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$\mathcal{L}_2$ gain analysis for switched systems with continuous-time and discrete-time subsystems

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In this paper, we study $\mathcal{L}_2$ gain property for a class of switched systems which are composed of both continuous-time LTI subsystems and discrete-time LTI subsystems. Under the assumption that all subsystems are Hurwitz/Schur stable and have the $\mathcal{L}_2$ gain less than γ, we discuss the $\mathcal{L}_2$ gain that the switched system could achieve. First, we consider the case where a common Lyapunov function exists for all subsystems in $\mathcal{L}_2$ sense, and show that the switched system has the $\mathcal{L}_2$ gain less than the same level γ under arbitrary switching. As an example in this case, we analyse switched symmetric systems and establish the common Lyapunov function explicitly. Next, we use a piecewise Lyapunov function approach to study the case where no common Lyapunov function exists in $\mathcal{L}_2$ sense, and show that the switched system achieves an ultimate (or weighted) $\mathcal{L}_2$ gain under an average dwell time scheme.

1. Introduction

In the last two decades, there has been increasing interest in stability analysis and controller design for switched systems; see the survey papers Liberzon and Morse (1999), DeCarlo *et al*. (2000), the recent book Liberzon (2003) and the references cited therein. The motivation for studying switched systems is from many aspects. It is known that many practical systems are inherently multimodal in the sense that several dynamical subsystems are required to describe their behaviour which may depend on various environmental factors. Since these systems are essentially switched systems, powerful analysis or design results of switched systems are helpful dealing with real systems. Another important observation is that switching among a set of controllers for a specified system can be regarded as a switched system (Hu *et al*. 2000), and that switching has been used in adaptive control to assure stability in situations where stability can not be proved otherwise (Fu and Barmish 1986, Morse *et al*. 1992), or to improve transient response of adaptive control systems (Narendra and Balakrishnan 1994a). Also, the methods of intelligent control system design are based on the idea of switching among different controllers (Morse 1996). Therefore, study of switched systems contributes significantly to switching controller and intelligent controller design.

When focusing on stability analysis of switched systems, there are many valuable results which have appeared in the last two decades. For example, Narendra and Balakrishnan (1994b) showed that when all subsystems are stable and commutative pairwise, the switched linear system is stable under arbitrary switching. Liberzon *et al*. (1999) extended this result from the commutation condition to a Lie-algebraic condition. Zhai *et al*. (2002a) showed that a class of switched symmetric systems are asymptotically stable under arbitrary switching since a common Lyapunov function, in the form of $V(x) = x^T x$, exists for all the subsystems. Wicks *et al*. (1994), Hespanha and Morse (1999), Zhai *et al*. (2001b), considered the stability analysis problem using piecewise Lyapunov functions.

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Pettersson and Lennartson (1997) considered a stabilization problem by dividing the state space associated with appropriate switching depending on state, and Wicks *et al*. (1998) considered quadratic stabilization for switched systems composed of a pair of unstable linear subsystems by using a linear stable combination of unstable subsystems. Zhai (2001), Zhai *et al*. (2002a, 2002b) extended the consideration to stability analysis problems for switched systems composed of discrete-time subsystems.

Motivated by the observation that all these references deal with switched systems composed of only continuous-time subsystems or only discrete-time ones, the authors considered in the recent papers (Zhai *et al*. 2004a, 2004b) the new type of switched systems which are composed of both continuous-time and discrete-time dynamical subsystems. It was pointed out there that we can easily find many applications involving such switched systems. A typical example is a continuous-time plant controlled either by a physically implemented regulator or by a digitally implemented one together with a switching rule between them.

Motivation Example: Consider the continuous-time LTI system described by $\dot{x}(t) = Ax(t) + Bu(t)$, where $x(t)$ is the continuous-time state, $u(t)$ is the control input in time domain, and A, B are constant matrices. Suppose that a stabilizing state feedback $u(t) = Kx(t)$ has been designed so that $A + BK$ is Hurwitz stable (all the eigenvalues of $A + BK$ are in the open left complex plane). It is known that in any computer-aided system, the controller is implemented in a discrete-time manner. When the sampling period is small enough, the closed-loop system can be viewed as a continuous-time system described by $\dot{x}(t) = (A + BK)x(t)$. When the sampling period does not have to be very small, we only need to deal with the value change on sampling points, and thus it is natural to consider the discrete-time system $x(k+1) = e^{(A+BK)\tau}x(k)$, where τ is the sampling period and $x(k) \triangleq x(k\tau)$. Although we used the same feedback gain K here for simplicity, we may want to design different gains for continuous-time domain and discrete-time one. Therefore, the entire system can be considered as a switched system composed of a continuous-time subsystem and a discrete-time one.

