Stability and $\mathcal{L}_2$ Gain Analysis for Discrete-Time LTI Systems with Controller Failures

GUISHENG ZHAI [†] XINKAI CHEN [‡] HAI LIN [§]

[†] Department of Opto-Mechatronics, Wakayama University
Sakaedani, Wakayama 640-8510, JAPAN

[‡] Department of Intelligent Systems, Kinki University
Nishimitani, Wakayama 649-6493, JAPAN

[§] Department of Electrical Engineering, University of Notre Dame
Notre Dame, Indiana 46556, USA

Abstract: - In this paper, we analyze stability and $\mathcal{L}_2$ gain properties for discrete-time linear time-invariant (LTI) systems controlled by a pre-designed dynamical output feedback controller which fails from time to time due to physical or purposeful reason. Our aim is to find conditions concerning controller failure time, under which the system's stability and $\mathcal{L}_2$ gain properties are preserved to a desired level. For stability, by using a piecewise Lyapunov function, we show that if the unavailability rate of the controller is smaller than a specified constant and the average time interval between controller failures (ATBCF) is large enough, then global exponential stability of the system is guaranteed. For $\mathcal{L}_2$ gain, also by using a piecewise Lyapunov function, we show that if the unavailability rate of the controller is smaller than a specified constant, then the system with an ATBCF achieves a reasonable weighted $\mathcal{L}_2$ gain level, and the weighted $\mathcal{L}_2$ gain approaches normal $\mathcal{L}_2$ gain when the ATBCF is sufficiently large.

Key-Words: - Linear time-invariant (LTI) system, dynamical output feedback, exponential stability, (weighted) $\mathcal{L}_2$ gain, controller failure, unavailability rate, average time between controller failures (ATBCF), piecewise Lyapunov function

1 Introduction

In this paper, we consider some quantitative properties for discrete-time linear time-invariant (LTI) control systems with controller failures. The motivation of studying such problem stems from the fact that controller failures always exist in any real control systems due to various environmental factors. For example, in a feedback control system which is composed of a system and a feedback controller, controller failures occur when the signals are not transmitted perfectly on routes among the system, the sensors and the actuators, or when the controller itself is not available sometimes due to some known or unknown reason. Another important motivation concerning controller failures is that we can think about "failure" in a positive way: "suspension", i.e., in the situation that economical issue or system life consideration is concerned, we desire to suspend the controller purposefully from time to time.

For feedback control systems, the problem of possessing integrity was considered in [1], where it was proposed to design a state feedback controller so that the closed-loop system remains stable even when some part of the controller fails. In [2], similar control systems were dealt with using the name of asynchronous dynamical systems (ADS), and two real systems, the control over asynchronous network and the parallelized algorithm, were discussed. In that context, a Lyapunov-based approach was proposed to construct the controller so that the system has desired properties. Ref. [3] stated similar control problems in the framework of networked control systems (NCS), where various information (reference input, plant output, control input, etc.) is exchanged through a network among control system components (sensors, controller, actuators, etc.), and thus packet dropouts occuring inevitably due to unreliable transmission paths lead to controller failures.

Encouraged by the above works, the authors considered in [4] a controller failure time analysis problem for exponential stability of LTI continuous-time systems. By using a piecewise Lyapunov function, we showed that if the unavailability rate of the controller is smaller than a specified constant and the average time interval between controller failures is large enough, then global exponential stability of the system is guaranteed. In [5], the result of [4] was extended to LTI discrete-time systems. Furthermore, the authors extended the consideration to $\mathcal{L}_2$ gain analysis for LTI continuous-time systems with controller failures in [6]. However, Refs. [4]-[6] dealt with only state feedback. Although it is quite easy to extend the results in [4]-[6] to the case of static output feedback, the extension to dynamical output feedback is not the case.

