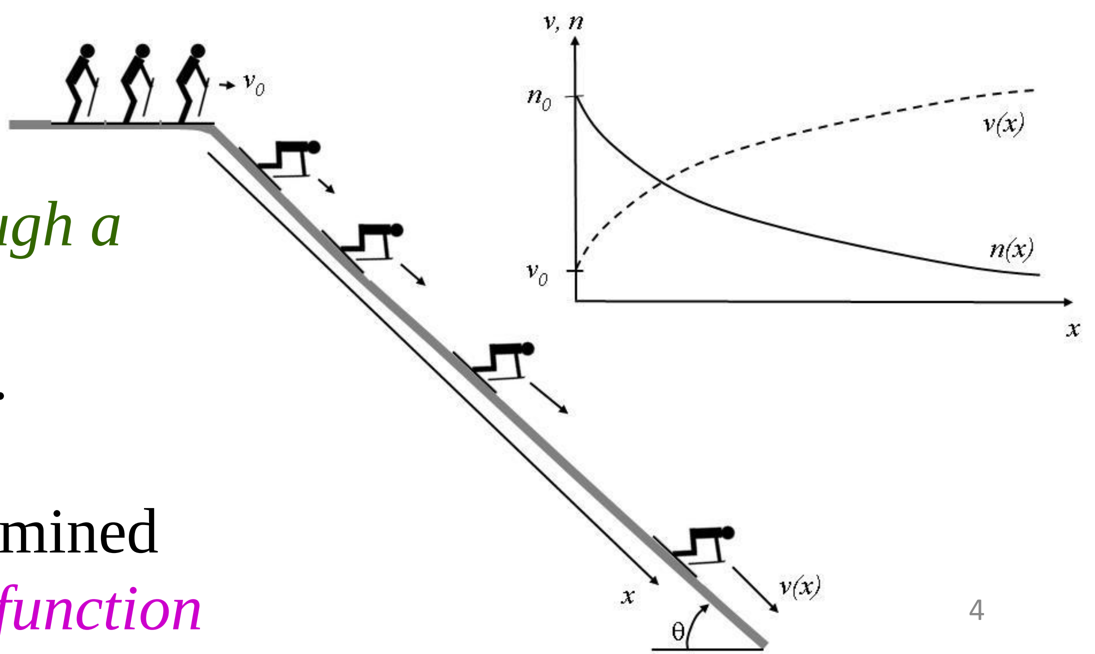
The idea here is to study a very simple model to develop intuition.

Simple model for rarefaction waves: Imagine skiers going downhill. Linear number density n_0 skiers per meter wait at the ski ramp, moving to the starting point with a speed v_0 .

Speed of skier: $v^2(x) = v_0^2 + 2\,g\,x\,\sin\theta$

Flux conservation: $n_0\,v_0 = n(x)\,v(x) \Rightarrow n(x) = n_0\,v_0 \Big/ \sqrt{v_0^2 + 2\,g\,x\,\sin\theta}$

Skiers keep changing, shape of rarefaction wave stays fixed. Analogously, *atoms move through a rarefaction, but the shape of a rarefaction wave remains fixed*.



Structure of rarefaction is determined entirely by the *form of the flux function*

<u>Simple model for shock waves</u>: Skiers reaching downhill with a high speed v_b run into a tree. They approach the bottom with number density n_b . At the pileup they will again be closely packed, number density n_0 .

The point where the pile-up occurs moves to the left with a speed "*s*". This is the shock front moving with a speed "s" to the left.

Locate yourself in the frame of the shock. Flux of skiers coming in from left : $n_b\left(v_b - s\right)$

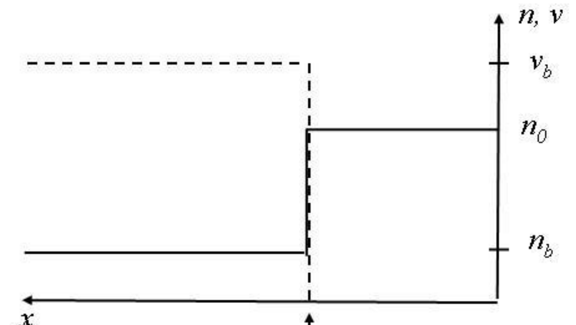
Flux of skiers leaving the plane of the shock to the right : $-n_0\ s$

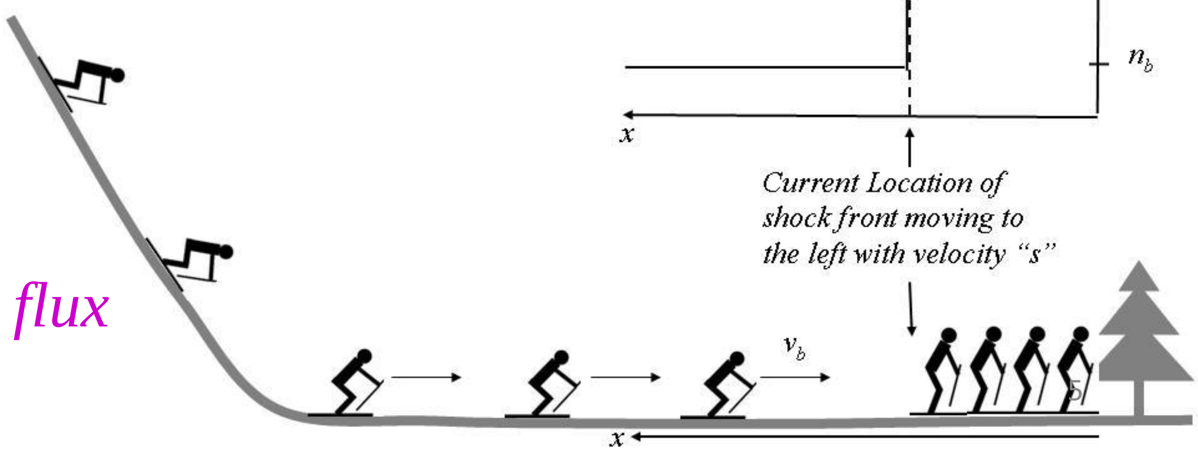
The two fluxes must balance: $n_b\left(v_b - s\right) = -n_0\ s$

As before, we use a conservation law.

Location of shock is not pinned to any one skier. *Skiers, like atoms in a fluid shock, move through the shock.*

Form of shock depends on *flux function.*





4.2.2) The Formation of Shocks and Rarefaction Waves

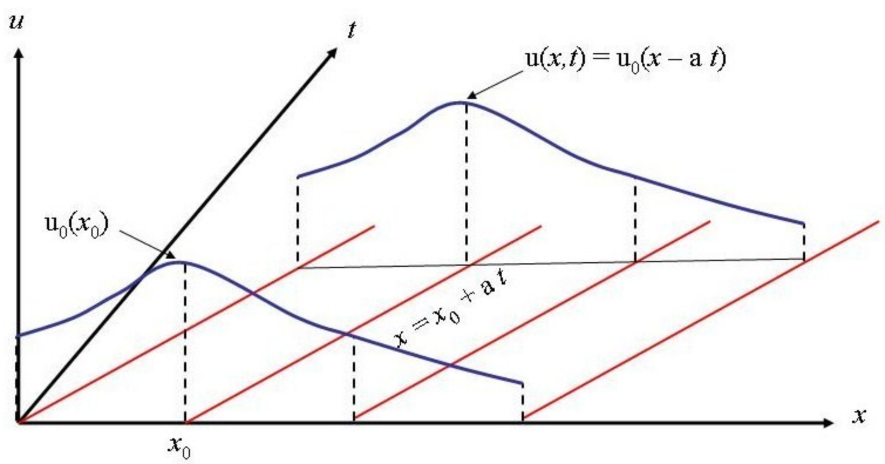
Two equivalent forms of Burgers eqn. : $u_t + \left(\frac{u^2}{2}\right)_x = 0 \quad \Leftrightarrow \quad u_t + u\,u_x = 0$

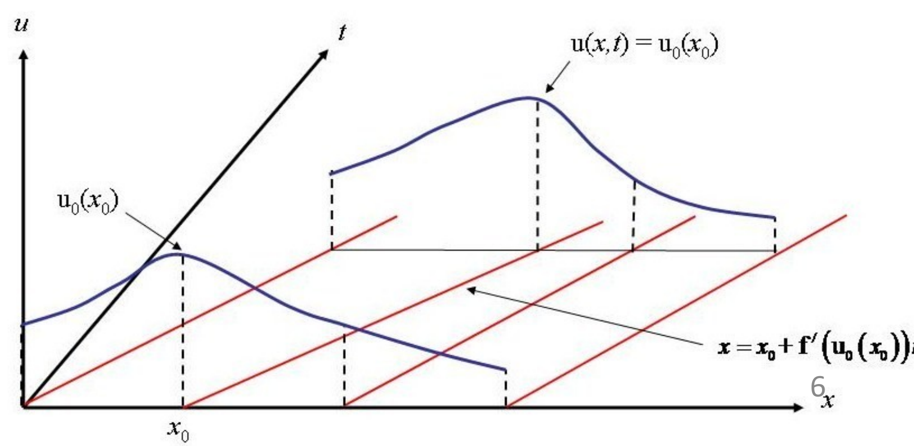
Compare with advection eqn. to see what it says : $u_t + a\,u_x = 0$

Let $u_0(x)$ be the initial condition. Let us compare the respective solutions pictorially and analytically.