For stability analysis of such mixed types of switched systems, Zhai *et al*. (2004a) gave some analysis and design results. For example, the case where commutation condition holds, and the case of switched symmetric systems, were dealt with there. This paper aims to extend the results of Zhai *et al*. (2004a) to $\mathcal{L}_2$ gain analysis for switched input–output systems composed of both continuous-time and discrete-time dynamical subsystems. There are a few results concerning $\mathcal{L}_2$ gain analysis for switched systems composed of continuous-time subsystems. Hespanha considered such a problem in his PhD dissertation (Hespanha 1998), by using a piecewise Lyapunov function approach. In Zhai *et al*. (2001a), a modified approach has been proposed for more general switched systems and more exact results have been obtained. In that context, it has been shown that when all subsystems are Hurwitz stable and have $\mathcal{L}_2$ gains smaller than a positive scalar γ_0, the switched system under an average dwell time scheme (Hespanha and Morse 1999) achieves a weighted $\mathcal{L}_2$ gain γ_0, and the weighted $\mathcal{L}_2$ gain approaches normal $\mathcal{L}_2$ gain if the average dwell time is chosen sufficiently large. Recently, Hespanha (2003) considered the computation of $\mathcal{L}_2$ gain for switched linear systems with large dwell time, and gave an algorithm by considering the separation between the stabilizing and antistabilizing solutions to a set of algebraic Riccati equations.

Parallel with the discussion in Zhai *et al*. (2001a), we study in this paper the $\mathcal{L}_2$ gain property for switched systems which are composed of both continuous-time LTI subsystems and discrete-time LTI subsystems. Under the assumption that all subsystems are Hurwitz/Schur stable and have the $\mathcal{L}_2$ gain less than γ, we discuss the $\mathcal{L}_2$ gain that the switched system could achieve. First, we consider the case where a common Lyapunov matrix (function) exists for all subsystems in $\mathcal{L}_2$ sense, and show that the switched system has the $\mathcal{L}_2$ gain less than the same level γ under arbitrary switching. As an example in this case, we analyse switched symmetric systems and derive the common Lyapunov function clearly. Next, we use a piecewise Lyapunov function approach for the case where no common Lyapunov function exists in $\mathcal{L}_2$ sense, and show that the switched system achieves an ultimate (or weighted) $\mathcal{L}_2$ gain under an average dwell time scheme.

2. Problem formulation and preliminaries

The switched system we consider in this paper is composed of a set of continuous-time subsystems

$$\begin{cases} \dot{x}(t) = A_{ci}x(t) + B_{ci}w(t) \\ z(t) = C_{ci}x(t) + D_{ci}w(t), \quad i = 1, \ldots, N_c \end{cases} \tag{1}$$

and a set of discrete-time subsystems

$$\begin{cases} x(k+1) = A_{dj}x(k) + B_{dj}w(k) \\ z(k) = C_{dj}x(k) + D_{dj}w(k), \quad j = 1, \ldots, N_d, \end{cases} \tag{2}$$

where $x(t)(x(k)) \in \mathcal{R}^n$ is the subsystem state, $w(t)(w(k)) \in \mathcal{R}^m$ is the input, $z(t)(z(k)) \in \mathcal{R}^p$ is the output. A_{ci}, B_{ci}, C_{ci}, D_{ci} $(i = 1, \ldots, N_c)$ and A_{dj}, B_{dj},

C_{dj}, D_{dj} $(j = 1, \ldots, N_d)$ are constant matrices of appropriate dimensions denoting the subsystems, and $N_c \geq 1$ and $N_d \geq 1$ are the numbers of continuous-time subsystems and discrete-time ones, respectively.

To discuss stability and $\mathcal{L}_2$ gain of the overall switched system, we assume without loss of generality that the sampling periods of all the discrete-time subsystems are of the same value $\tau > 0$ (the discussion can be easily extended to the case where the discrete-time subsystems have different sampling periods). Since the states/inputs/outputs of the discrete-time subsystems can be viewed as piecewise constant vectors between sampling points, we can consider the value of the system states/inputs/outputs in the continuous-time domain. Therefore, although $x(t)/w(t)/z(t)$ is not continuous with respect to time t due to existence of discrete-time subsystems, the vectors of $x(t)$ and $z(t)$ are uniquely defined at all time instants for given disturbance vector $w(t)$, and thus stability and $\mathcal{L}_2$ properties can be discussed in the continuous-time domain.

Focusing on $\mathcal{L}_2$ gain analysis, we give the following definition.

Definition 1: The switched system is said to have $\mathcal{L}_2$ gain less than γ if

$$\int_0^t z^T(s)z(s)ds \leq \gamma^2 \int_0^t w^T(s)w(s)ds \tag{3}$$

holds for any $t > 0$ when the initial state is zero.

The above definition is given in the continuous-time domain form. On the time interval where discrete-time subsystems are activated, the two integral terms are understood as impulsive forms like $\sum_{k_1}^{k_1+m\tau} z^T(k)z(k)$ and $\sum_{k_1}^{k_1+m\tau} w^T(k)w(k)$.

We now list two well known bounded real lemmas dealing with $\mathcal{L}_2$ gain analysis of continuous-time system and discrete-time system, respectively.