Motivated by the above observations, we in this paper extend the results in [4, 6] to dynamical output feedback case. The system we consider is described by equations of the form

$$\left\{\begin{array}{rcl} x(k+1) &=& Ax(k)+B_1w(k)+B_2u(k) \\ z(k) &=& C_1x(k)+Du(k) \\ y(k) &=& C_2x(k)\,, \end{array}\right. \tag{1}$$

where $x(k)\in\Re^n$ is the state, $u(k)\in\Re^m$ is the control input, $w(k)\in\Re^r$ is the disturbance input, $y(k)\in\Re^p$ is the measurement output, $z(k)\in\Re^q$ is the controlled output, and A, B_1, B_2, C_1, C_2, D are constant matrices of appropriate dimension. Throughout this paper, we assume (i) A is not stable; (ii) the triple (A, B_2, C_2) is stabilizable and detectable; (iii) a dynamical output feedback controller

$$\left\{\begin{array}{rcl} x_c(k+1) &=& A_cx_c(k)+B_cy(k) \\ u(k) &=& C_cx_c(k)+D_cy(k) \end{array}\right. \tag{2}$$

has been designed so that the closed-loop system composed of (1) and (2) has desired property (exponential stability with certain decay rate or certain $\mathcal{L}_2$ gain level), where $x_c(k)\in\Re^{n_c}$ is the controller's state, n_c is the controller's order, and A_c, B_c, C_c, D_c are constant matrices. However, due to physical or purposeful reason, the designed controller sometimes fails with a (not constant necessarily) time interval until we recover it. In this setting, we derive the condition of controller failure time, under which the system's exponential stability or its $\mathcal{L}_2$ gain property is preserved to a desired level. As in [4, 5, 6], we use the word "controller failure" in this paper to mean complete breakdown of the controller ($u=0$) on certain time interval, neither as the one in [1] that part of the controller fails, nor as the one in [2] that the controller decays slowly at a rate.

To proceed, we first introduce some notation. For any given $k>1$, we denote by $T_u(k)$ the total time interval of controller failures during $[0,k)$, and call the ratio $\frac{T_u(k)}{k}$ the "unavailability rate" of the controller in the system. We denote by N_k the number of times of controller failures during $[0,k)$. If for some constant $T_f>0$, the inequality $N_k\le\frac{k}{T_f}$ holds for any $k>1$, then T_f is called as a lower bound of the "average time between controller failures" (ATBCF). The idea is that the average time interval between subsequent controller failures is not less than T_f according to the equivalent inequality $\frac{k}{N_k}\ge T_f$.

For stability analysis, we will prove in Section 2 that if the unavailability rate of the controller is smaller than a specified constant and the ATBCF is large enough, then global exponential stability of the system is guaranteed. For $\mathcal{L}_2$ gain analysis, we show in Section 3 that if the unavailability rate of the controller is smaller than a specified constant, then the system with an ATBCF achieves a reasonable weighted $\mathcal{L}_2$ gain level, and the weighted $\mathcal{L}_2$ gain approaches normal $\mathcal{L}_2$ gain if the ATBCF is sufficiently large. Finally, Section 4 concludes the paper.

2 Stability Analysis

In this section, we set $w(k)\equiv 0$ in the system (1) to analyze stability for the system with controller failures. More precisely, we assume that the controller (2) has been designed so that the closed-loop system

$$\tilde{x}(k+1)=A_s\tilde{x}(k)\,,\ A_s=\left[\begin{array}{cc} A+B_2D_cC_2 & B_2C_c \\ B_cC_2 & A_c \end{array}\right] \tag{3}$$

is exponentially stable, where $\tilde{x}(k)=[x^T(k)\ \ x_c^T(k)]^T$ is the state of the closed-loop system.

We first give a definition concerning exponential stability of an autonomous system quantitatively.

Definition 1. The system $x(k+1)=f(x(k))$ with $f(0)=0$ is said to be *globally exponentially stable with decay rate* $0<\lambda<1$ if $\|x(k)\|\le c\lambda^k\|x(0)\|$ holds for any x_0, any $k\ge 1$ and a constant $c>0$. ■

We suppose that the designed controller (2) sometimes fails and we need a (not definitely constant) time interval to recover it. Obviously, when the controller fails, the closed-loop system assumes the form of

$$\tilde{x}(k+1)\ =\ A_u\tilde{x}(k)\,,\quad A_u=\left[\begin{array}{cc} A & 0 \\ B_cC_2 & A_c \end{array}\right]\,, \tag{4}$$

which is obtained by substituting $u(k)=0$ in (1). Hence, the performance of the entire system is dominated by the following piecewise difference equation:

$$\tilde{x}(k+1)=\left\{\begin{array}{ll} A_s\tilde{x}(k) & \text{when the controller works} \\ A_u\tilde{x}(k) & \text{when the controller fails}\,. \end{array}\right. \tag{5}$$