Solution to Burgers equation : $u(x,t) = u_0(x_0)$ where $x_0 = x - f'(u_0(x_0))\,t$

Solution to advection equation : $u(x,t) = u_0(x_0)$ where $x_0 = x - a\,t$





Similarity: Both equations tell us that the solution at any space-time point (x,t) is obtained by *following the characteristic* through this point backward in time to the x-axis.

Difference: *Characteristics are parallel* for advection equation, not so for Burgers. *Characteristics are solution-dependent* for Burgers equation.

As a result, solution of Burgers equation becomes transcendental.
Question: Can you show this?

Depending on slope of initial conditions, *characteristics converge or diverge* for Burgers equation. Solution steepens when characteristics converge; becomes less steep where characteristics diverge. At some point in space-time, the characteristics might intersect. Let us find the first time the characteristics intersect, called the *breaking time*.

$$x = x_0 + f'\left(u_0\left(x_0\right)\right)t \quad \text{and} \quad x = x_0 + \Delta x_0 + f'\left(u_0\left(x_0 + \Delta x_0\right)\right)t$$

$$\text{Characteristics intersect when}: t = -1\Big/\left[f''\left(u_0\left(x_0\right)\right)\, u_0'\left(x_0\right)\right]$$

$$\text{Braking time, i.e. first time they intersect}: T_{break} = -1\Big/\min_x\left[f''\left(u_0\left(x\right)\right)\, u_0'\left(x\right)\right]$$

$$x = x_0 + f'\left(u_0\left(x_0\right)\right)t \quad \text{and} \quad x = x_0 + \Delta x_0 + f'\left(u_0\left(x_0 + \Delta x_0\right)\right)t$$

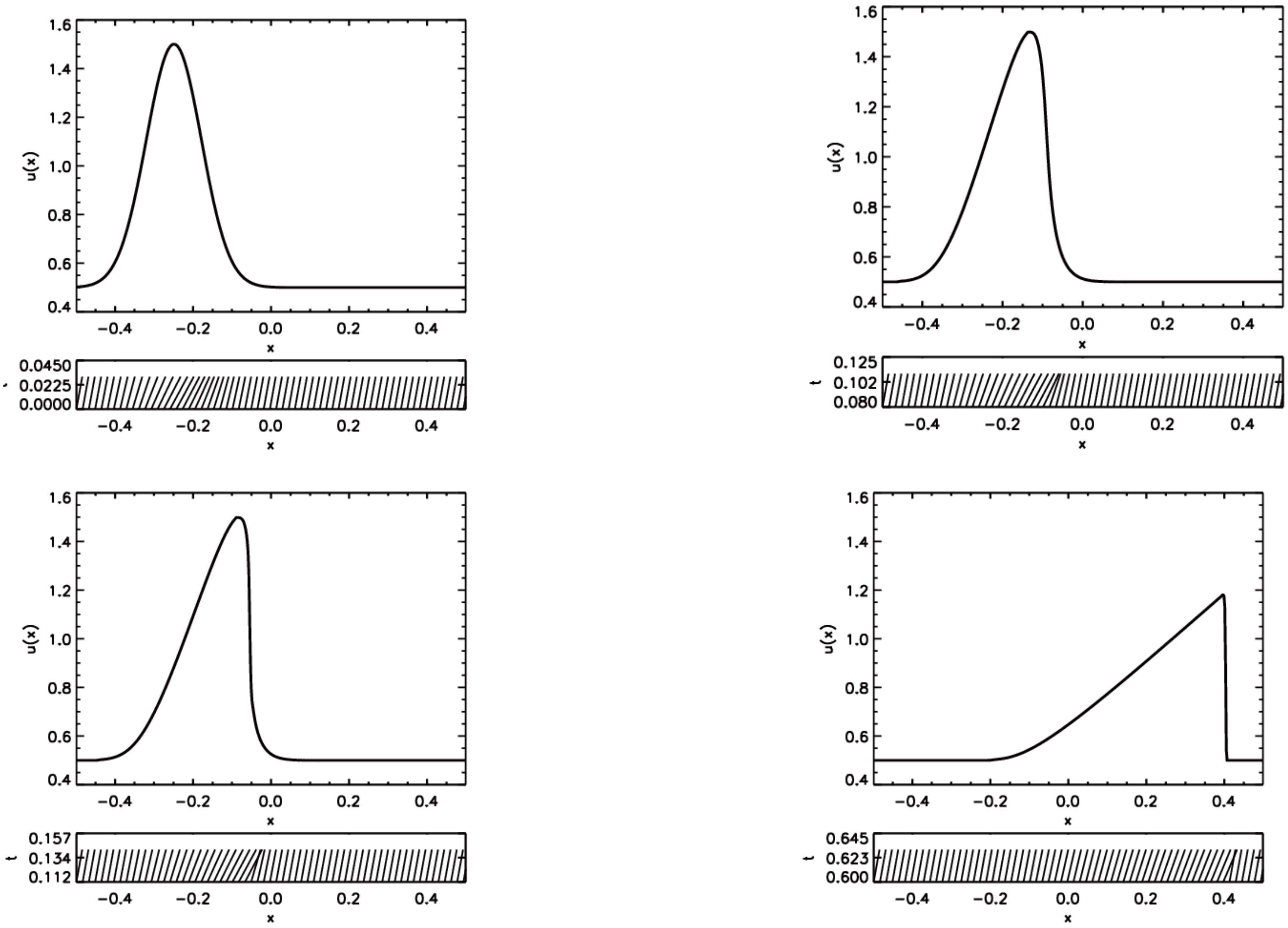
Characteristics intersect when : $t = -1 \Big/ \left[f''\left(u_0\left(x_0\right)\right)\, u_0'\left(x_0\right) \right]$

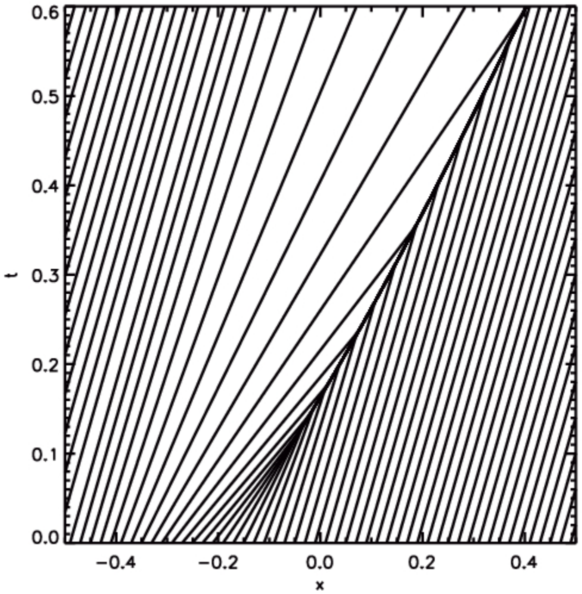
Braking time, i.e. first time they intersect : $T_{break} = -1 \Big/ \min_x \left[f''\left(u_0\left(x\right)\right)\, u_0'\left(x\right) \right]$

4.2.3) Shock and Rarefaction Wave Solutions from Burgers Equation

Burgers equation with initial condition : $u_0(x) = 0.5 + \exp\left(-100(x+0.25)^2\right)$

Solution shown at t=0, 0.08, 0.1116, 0.6 . Can find that $T_{break} = 0.1116$





This figure shows the characteristics in space-time. Notice ***compressional*** *(converging characteristics) and* ***rarefaction*** *(diverging characteristics) waves*

The ***shock*** forms when the *characteristics intersect*. The position of the shock is shown by the thick line at which the characteristics intersect.

We also observe that the *characteristics diverge* at the location of the ***rarefaction wave***.

Think of the characteristics carrying information. The *information is destroyed when the characteristics flow into a shock*. I.e., if we try to retrieve initial conditions, we can't! different initial conditions can give rise to the same shock. *Information destruction* ➔ *entropy generation*.