Since A_s is stable and A_u is unstable, we can always find two positive scalars $\lambda_s>1$ and $\lambda_u>1$ such that λ_sA_s remains stable and $\lambda_u^{-1}A_u$ becomes stable. As can be seen later, large λ_s and small λ_u are desirable in real problems. Then, there are two matrices $P_s>0$ and $P_u>0$ such that

$$\lambda_s^2A_s^TP_sA_s-P_s<0\,,\ \ \lambda_u^{-2}A_u^TP_uA_u-P_u<0\,. \tag{6}$$

Note that the above inequalities are LMIs [9] with respect to P_s, P_u, and thus are easily solved using any of the existing softwares, such as the LMI Control Toolbox.

Using the solutions P_s and P_u of (6), we define the following *piecewise Lyapunov function* candidate

$$V(k)=V_{\sigma(k)}(\tilde{x}(k))=\tilde{x}^T(k)P_{\sigma(k)}\tilde{x}(k) \tag{7}$$

for the system. Here, $P_{\sigma(k)}$ is a two-valued piecewise constant matrix function as

$$P_{\sigma(k)} = \begin{cases} P_s & \text{when the controller works} \\ P_u & \text{when the controller fails}, \end{cases} \quad (8)$$

and $V_{\sigma(k)}(\tilde{x}(k))$ is defined correspondingly. Then, the following properties of (7) are obtained:

(i) $V_s(\tilde{x}(k)) = \tilde{x}^T(k)P_s\tilde{x}(k)$, $V_u(\tilde{x}(k)) = \tilde{x}^T(k)P_u\tilde{x}(k)$ are continuous and their differences along solutions of the corresponding system satisfy

$$\begin{aligned} V_s(\tilde{x}(k+1)) &\le \lambda_s^{-2}V_s(\tilde{x}(k)) \\ V_u(\tilde{x}(k+1)) &\le \lambda_u^{2}V_u(\tilde{x}(k)); \end{aligned} \quad (9)$$

(ii) There exist constant scalars $\alpha_2 \ge \alpha_1 > 0$ such that

$$\alpha_1\|\tilde{x}\|^2 \le \{V_s(\tilde{x}), V_u(\tilde{x})\} \le \alpha_2\|\tilde{x}\|^2, \ \forall\tilde{x}; \quad (10)$$

(iii) There exists a constant scalar $\mu \ge 1$ such that

$$V_s(\tilde{x}) \le \mu V_u(\tilde{x}), \ V_u(\tilde{x}) \le \mu V_s(\tilde{x}), \ \forall\tilde{x}. \quad (11)$$

The first property is a straightforward consequence of (6) while the second and third properties hold, for example, with $\alpha_1 = \min\{\lambda_m(P_s), \lambda_m(P_u)\}$, $\alpha_2 = \max\{\lambda_M(P_s), \lambda_M(P_u)\}$, and $\mu = \frac{\alpha_2}{\alpha_1}$, respectively. Here, $\lambda_M(\cdot)$ ($\lambda_m(\cdot)$) denotes the largest (smallest) eigenvalue of a symmetric matrix.

Now, without loss of generality, we assume that the designed controller works during $[k_{2j}, k_{2j+1})$, and the controller fails during $[k_{2j+1}, k_{2j+2})$, $j = 0, 1, \cdots$, where $k_0 = 0$. Then, for any $k > 1$, we obtain from (9) that

$$V(k) \le \begin{cases} \lambda_s^{-2(k-k_{2j})}V(k_{2j}) & k_{2j} \le k < k_{2j+1}, \\ \lambda_u^{2(k-k_{2j+1})}V(k_{2j+1}) & k_{2j+1} \le k < k_{2j+2}. \end{cases} \quad (12)$$