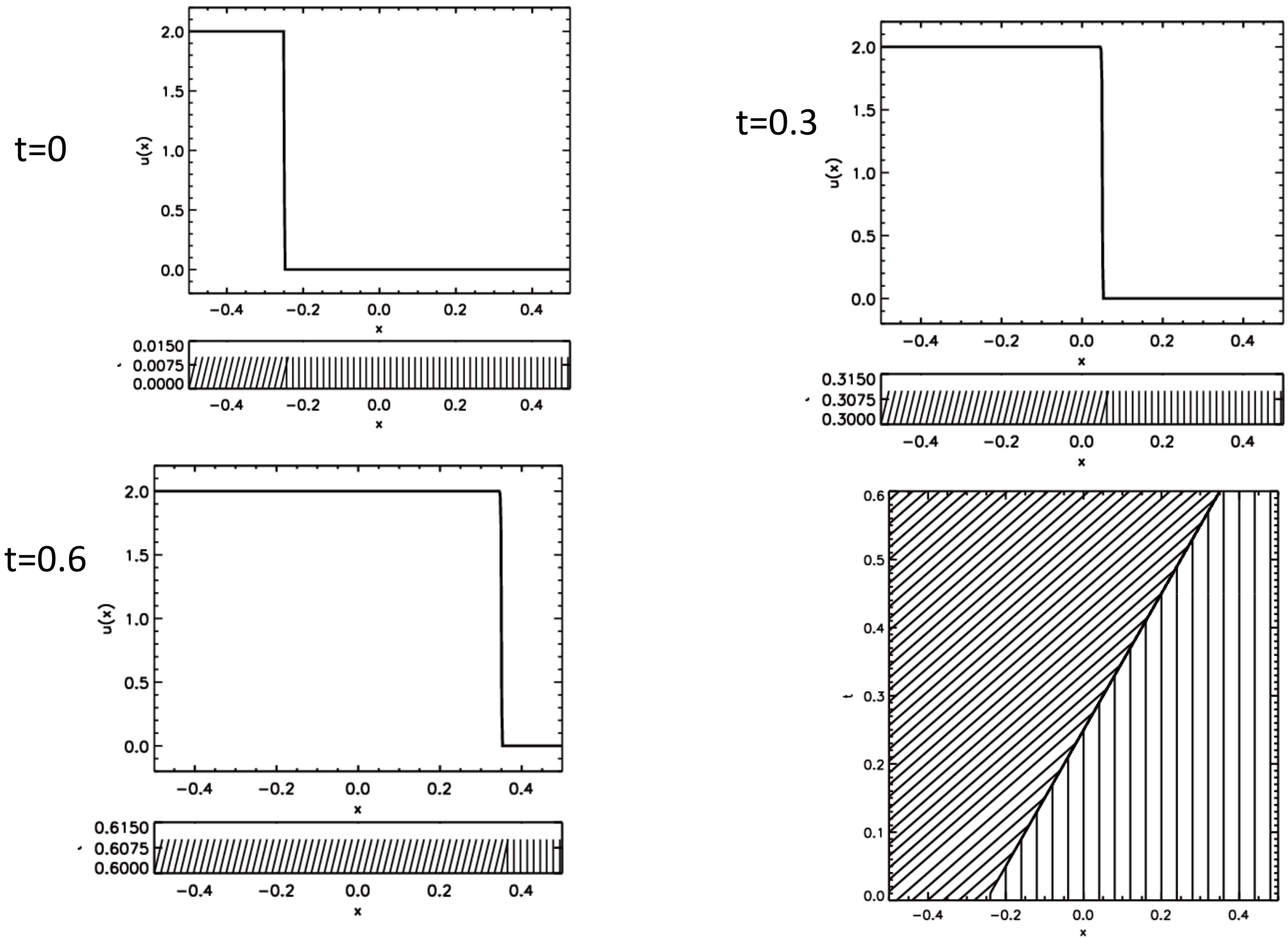
This can also be seen in hydrodynamic shocks where there is a clearly available *entropy function*.

Availability of entropy function is very useful for designing schemes.

4.2.4) Simple Wave Solutions of the Burgers Equation

Consider left and right states with : $u_0(x) = 2$ for $x < -0.25$; $u_0(x) = 0$ for $x > -0.25$

$f'(2) = 2$, $f'(0) = 0$ $\Rightarrow$ characteristics flow into initial discontinuity.



Notice, characteristics from either side are *flowing into (converging to)* the initial discontinuity

Discontinuity is form-preserving, i.e. *self-similar*. Self-similarity will become a very important concept later in this chapter. Known as ***isolated shock wave***.

Their *propagation speed* depends on the *form of the flux function*.

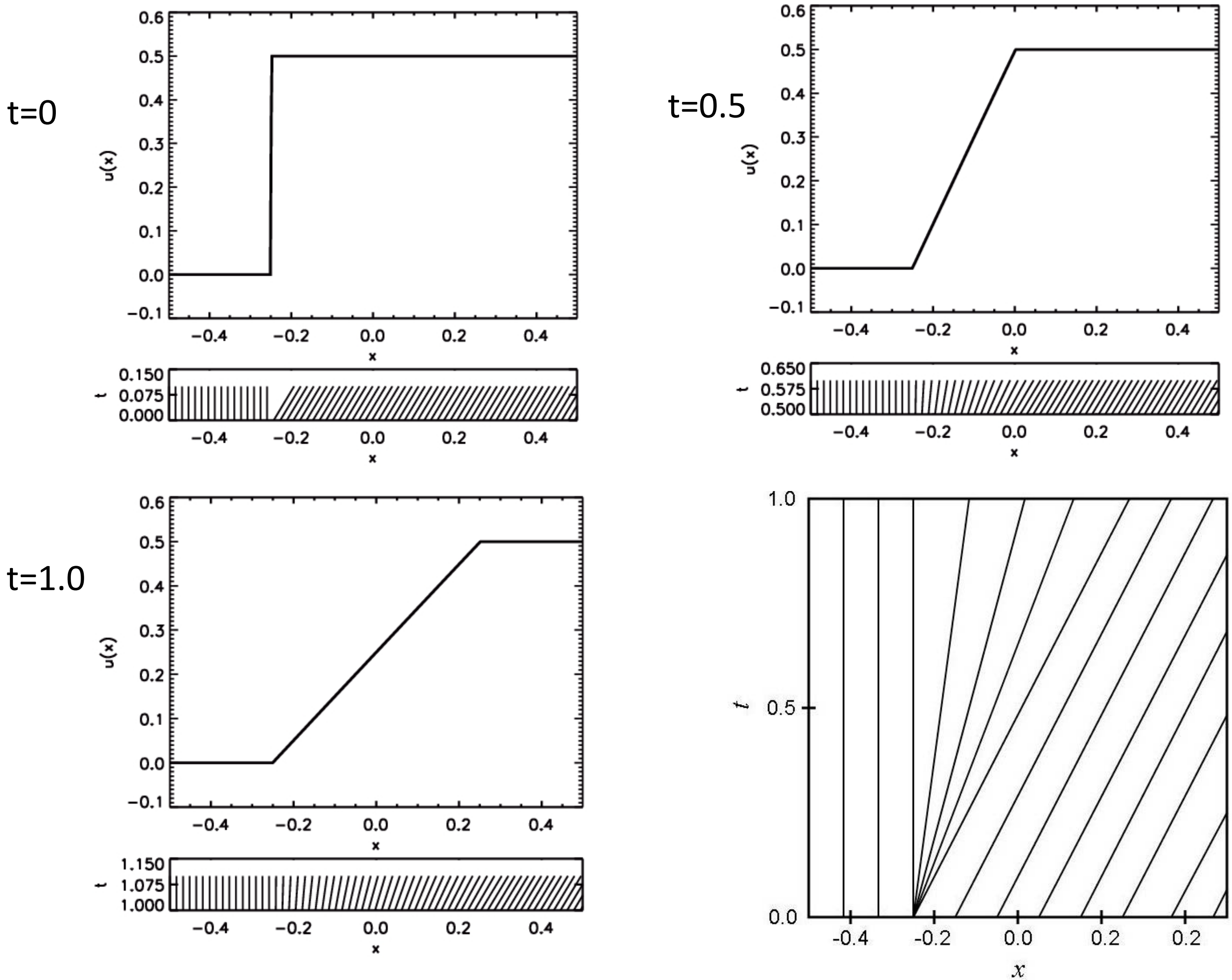
Similarity: Shocks are analogues of the *simple waves* studied in the previous chapter on linear hyperbolic systems.