Noting that $V(k_j) \le \mu V(k_j^-)$ holds for any $j > 0$ accroding to (11), we obtain from (12) that for $k_{2j} \le k < k_{2j+1}$,

$$\begin{aligned} V(k) &\le \lambda_s^{-2(k-k_{2j})}V(k_{2j}) \le \mu\lambda_s^{-2(k-k_{2j})}V(k_{2j}^-) \\ &\le \mu^2\lambda_s^{-2(k-k_{2j})}\lambda_u^{2(k_{2j}-k_{2j-1})}V(k_{2j-1}) \end{aligned} \quad (13)$$

and then by induction that

$$V(k) \le \mu^{2N_k}\lambda_s^{-2(k-T_u(k))}\lambda_u^{2T_u(k)}V(0). \quad (14)$$

It is easy to confirm that the above inequality is also true for $k_{2j+1} \le k < k_{2j+2}$. Since $V(k) \ge \alpha_1\|\tilde{x}(k)\|^2$ and $V(0) \le \alpha_2\|\tilde{x}(0)\|^2$, we obtain from the above inequality that

$$\|\tilde{x}(k)\| \le \sqrt{\frac{\alpha_2}{\alpha_1}}\mu^{N_k}\lambda_s^{-(k-T_u(k))}\lambda_u^{T_u(k)}\|\tilde{x}(0)\|. \quad (15)$$

From now on, we consider the convergence property of $\tilde{x}(k)$ in (15) for two different cases.

First, when $\mu = 1$, we obtain from (15) that

$$\|\tilde{x}(k)\| \le \sqrt{\frac{\alpha_2}{\alpha_1}}\lambda_s^{-(k-T_u(k))}\lambda_u^{T_u(k)}\|\tilde{x}(0)\|. \quad (16)$$

If there exists a positive scalar λ satisfying $\lambda < 1$ such that

$$\frac{T_u(k)}{k} \le \frac{\ln\lambda_s + \ln\lambda}{\ln\lambda_s + \ln\lambda_u}, \quad (17)$$

which is a condition on the unavailability rate of the controller, then we obtain easily from (17) that

$$(\lambda_s\lambda_u)^{T_u(k)} \le (\lambda_s\lambda)^k \Longleftrightarrow \lambda_s^{-(k-T_u(k))}\lambda_u^{T_u(k)} \le \lambda^k \quad (18)$$

and thus

$$\|\tilde{x}(k)\| \le \sqrt{\frac{\alpha_2}{\alpha_1}}\lambda^k\|\tilde{x}(0)\|. \quad (19)$$

This implies that the entire system is globally exponentially stable with decay rate λ.

Secondly, when $\mu > 1$, in addition to (17), if there exists a scalar $\lambda^* \in (\lambda, 1)$ such that

$$N_k \le \frac{k}{T_f}, \quad T_f = \frac{\ln\mu}{\ln(\lambda^*) - \ln(\lambda)} \quad (20)$$

holds for all $k > 1$, then we obtain easily

$$\mu^{N_k}\lambda^k \le (\lambda^*)^k, \quad (21)$$

and thus

$$\|\tilde{x}(k)\| \le \sqrt{\frac{\alpha_2}{\alpha_1}}(\lambda^*)^k\|\tilde{x}(0)\|. \quad (22)$$

This means that the entire system is globally exponentially stable with decay rate λ^*.

We observe that the condition (20) is the requirement on the ATBCF. More precisely, if the ATBCF in the system (1) is larger than or equal to T_f given in (20), then (21) and (22) hold as the same and thus the system's exponential stability is guaranteed.

The above discussions indicate that (7) with (6) constitutes a piecewise Lyapunov function for the system (1) with controller failures satisfying (17) and (20). We state this result in the following theorem.

Theorem 1. *If the unavailability rate of the controller in the system* (1) *is small in the sense of satisfying* (17) *for some positive scalar* $\lambda < 1$, *then for any positive scalar* λ^* *satisfying* $\lambda < \lambda^* < 1$ *there exists a finite constant* T_f *such that the system* (1) *is globally exponentially stable with decay rate* λ^*, *for any ATBCF larger than or equal to* T_f. ■

The next two remarks give more precise discussion about the conditions (17) and (20).

Remark 1. The condition (17) implies that if we expect the entire system has the potentiality of decay rate close to λ_s^{-1} (i.e., $\lambda \to \lambda_s^{-1}$), where λ_s^{-1} is known to be kind of decay rate of the closed-loop system (3), we should restrict the total controller failure time small enough (i.e., $T_u(k) \to 0$). This is reasonable when we consider the control system with the designed controller breaks down very occasionally and we can recover it very soon. In this case, we definitely expect that the system stability does not degenerate greatly.