Difference: However, the *speed of propagation* has become *solution-dependent* in the non-linear case. This is an important point of difference between linear and non-linear hyperbolic conservation laws.

Question: When considering linear hyperbolic systems: If the strength of a simple wave changed, did that also cause a change in its speed?

Consider left and right states with : $u_0(x) = 0$ for $x < -0.25$; $u_0(x) = 0.5$ for $x > -0.25$

$f'(0) = 0$, $f'(0.5) = 0.5$ $\Rightarrow$ characteristics flow away from initial discontinuity.



Notice, characteristics from either side are *flowing away from (diverging from)* the initial discontinuity.

Discontinuity is form-preserving, i.e. *self-similar*. Known as ***isolated rarefaction fan***. Their propagation also depends on the *form of the flux function*.

Similarity: Isolated rarefaction fans are analogues of the *simple waves* studied in the previous chapter on linear hyperbolic systems.

Difference: However, the *structure of the rarefaction* has become *solution-dependent* in the non-linear case. This is an important point of difference between linear and non-linear hyperbolic conservation laws.

We get the important insight that : *Piecewise constant initial conditions with a single discontinuity in them can give rise to isolated shocks or rarefaction fans (**self-similar simple waves**) depending on whether the characteristics **converge** into the discontinuity or **diverge** away from it*.

What happens when the conservation law has a *non-convex flux*?

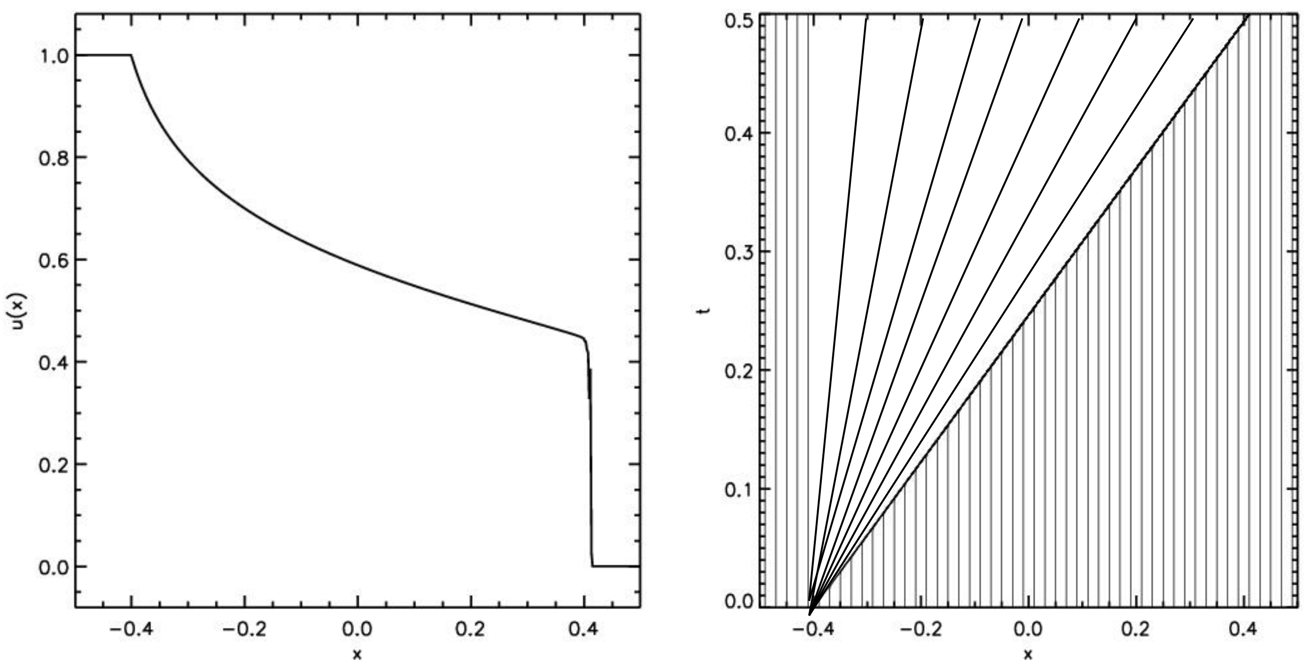
Consider the Buckley-Leverett eqn. with flux : $f(u) = \frac{4u^2}{4u^2 + (1-u)^2}$

Observe that $f'(u) = \frac{8u(1-u)}{\left[4u^2 + (1-u)^2\right]^2}$ so that $f'(0) = f'(1) = 0$ and $f'(u) > 0$ for $0<u<1$

i.e. the flux is non-convex

The result of the initial conditions shown below is a rarefaction fan stuck to a shock – a ***compound wave***. Questions: Why would a compound shock never arise for Burgers equation? Which other systems should we watch out for?

Notice that the characteristics aren't flowing into the shock from the left.



4.3) Isolated Shock Waves:

4.3.1) Shocks as Solutions of Viscous Equations in the Inviscid Limit

Consider a gas dynamics problem that forms a ***shock***. *Atoms* that make up the gas *undergo collisions*.

Collisions ➔ *non-ideal processes*, i.e. viscosity & conductivity, dominate on the smallest scales of the problem. Thus, starting with the Euler equations with a shock, we automatically reach the Navier Stokes limit if we look on *small enough scales*. On those smaller scales, the *problem is parabolic*.

Shock's profile is smoothed out on the viscous scales because the problem has been turned into a parabolic problem.

The viscous terms operating at a shock *raise the entropy* of parcels of fluid that flow into a shock.

The shock is only a discontinuity if we choose to *simultaneously ignore the viscous terms* in our governing equations as well as the *viscous length scales* in the physical problem.

$$u_t + \left(\frac{u^2}{2}\right)_x = \eta\, u_{xx} \quad \leftarrow$$ Viscous Burger's equation. Parabolic; Does not form discontinuities.

Viscous "shock" solutions : $u(x,t) = u_R + \frac{1}{2}(u_L - u_R)\left\{1 - \tanh\left[\frac{(u_L - u_R)}{4\eta}\left(x - \frac{1}{2}(u_L + u_R)t\right)\right]\right\}$

Solutions shown for : $\eta = 0.5,\ 0.2,\ 0.05,\ \eta \to 0$

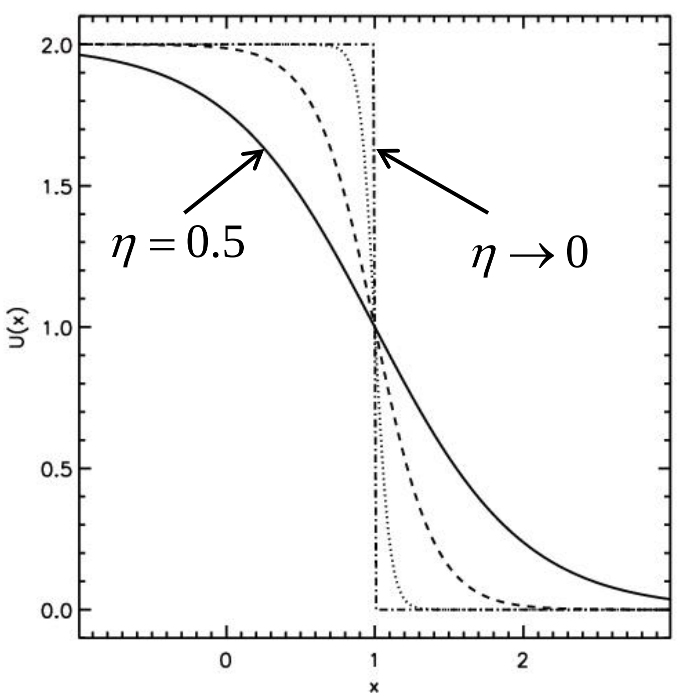
1) We see a competition between non-linear terms trying to steepen and diffusion terms trying the smooth out the shock profile. Similar competing effects operate on viscid hydrodynamical shocks.

2) Width of shock : $4\eta/(u_L - u_R)$. Decreases with decreasing η *and* increasing size of jump.

The same trend is seen for viscous shocks in the Navier Stokes equations.

3) Viscous shock moves with the same speed as inviscid shock : $\frac{1}{2}(u_L + u_R)$. The same trend is seen for viscous shocks in the Navier Stokes equation.

Question: So why don't we just study viscous shocks and forget about inviscid shocks?



The previous question compellingly shows that we have to represent shocks as isolated discontinuities in many science and engineering problems. Representing each and every viscous shock profile on a practical computing mesh is not an option.

<u>Question</u>: Can you make this claim concrete for a shock going through air in this room?

4.3.2) Shocks as Weak Solutions of a Hyperbolic Equation

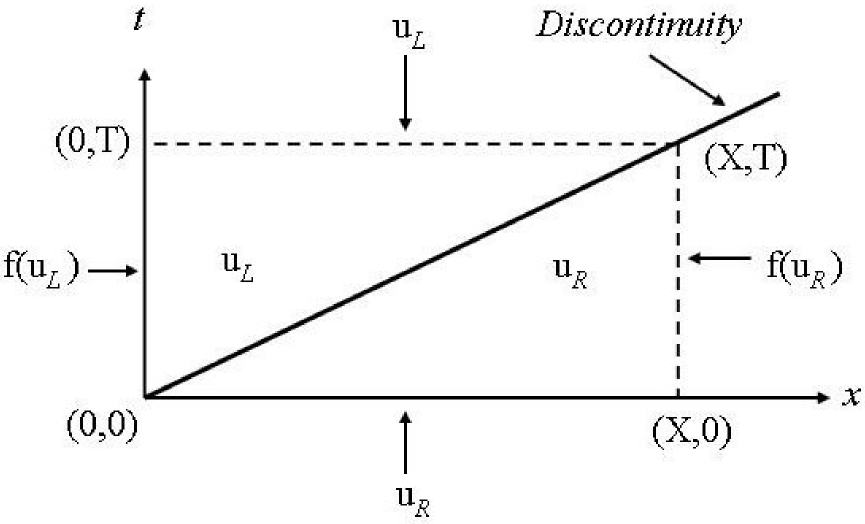
We have seen in the previous chapter that treating discontinuities, i.e. obtaining *weak solutions*, requires working with the PDE in *integral form*. The *self-similarity* of the problem ensures that the discontinuity follows a linear, self-similar, trajectory in space-time.

Integrate $u_t + f(u)_x = 0$ over rectangle

$$u_L X - u_R X + f(u_R) T - f(u_L) T = 0 \Leftrightarrow$$

$$f(u_R) - f(u_L) = \frac{X}{T} (u_R - u_L)$$

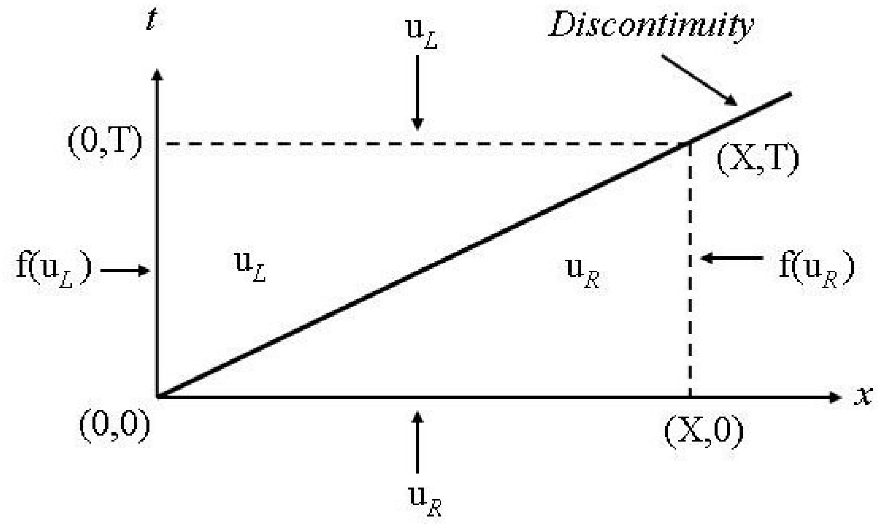
Shock speed : $$s = \frac{f(u_R) - f(u_L)}{u_R - u_L} = \frac{[f(u)]}{[u]}$$



Integrate $u_t + f(u)_x = 0$ over rectangle

$$u_L X - u_R X + f(u_R)T - f(u_L)T = 0 \quad \Leftrightarrow$$

$$f(u_R) - f(u_L) = \frac{X}{T}(u_R - u_L)$$



Shock speed : $s = \dfrac{f(u_R) - f(u_L)}{u_R - u_L} = \dfrac{[f(u)]}{[u]}$

We define the jumps as : $[u] = u_R - u_L$ and $[f(u)] = f(u_R) - f(u_L)$

The above equations are known as the *Rankine-Hugoniot jump conditions*. Even hyperbolic systems of conservation laws have similar jumps.

We have now proved that the form of the shock speed depends on the flux function.

Now see that the inviscid shock speed for Burgers equation is $(u_L+u_R)/2$

The above expression also holds for *non-convex* shocks. Question: What is the caveat though when applying the above formula to non-convex shocks?

The above derivation also highlights the importance of flux conservative forms in computations.

4.4) Isolated Rarefaction Fans

4.4.1) The Structure of an Isolated Rarefaction Fan

From previous examples, we see that other forms of self-similar solutions are possible – the ***rarefaction fans***.

Two important properties about our rarefaction fan solutions : A) They are *self-similar(depend on x/t)*. B) *inside* a rarefaction fan (i.e. excluding its end points), the solution is *differentiable*.

Start with Initial Conditions: $\mathrm{u}_0(x) = \mathrm{u}_L$ for $x < 0$; $\mathrm{u}_0(x) = \mathrm{u}_R$ for $x > 0$

Consider a convex flux with $\mathrm{f}'(\mathrm{u}_L) < \mathrm{f}'(\mathrm{u}_R)$

Assert a self-similar solution that is centered at the origin:

$\mathrm{u}(x,t) = \tilde{\mathrm{u}}(\xi) = \tilde{\mathrm{u}}(x/t)$ where $\xi \equiv x/t$ is the self-similarity variable.

Define $\tilde{\mathrm{u}}'(\xi) \equiv d\,\tilde{\mathrm{u}}(\xi)/d\xi$ to get

$$\mathrm{u}_t(x,t) = -\frac{x}{t^2}\tilde{\mathrm{u}}'(\xi) \quad \text{and} \quad \mathrm{f}_x(x,t) = \frac{1}{t}\,\mathrm{f}'(\tilde{\mathrm{u}}(\xi))\,\tilde{\mathrm{u}}'(\xi)$$

Start with Initial Conditions: $u_0(x) = u_L$ for $x < 0$; $u_0(x) = u_R$ for $x > 0$

Assert a self-similar solution that is centered at the origin:

$u(x,t) = \tilde{u}(\xi) = \tilde{u}(x/t)$ where $\xi \equiv x/t$ is the self-similarity variable.

Define $\tilde{u}'(\xi) \equiv d\,\tilde{u}(\xi)/d\xi$ to get

$$u_t(x,t) = -\frac{x}{t^2}\tilde{u}'(\xi) \quad \text{and} \quad f_x(x,t) = \frac{1}{t} f'(\tilde{u}(\xi))\,\tilde{u}'(\xi)$$

$$f'(\tilde{u}(\xi)) = \xi \quad \text{for} \quad f'(u_L) < \xi < f'(u_R) \quad \text{where } \xi = \frac{x}{t}$$

Substituting above derivatives in $u_t + f'(u)\, u_x = 0$ gives :

$$f'(\tilde{u}(\xi)) = \xi \quad \text{for} \quad f'(u_L) < \xi < f'(u_R) \quad \text{where } \xi = \frac{x}{t}$$

At $x = f'(u_L)\, t$ and $x = f'(u_R)\, t$ the rarefaction fan joins the constant left and right states.

Physically, the characteristics are straight lines in space-time. The solution is constant along each of those characteristics. At its end points, the speeds match those of the constant states on either side.

Example : $f'(u) = u$ for Burgers equation. Solution is $u(x,t) = \dfrac{x}{t}$

At $x = u_L\, t$ and $x = u_R\, t$ the rarefaction fan joins the constant left and right states.