Concerning the other two stability indices λ_s and λ_u, we observe that according to the unavailability rate condition (17), comparatively long controller failure time $T_u(k)$ is tolerable for large λ_s and small λ_u. This is reasonable since the closed-loop system has large decay rate (thus good stability property) when the controller works with large λ_s, and the open-loop system does not diverge greatly when the controller fails with small λ_u. Therefore, if we concentrate on stability property of the system, we should design the original output feedback controller so that a large λ_s can be obtained. ■

Remark 2. While the unavailability rate condition (17) of the controller is easy to imagine, the ATBCF condition (20) is not so straightforward. The key point is that if the open-loop system (when the controller fails) has poor stability property and the controller failures occur very frequently, then the entire system will not perform well even when the total controller failure time interval is not long. If we expect that the entire system has decay rate close to λ, we should require T_f to be large enough and thus N_k to be small enough, which means that the controller does not fail very frequently. Therefore, the condition (20) is a balanced requirement of decay rate and the number of controller failure times. ■

Remark 3. Although we concentrated on the case of complete controller breakdown ($u = 0$) in this paper, it is an easy matter to extend the discussion to the case where due to various reason the output feedback controller (2) (write as $u = G(z)y$ shortly) decays in the sense of $u \to \alpha u$ with α being a fixed constant satisfying $0 \le \alpha < 1$. In that case, if the closed-loop system composed of (1) and $u = \alpha G(z)y$ is unstable, the discussions up to now are the same by making some notation modification. If this is not the case, then the entire system can be considered as a switched system composed of two stable subsystems, and thus it is globally exponential stable if the ATBCF is large enough; see detailed discussions in [5, 7, 8]. ■

3 $\mathcal{L}_2$ Gain Analysis

In this section, we assume that the dynamical output feedback controller (2) has been designed so that the closed-loop system

$$\begin{cases} \tilde{x}(k+1) = A_s\tilde{x}(k) + \tilde{B}_1 w(k) \\ z(k) = C_s\tilde{x}(k)\,, \end{cases} \tag{23}$$

is stable and the $\mathcal{L}_2$ gain of the transfer function from w to z in (23) is smaller than a prespecified constant γ, where $\tilde{B}_1 = \begin{bmatrix} B_1^T & 0 \end{bmatrix}^T$, $C_s = [C_1 + DD_cC_2 \;\; DC_c]$. Since our interest here is to analyze $\mathcal{L}_2$ gain property of the system, we assume $\tilde{x}(0) = 0$ in (23).

Also, we suppose that the designed controller (2) sometimes fails and we need a (not constant necessarily) time interval to recover it. When the controller fails, the closed-loop system assumes the form of

$$\begin{cases} \tilde{x}(k+1) = A_u\tilde{x}(k) + \tilde{B}_1 w(k) \\ z(k) = C_u\tilde{x}(k) \end{cases} \tag{24}$$

where $C_u = [C_1 \;\; 0]$. Then, the behavior of the entire system is dominated by the piecewise LTI system: the system (23) when the controller works and the system (24) when the controller fails.

Since A_s is stable and the $\mathcal{L}_2$ gain of the transfer function from w to z in (23) is smaller than γ, according to the well known Bounded Real Lemma for discrete-time LTI systems [10], there exists $P_s > 0$ such that

$$\begin{bmatrix} -P_s & P_sA_s & P_s\tilde{B}_1 & 0 \\ A_s^TP_s & -P_s & 0 & C_s^T \\ \tilde{B}_1^TP_s & 0 & -\gamma I & 0 \\ 0 & C_s & 0 & -\gamma I \end{bmatrix} < 0\,, \tag{25}$$

which is equivalent to

$$\begin{bmatrix} A_s^TP_sA_s - P_s + \frac{1}{\gamma}C_s^TC_s & A_s^TP_s\tilde{B}_1 \\ \tilde{B}_1^TP_sA_s & \tilde{B}_1^TP_s\tilde{B}_1 - \gamma I \end{bmatrix} < 0\,. \tag{26}$$