4.4.2)The Role of Entropy in Arbitrating the Evolution of Discontinuities

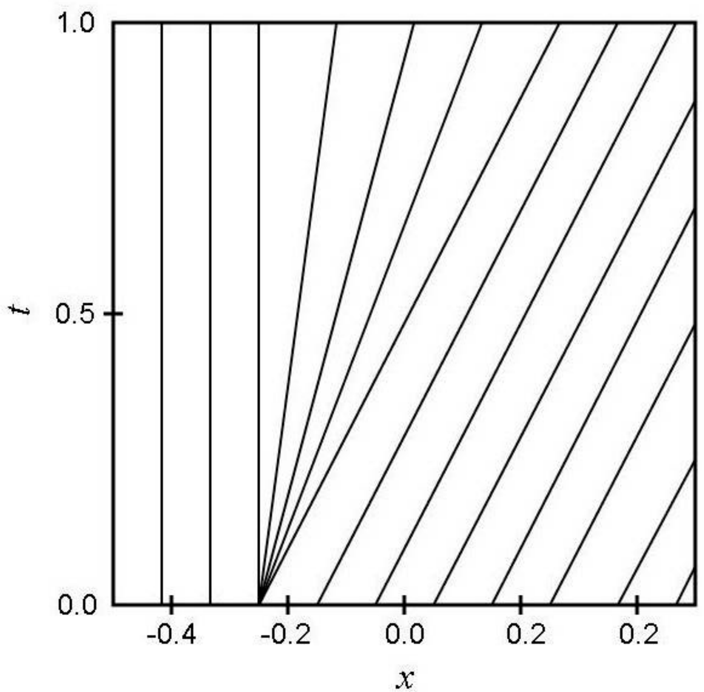
Question: *But what extra bit of physics determines which discontinuity becomes a shock and which one becomes a rarefaction?*

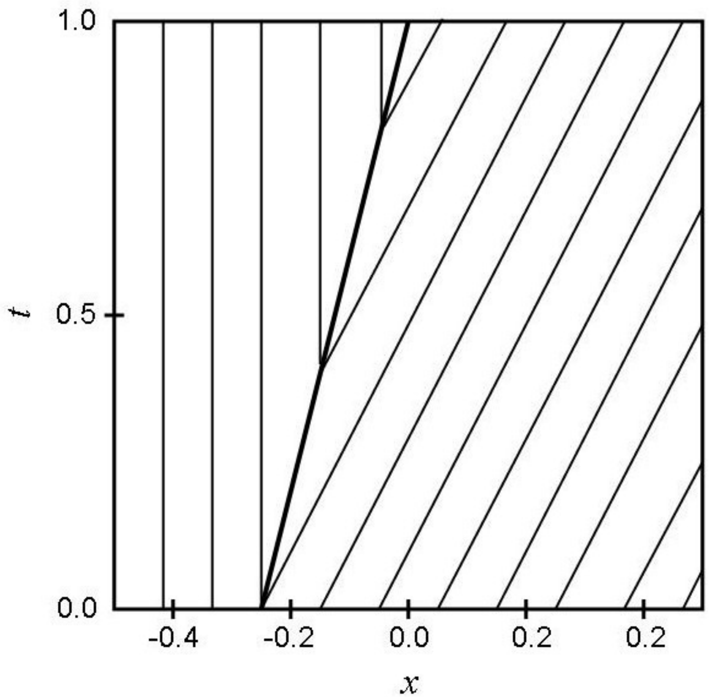
Surely, we can assert a shock jump condition and a shock speed for any initial discontinuity. Or would we be violating some other principle?

Consider left and right states with : $u_0(x) = 0$ for $x < -0.25$; $u_0(x) = 0.5$ for $x > -0.25$

$f'(0) = 0$, $f'(0.5) = 0.5$ $\Rightarrow$ characteristics flow away from initial discontinuity. s=0.25.

Which of the plots below is physical?





The physical principle here is *entropy generation*.

Characteristics carry information about the solution. (Think of *information entropy*.)

The shock solution to the right is called a ***rarefaction shock***. New information is generated at the rarefaction shock, because characteristics come out of it. This is *unphysical*.

Nature provides a *physical entropy for the Euler equations*, and several other systems.

The solution to the left satisfies an entropy condition. The solution to the right does not. We call the one to the left an *entropy-satisfying physical solution*. We want *numerical schemes that find the physical solution*.

For equations like Burgers or Buckley-Leverett, mathematicians have to formulate ***entropy conditions***, also known as *admissibility conditions*.

Lax showed that in order for a discontinuity to be physical for a scalar conservation law with a convex flux, we have the entropy condition:

$$f'(u_L) > s > f'(u_R)$$

This is *Lax's entropy condition for convex fluxes*. Excludes entropy violating shocks! Lax's entropy condition closely parodies the flow of characteristics into a hydrodynamical shock as we will see in the next chapter.

Question: Can you apply it to the previous two plots to pick out the one that is physical? I.e. show that rarefaction shocks are unphysical.

There exist other entropy conditions by Oleinik that pertain to convex and non-convex fluxes.

A result by Harten says that symmetrizable hyperbolic systems admit a numerically-motivated entropy condition. For example, the Euler system is symmetrizable.

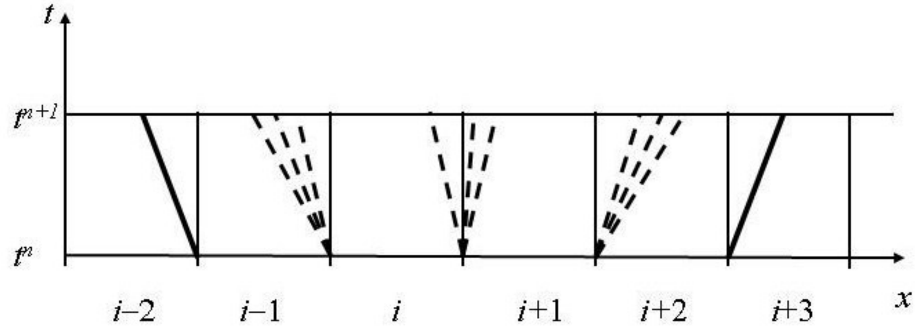
4.5) The Entropy Fix and Approximate Riemann Solvers

This section is all about obtaining a numerical flux and doing it in the simplest/fastest way possible without relinquishing physical solutions.

4.5.1) The Entropy Fix

Consider the Godunov scheme that is schematically shown below. To find the numerical flux at the zone boundaries, we first need to obtain the resolved state that overlies the zone boundary. Question: How do we obtain the resolved state at zone boundaries $i-3/2$, $i-1/2$, $i+1/2$ and $i+3/2$?

At zone boundary i+1/2 we have to do something special. We have to solve for the *interior structure of the rarefaction fan*. While this is inexpensive for Burgers equation, this can in general be quite expensive. We wish to find *inexpensive alternatives*.



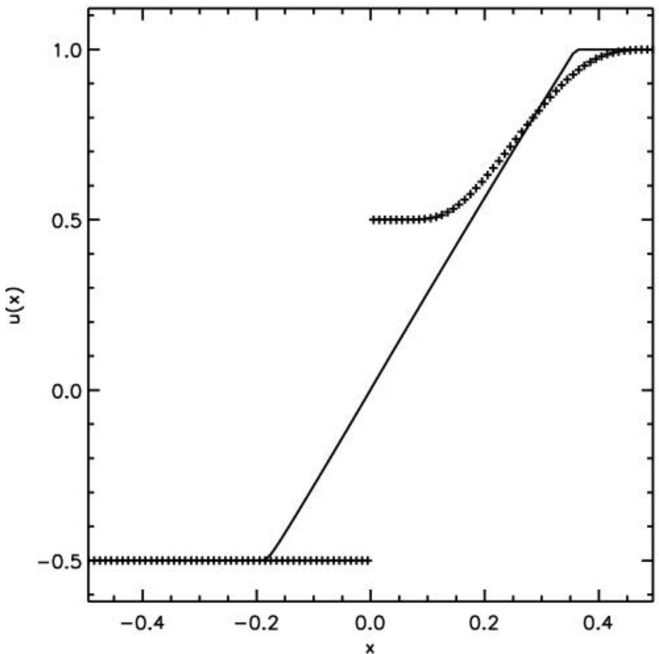
When the boundary is not straddled by a rarefaction fan, the resolved state, and numerical flux, are easy to find.

We'd like to cut corners with the one case that is difficult – the case where the rarefaction straddles the zone boundary. But can we cut corners?

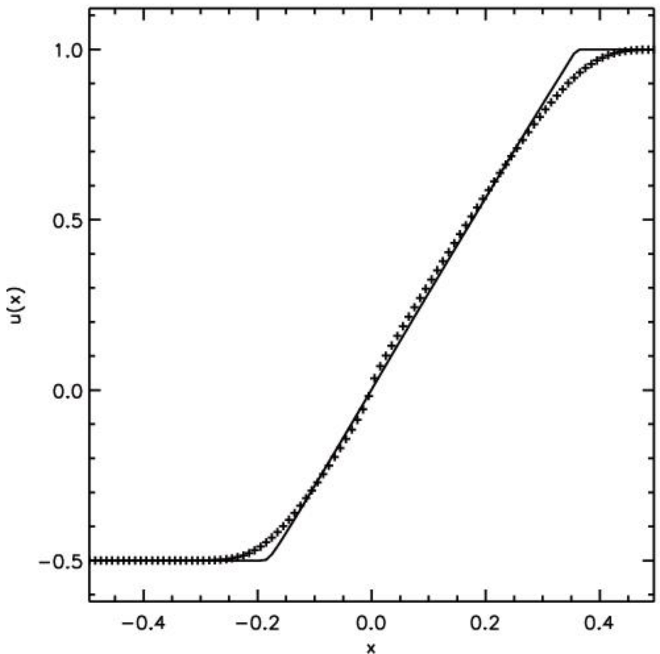
Our first attempt: *Replace the rarefaction fan by a rarefaction shock*.