Thus, there always exists a scalar $\zeta_s > 1$ such that

$$\begin{bmatrix} A_s^TP_sA_s - \zeta_s^{-2}P_s + \frac{1}{\gamma}C_s^TC_s & A_s^TP_s\tilde{B}_1 \\ \tilde{B}_1^TP_sA_s & \tilde{B}_1^TP_s\tilde{B}_1 - \gamma I \end{bmatrix} < 0\,. \tag{27}$$

Now we consider the case when the controller fails. In this case, we can always find a scalar $\zeta_u > 1$ such that $\zeta_u^{-1}A_u$ is Schur stable and the $\mathcal{L}_2$ gain of the transfer function $\left(\zeta_u^{-1}A_u\,,\ \tilde{B}_1\,,\ \zeta_u^{-1}C_u\right)$ is smaller than γ. Thus, there exists $P_u > 0$ such that

$$\begin{bmatrix} -P_u & P_u(\zeta_u^{-1}A_u) & P_u\tilde{B}_1 & 0 \\ (\zeta_u^{-1}A_u)^TP_u & -P_u & 0 & (\zeta_u^{-1}C_u)^T \\ \tilde{B}_1^TP_u & 0 & -\gamma I & 0 \\ 0 & (\zeta_u^{-1}C_u) & 0 & -\gamma I \end{bmatrix} < 0\,, \tag{28}$$

or equivalently,

$$\begin{bmatrix} A_u^TP_uA_u - \zeta_u^2P_u + \frac{1}{\gamma}C_u^TC_u & A_u^TP_u\tilde{B}_1 \\ \tilde{B}_1^TP_uA_u & \tilde{B}_1^TP_u\tilde{B}_1 - \gamma I \end{bmatrix} < 0\,. \tag{29}$$

Note that the above inequalities are LMIs [9] with respect to P_s and P_u, and thus can be easily solved.

Using the solutions P_s and P_u, we define the same *piecewise Lyapunov function* candidate (7) for the system (1), and consider the difference of the piecewise Lyapunov function candidate along the trajectories of the system (23) or (24). When the controller works,

$$\begin{aligned} V_s(k+1) - V_s(k) &= \tilde{x}^T(k+1)P_s\tilde{x}(k+1) - \tilde{x}^T(k)P_s\tilde{x}(k) \\ &= \begin{bmatrix} \tilde{x}^T(k) & w^T(k) \end{bmatrix} \begin{bmatrix} A_s^TP_sA_s - P_s & A_s^TP_s\tilde{B}_1 \\ \tilde{B}_1^TP_sA_s & \tilde{B}_1^TP_s\tilde{B}_1 \end{bmatrix} \begin{bmatrix} \tilde{x}(k) \\ w(k) \end{bmatrix} \\ &\le \begin{bmatrix} \tilde{x}^T(k) & w^T(k) \end{bmatrix} \begin{bmatrix} -\frac{1}{\gamma}C_s^TC_s + (\zeta_s^{-2} - 1)P_s & 0 \\ 0 & \gamma I \end{bmatrix} \begin{bmatrix} \tilde{x}(k) \\ w(k) \end{bmatrix} \\ &= -\frac{1}{\gamma}\Gamma(k) + (\zeta_s^{-2} - 1)V_s(k)\,, \end{aligned} \tag{30}$$

where $\Gamma(k) \triangleq z^T(k)z(k) - \gamma^2 w^T(k)w(k)$ and (27) was used to obtain the inequality. Therefore, in the case where the designed controller works, we obtain

$$V_s(k+1) \le \zeta_s^{-2} V_s(k) - \frac{1}{\gamma}\Gamma(k)\,. \tag{31}$$

In a similar manner, when the controller fails, we obtain

$$V_u(k+1) \le \zeta_u^{2} V_u(k) - \frac{1}{\gamma}\Gamma(k)\,. \tag{32}$$

Now, without loss of generality, we assume that the designed controller works during $[k_{2j}, k_{2j+1})$, and the controller fails during $[k_{2j+1}, k_{2j+2})$, $j = 0, 1, \cdots$, where $k_0 = 0$. Then, for any $k \ge 1$ in the interval $[k_{2j}, k_{2j+1})$, we obtain easily from (31) that

$$V(k) \le \zeta_s^{-2(k-k_{2j})} V(k_{2j}) - \frac{1}{\gamma}\sum_{m=k_{2j}}^{k-1} \zeta_s^{-2(k-1-m)}\Gamma(m)\,, \tag{33}$$

and similarly for any $k \in [k_{2j+1}, t_{2j+2})$,

$$V(k) \le \zeta_u^{2(k-k_{2j+1})} V(k_{2j+1}) - \frac{1}{\gamma}\sum_{m=k_{2j+1}}^{k-1} \zeta_u^{2(k-1-m)}\Gamma(m) \tag{34}$$

according to (32).

Using the fact $V(k_i) \le \mu V(k_i^-)$, we obtain by induction that

$$\begin{aligned} &V(k) \\ &\le \mu^{2N_k}\lambda_s^{-2(k-T_u(k))}\lambda_u^{2T_u(k)}V(0) - \sum_{m=0}^{k-1}\mu^{2(N_{k-1}-N_m)} \\ &\quad \times \zeta_s^{-2(k-1-m-(T_u(k-1)-T_u(m)))}\zeta_u^{2(T_u(k-1)-T_u(m))}\Gamma(m)\,. \end{aligned} \tag{35}$$

When $\mu = 1$, we obtain from (35) with $x(0) = 0$ and $V(k) \ge 0$ that

$$\sum_{m=0}^{k-1} \zeta_s^{-2(k-1-m-(T_u(k-1)-T_u(m)))}\zeta_u^{2(T_u(k-1)-T_u(m))}\Gamma(m) \le 0\,. \tag{36}$$

Note that the summation term before $\Gamma(m)$ in the above inequality is the transition matrix from time instant m to $k-1$. Then, according to the stability analysis result in the previous section, the inequality

$$\begin{aligned} &|\zeta_s^{-2(k-1-m-(T_u(k-1)-T_u(m)))}\zeta_u^{2(T_u(k-1)-T_u(m))}| \\ &\quad \le c^2\zeta^{2(k-1-m)} \end{aligned} \tag{37}$$

holds with $c = \sqrt{\frac{\alpha_2}{\alpha_1}}$, under the assumption that there exists a positive scalar $\zeta < 1$ such that

$$\frac{T_u(m)}{m} \le \frac{\ln(\zeta_s) + \ln(\zeta)}{\ln(\zeta_s) + \ln(\zeta_u)}\,, \tag{38}$$

for any $m > 1$, which is the condition on the unavailability rate of the controller. Combining (36) and (37), we obtain

$$\begin{aligned} &\sum_{m=0}^{k-1} \zeta_s^{-2(k-1-m)} z^T(m)z(m) \\ &\quad \le c^2\gamma^2 \sum_{m=0}^{k-1} \zeta^{2(k-1-m)} w^T(m)w(m)\,. \end{aligned} \tag{39}$$

We sum both sides of the above inequality from $k = 1$ to $k = +\infty$ to obtain (by rearranging the double-summation area)

$$\frac{1}{1-\zeta_s^{-2}}\sum_{m=0}^{\infty} z^T(m)z(m) \le \frac{c^2\gamma^2}{1-\zeta^2}\sum_{m=0}^{\infty} w^T(m)w(m)\,, \tag{40}$$

which means the $\mathcal{L}_2$ gain level $\sqrt{\frac{1-\zeta_s^{-2}}{1-\zeta^2}}c\gamma$ is achieved under the unavailability rate condition (38).

Next, when $\mu > 1$, we rewrite (35) as

$$\begin{aligned} &\sum_{m=0}^{k-1} \mu^{-2N_m}\zeta_s^{-2(k-1-m-(T_u(k-1)-T_u(m)))} \\ &\quad \times \zeta_u^{2(T_u(k-1)-T_u(m))}\Gamma(m) \le 0\,. \end{aligned} \tag{41}$$

In this case, if in addition to (38) there exists a positive scalar $\zeta^* \in (\zeta, 1)$ such that

$$N_m \le \frac{m}{T_f^*}\,, \qquad T_f^* = \frac{\ln(\mu)}{\ln(\zeta^*) - \ln(\zeta)} \tag{42}$$

holds for all $m > 1$, then we know

$$\mu^{N_m}\zeta^m \le (\zeta^*)^m \Longleftrightarrow \mu^{-N_m}\zeta^{-m} \ge (\zeta^*)^{-m} \tag{43}$$

holds for any $m > 1$.