<u>Big Question</u>: Does the numerical scheme still produce physical results?

RS with Rarefaction shocks (unphysical)



RS with rarefaction fans(physical)



Thus our first attempt fails. *It is **not** possible to replace a rarefaction wave with a rarefaction shock!*

The Riemann solver ***must*** build in some knowledge that the rarefaction fan opens up.

The fix that is introduced into a Riemann solver to enable it to recognize the presence of a rarefaction fan that straddles a zone boundary is called the ***entropy fix***.

The solution of the exact Riemann problem becomes increasingly difficult as the system becomes larger and/or more complicated.

In all such situations we wish to build ***approximate Riemann solvers***.

The approximate Riemann solvers must also incorporate some notion of an *entropy fix*.

4.5.2) Approximate Riemann Solvers

Realize that the Riemann problem is a *self-similar solution*. Thus we can replace the actual wave structure by a *wave model*. This is a proxy for the actual, self-similar wave structure in a Riemann problem.

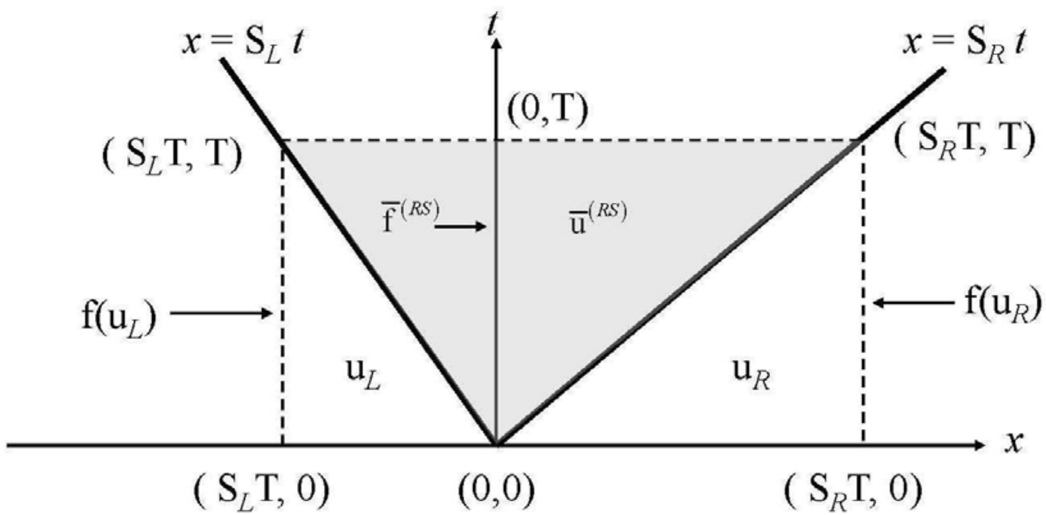
We still wish to *avoid* a complete and exact solution of the *internal structure of a rarefaction fan*.

When we have a rarefaction fan, the extremal speeds are easy to find:

$$S_L = f'(u_L) < 0 \text{ and } S_R = f'(u_R) > 0$$

Consider a *constant*, resolved state $\overline{u}^{(RS)} \leftarrow$ an approximation!

Let the corresponding resolved flux be $\overline{f}^{(RS)}$



Our goal: Find $\overline{u}^{(RS)}$ and $\overline{f}^{(RS)}$

To obtain resolved state of the approximate HLL Riemann solver:

Integrate conservation law in weak form over dashed rectangle as follows

$$\overline{u}^{(RS)}\ (S_R - S_L)\ T - u_R\ S_R\ T + u_L\ S_L\ T + f(u_R)\ T - f(u_L)\ T = 0$$

to get:

$$\overline{u}^{(RS)} = \frac{S_R\ u_R - S_L\ u_L - (f(u_R) - f(u_L))}{(S_R - S_L)}$$

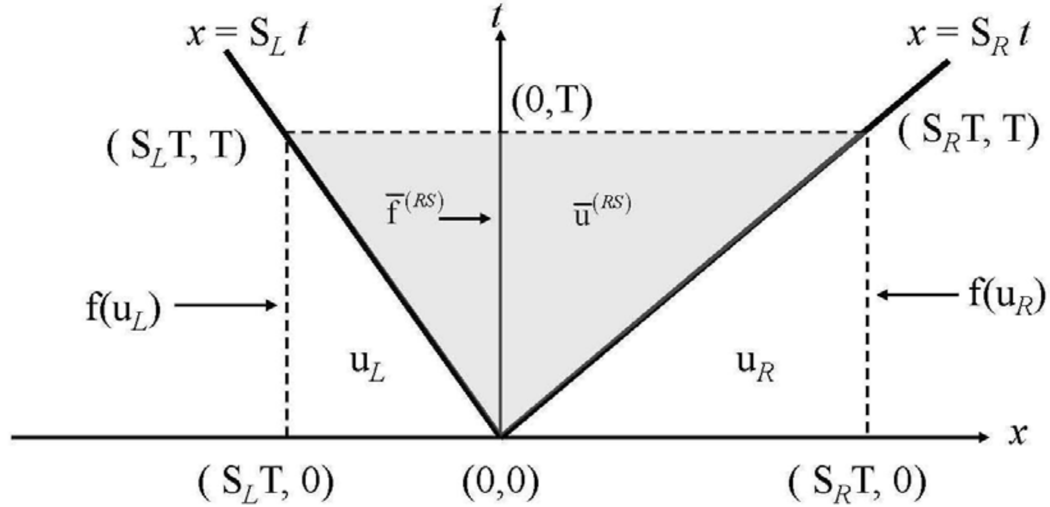
To obtain resolved flux of the approximate Riemann solver: Integrate conservation law in weak form over x>0 part of dashed rectangle as follows

$$\overline{u}^{(RS)}\ S_R\ T - u_R\ S_R\ T + f(u_R)\ T - \overline{f}^{(RS)}\ T = 0$$

to get the HLL flux :

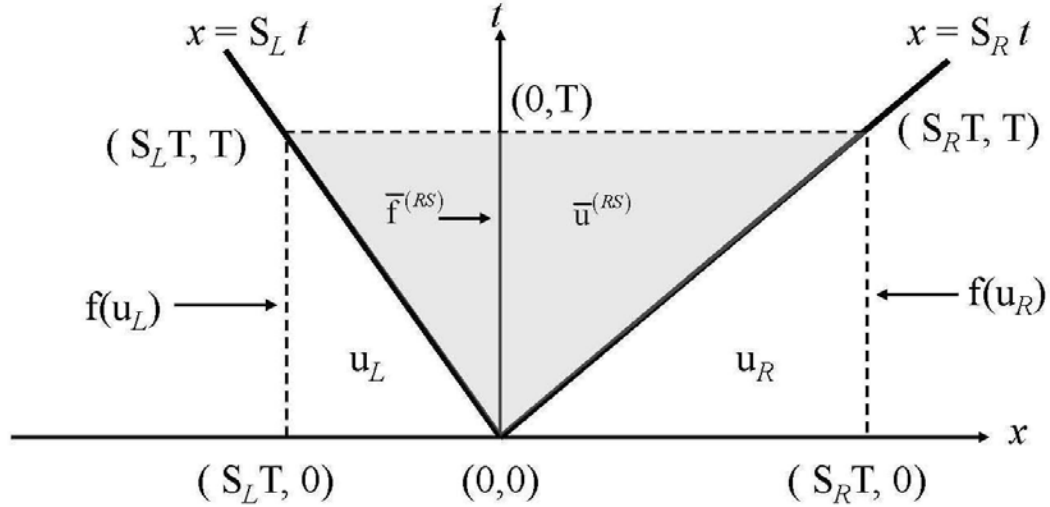
$$\overline{f}^{(RS)} = \left[\frac{S_R}{S_R - S_L}\right] f(u_L) - \left[\frac{S_L}{S_R - S_L}\right] f(u_R) + \left[\frac{S_R\ S_L}{S_R - S_L}\right](u_R - u_L)$$

$$\int_{t=0}^{t=T} \int_{x=S_L T}^{x=S_R T} \left(u_t + f_x \right) dx\, dt = 0$$



$$\overline{u}^{(RS)} = \frac{S_R\, u_R - S_L\, u_L - \left(f\left(u_R\right) - f\left(u_L\right)\right)}{\left(S_R - S_L\right)}$$

$$\int_{t=0}^{t=T} \int_{x=0}^{x=S_R T} \left(u_t + f_x \right) dx \, dt = 0$$



$$\overline{f}^{(RS)} = \left[\frac{S_R}{S_R - S_L} \right] f\left(u_L\right) - \left[\frac{S_L}{S_R - S_L} \right] f\left(u_R\right) + \left[\frac{S_R\ S_L}{S_R - S_L} \right] \left(u_R - u_L\right)$$

The ***HLL Riemann solver***, detailed above, extends naturally to systems. It is a standard ingredient of the computationalist's toolkit.