Using this inequality in (41), we obtain

$$\begin{aligned} &\sum_{m=0}^{k-1} \left(\frac{\zeta^*}{\zeta}\right)^{-2m} \zeta_s^{-2(k-1-m)} z^T(m)z(m) \\ &\quad \le c^2\gamma^2 \sum_{m=0}^{k-1} \zeta^{-2(k-1-m)} w^T(m)w(m)\,. \end{aligned} \tag{44}$$

Summing both sides of the above inequality from $k = 1$ to $k = \infty$ yields

$$\begin{aligned} &\frac{1}{1-\zeta_s^{-2}}\sum_{m=0}^{\infty} \left(\frac{\zeta^*}{\zeta}\right)^{-2m} z^T(m)z(m) \\ &\quad \le \frac{c^2\gamma^2}{1-\zeta^2}\sum_{m=0}^{\infty} w^T(m)w(m)\,, \end{aligned} \tag{45}$$

which means a weighted $\mathcal{L}_2$ gain level $\sqrt{\frac{1-\zeta_s^{-2}}{1-\zeta^2}}c\gamma$ is achieved.

We observe that the condition (42) is the requirement on the ATBCF. More precisely, if the ATBCF in the system (1) is larger than or equal to T_f^* given in (42), then (45) holds as the same and thus the system achieves the same weighted $\mathcal{L}_2$ gain level. We summarize the above discussions in the following theorem.

Theorem 2. *If the unavailability rate of the controller in the system* (1) *is small in the sense of satisfying* (38) *for some positive scalar* $\zeta < 1$*, then there exists a finite constant* T_f^* *such that the system* (1) *achieves a weighted* $\mathcal{L}_2$ *gain level* $\sqrt{\frac{1-\zeta_s^{-2}}{1-\zeta^2}}c\gamma$ *in the sense of* (45)*, for any ATBCF larger than or equal to* T_f^* *.* ■

Remark 4. The inequality (45) describes a weighted $\mathcal{L}_2$ gain level due to the existence of $\left(\frac{\zeta^*}{\zeta}\right)^{-2m}$. When ζ^* is close enough to ζ, which means the ATBCF is sufficiently large according to (42), obviously the inequality (45) approaches a normal $\mathcal{L}_2$ gain. ■

Remark 5. Same as in the previous section, we can easily extend the discussion here to the case where the output feedback controller (2) (write as $u = G(z)y$ shortly) decays in the sense of $u \to \alpha u$ with α being a fixed constant satisfying $0 \le \alpha < 1$. In that case, if the closed-loop system composed of (1) and $u = \alpha G(z)y$ is unstable, the discussions up to now are the same by making some notation change. If this is not the case, then the entire system can be viewed as a switched system composed of two stable subsystems, and thus a weighted $\mathcal{L}_2$ gain level is achieved under an ATBCF scheme (without consideration of unavailability rate of the controller) and the achieved weighted $\mathcal{L}_2$ gain level approaches normal $\mathcal{L}_2$ gain level if the ATBCF is large enough; refer to [11, 12] for detailed discussions. ■

4 Conclusion

We have studied stability and $\mathcal{L}_2$ gain properties for discrete-time LTI control systems controlled by a pre-designed dynamical output feedback controller which fails from time to time due to physical or purposeful reason. For stability, by using a piecewise Lyapunov function, we have shown that if the unavailability rate of the controller is smaller than a specified constant and the average time interval between controller failures (ATBCF) is large enough, then global exponential stability of the system is guaranteed. For $\mathcal{L}_2$ gain, also by using a piecewise Lyapunov function, we have shown that if the unavailability rate of the controller is smaller than a specified constant, then the system with an ATBCF achieves a reasonable weighted $\mathcal{L}_2$ gain level and the weighted $\mathcal{L}_2$ gain approaches normal $\mathcal{L}_2$ gain if the ATBCF is sufficiently large. We suggest that the methodology here also applies for other performance specification analysis of control systems with control failures [13, 14].

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