It is always good to have it as one of the options for a Riemann solver in any code for solving hyperbolic conservation laws.

We have still to specify the extremal speeds that need to be used:

$$S_L = \min\left(f'\left(u_L\right), s, 0\right) \qquad S_R = \max\left(f'\left(u_R\right), s, 0\right)$$

By analogy with Euler flow, when S_L and S_R have same sign, we call it *supersonic*. When the signs are opposite, it is a *subsonic* situation.

Question: How would we prove that the HLL flux is *consistent*?

Question: Can you show that the above choice always gives us properly *upwinded* fluxes in the *supersonic situations*?
How does the HLL RS generates *dissipation at subsonic rarefaction fans*?

For the subsonic case, we can write the HLL flux as:

$$\overline{f}^{(RS)}(u_L, u_R) = f^+(u_L) + f^-(u_R) \quad \text{with}$$

$$f^+(u_L) \equiv \left[\frac{S_R}{S_R - S_L}\right]\left[f(u_L) - S_L u_L\right] \quad \text{and} \quad f^-(u_R) \equiv -\left[\frac{S_L}{S_R - S_L}\right]\left[f(u_R) - S_R u_R\right]$$

Question: What is the real insight we gain from writing it this way?

This form of the flux is known as a *flux vector splitting*. Question: Why is this name appropriate.

Flux vector splittings can also be obtained for systems of conservation laws.

Another useful form of numerical flux is obtained from the ***Rusanov or local Lax-Friedrichs (LLF) flux***:

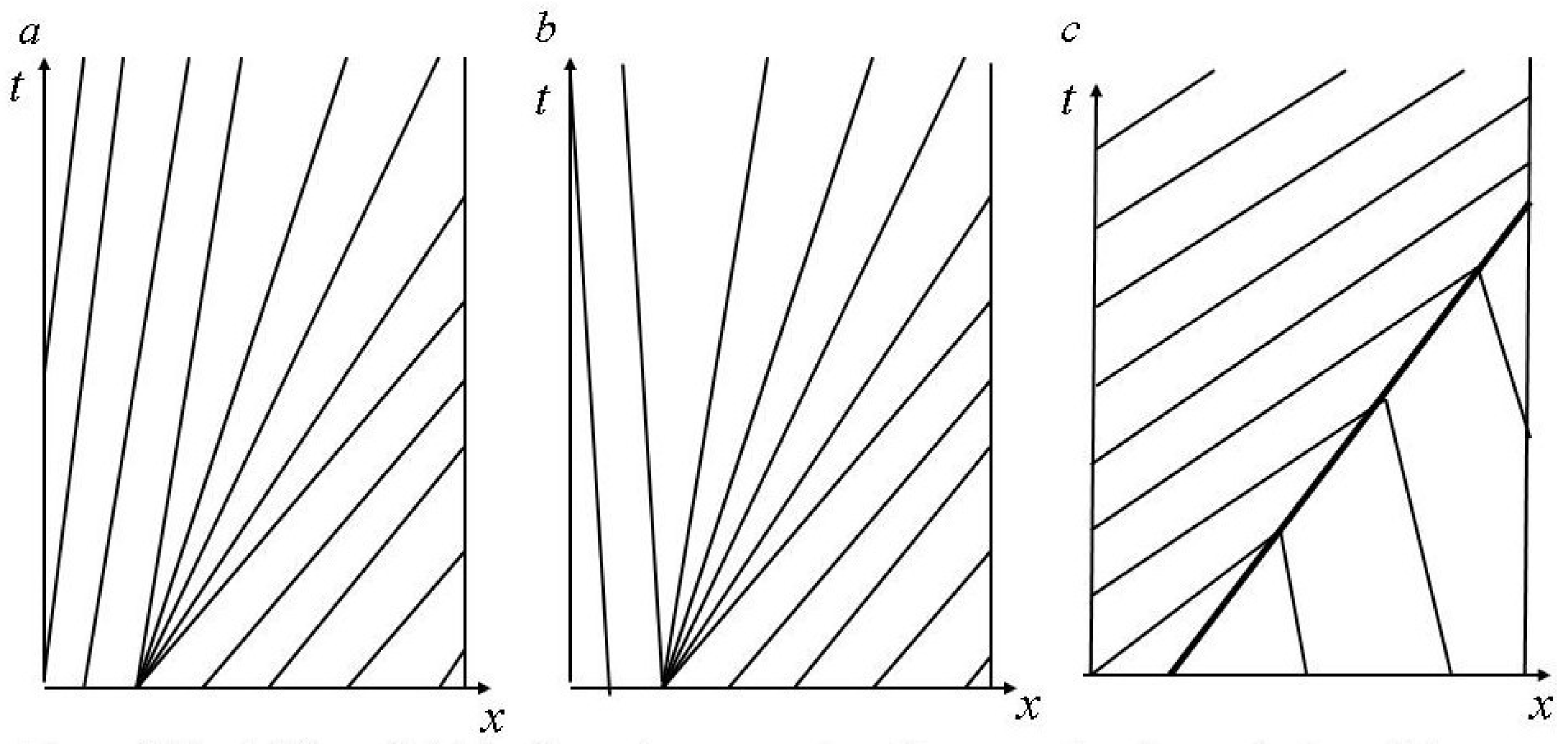
$$\overline{f}^{(RS)} = \frac{1}{2}\left(f(u_L) + f(u_R) - S_{Max}(u_R - u_L)\right) \quad \text{where} \quad S_{Max} \equiv \max\left(\left|f'(u_L)\right|, |s|, \left|f'(u_R)\right|\right)$$

Question: Compare and contrast the HLL and LLF numerical fluxes.

4.6) Boundary Conditions

For linear equations, the characteristic matrix is fixed. As a result, the kinds of boundary conditions that one imposes at each boundary are fixed.

For non-linear problems, the wave speed becomes solution-dependent and can change from one location to the other as well as from one time to the next as the solution evolves. As a result, the boundary conditions may also have to adapt as the solution changes at the boundaries. Question: What can you say about the boundary conditions for the three figures below?



When variables are initialized at ghost boundaries, they may cause waves to propagate with signal speeds that can be larger than the signal speed that is represented in the interior of the computational domain. In such situations, which usually occur when a strong shock propagates in from a boundary, the timestep should be reduced to satisfy the CFL condition in the ghost zones.

4.7) Numerical Methods for Scalar Conservation Laws

Lax-Wendroff theorem : The problem should be discretized on a computational mesh using a *consistent, stable* and *conservative* method if weak solutions (i.e. shocks and rarefactions) are to be *convergent* as the mesh is refined.

Runge-Kutta methods go over exactly as before:

Step 1: We have to obtain the *undivided differences* of the conserved variables.

Step 2: Obtain the *left and right states at the zone boundary.*

Step 3: Treat the Riemann solver as a machine that accepts two states and spits out a flux. Feed the above left and right states into the Riemann solver and obtain a properly *upwinded flux*.

Predictor-Corrector methods also go over much as before:

Step 1: We have to obtain the *undivided differences* of the conserved variables.

Step 2: Obtain the *left and right predicted states at the zone boundary.*

$$u_{L;i+1/2}^{n+1/2} = \overline{u}_i^n + \frac{1}{2}\,\overline{\Delta u}_i^n - \frac{1}{2}\,\frac{\Delta t}{\Delta x}\,f'\left(\overline{u}_i^n\right)\overline{\Delta u}_i^n$$

$$u_{R;i+1/2}^{n+1/2} = \overline{u}_{i+1}^n - \frac{1}{2}\,\overline{\Delta u}_{i+1}^n - \frac{1}{2}\,\frac{\Delta t}{\Delta x}\,f'\left(\overline{u}_{i+1}^n\right)\overline{\Delta u}_{i+1}^n$$

Step 3: Treat the Riemann solver as a machine that accepts two states and spits out a flux. Feed the above left and right states into the Riemann solver and obtain a properly *upwinded flux*.

Step 4: Make a single step *corrector update*.