Lemma 1 (Boyd *et al.* 1994, Iwasaki *et al.* 1998)**:** *Consider the continuous-time system*

$$\begin{cases} \dot{x}(t) = A_c x(t) + B_c w(t) \\ z(t) = C_c x(t) + D_c w(t), \end{cases} \tag{4}$$

where $x(t)$, $w(t)$ *and* $z(t)$ *are the same as in* (1), *and* A_c, B_c, C_c, D_c *are constant matrices of appropriate dimensions. The system* (4) *is Hurwitz stable and has* $\mathcal{L}_2$ *gain less than* γ *if and only if there exists* $P_c > 0$ *satisfying the LMI*

$$\begin{bmatrix} A_c^T P_c + P_c A_c & P_c B_c & C_c^T \\ B_c^T P_c & -\gamma I & D_c^T \\ C_c & D_c & -\gamma I \end{bmatrix} < 0 \tag{5}$$

or equivalently

$$\begin{bmatrix} A_c^T P_c + P_c A_c + \dfrac{1}{\gamma} C_c^T C_c & P_c B_c + \dfrac{1}{\gamma} C_c^T D_c \\ B_c^T P_c + \dfrac{1}{\gamma} D_c^T C_c & -\gamma I + \dfrac{1}{\gamma} D_c^T D_c \end{bmatrix} < 0. \tag{6}$$

Lemma 2 (Boyd *et al.* 1994, Iwasaki *et al.* 1998)**:** *Consider the discrete-time system*

$$\begin{cases} x(k+1) = A_d x(k) + B_d w(k) \\ z(k) = C_d x(k) + D_d w(k), \end{cases} \tag{7}$$

where $x(k)$, $w(k)$ *and* $z(k)$ *are the same as in* (2), *and* A_d, B_d, C_d, D_d *are constant matrices of appropriate dimensions. The system* (7) *is Schur stable and has* $\mathcal{L}_2$ *gain less than* γ *if and only if there exists* $P_d > 0$ *satisfying the LMI*

$$\begin{bmatrix} -P_d & P_d A_d & P_d B_d & 0 \\ A_d^T P_d & -P_d & 0 & C_d^T \\ B_d^T P_d & 0 & -\gamma I & D_d^T \\ 0 & C_d & D_d & -\gamma I \end{bmatrix} < 0 \tag{8}$$

or equivalently

$$\begin{bmatrix} A_d^T P_d A_d - P_d + \dfrac{1}{\gamma} C_d^T C_d & A_d^T P_d B_d + \dfrac{1}{\gamma} C_d^T D_d \\ B_d^T P_d A_d + \dfrac{1}{\gamma} D_d^T C_d & -\gamma I + \dfrac{1}{\gamma} D_d^T D_d + B_d^T P_d B_d \end{bmatrix} < 0. \tag{9}$$

3. Analysis using common Lyapunov function

In this section, we consider the case where all the subsystems are Hurwitz/Schur stable and have $\mathcal{L}_2$ gain less than γ in the sense that there exists a common solution $P > 0$ satisfying (6) and (9) for all the subsystems. More precisely,

$$\begin{bmatrix} A_{ci}^T P + P A_{ci} + \dfrac{1}{\gamma} C_{ci}^T C_{ci} & P B_{ci} + \dfrac{1}{\gamma} C_{ci}^T D_{ci} \\ B_{ci}^T P + \dfrac{1}{\gamma} D_{ci}^T C_{ci} & -\gamma I + \dfrac{1}{\gamma} D_{ci}^T D_{ci} \end{bmatrix} < 0 \tag{10}$$

holds for all $i = 1, \ldots, N_c$, and

$$\begin{bmatrix} A_{dj}^T P A_{dj} - P + \frac{1}{\gamma} C_{dj}^T C_{dj} & A_{dj}^T P B_{dj} + \frac{1}{\gamma} C_{dj}^T D_{dj} \\ B_{dj}^T P A_{dj} + \frac{1}{\gamma} D_{dj}^T C_{dj} & -\gamma I + \frac{1}{\gamma} D_{dj}^T D_{dj} + B_{dj}^T P B_{dj} \end{bmatrix} < 0 \tag{11}$$

holds for all $j = 1, \ldots, N_d$. It is easy to understand that in this situation there is a common (quadratic) Lyapunov function $V(x) = x^T P x$ for all the subsystems in $\mathcal{L}_2$ sense. Also, it is noted that the existence of common Lyapunov function is readily checked by solving the LMIs (10) and (11) with respect to the matrix variable $P > 0$.

We state and prove the main result in this section.

Theorem 1: *If all the subsystems are stable and have $\mathcal{L}_2$ gain less than γ in the sense that a common solution $P > 0$ satisfies* (10) *for all i's and* (11) *for all j's, then the switched system is stable and has $\mathcal{L}_2$ gain less than γ under arbitrary switching.*

Proof: Since the stability part can be referred to in Zhai *et al.* (2004a), we focus our attention on $\mathcal{L}_2$ gain analysis.

Without loss of generality, we assume that before any given time instant $t > 0$, subsystem A_{c1} was activated during $[0 = t_0, t_1)$, subsystem A_{d1} was activated during $[t_1, t_2 = t_1 + m_1\tau)$, and subsystem A_{c2} is now being activated from t_2. It can be seen that any other case can be analyzed in the same way.

During the time interval $[t_2, t]$, we compute the derivative of $V(x) = x^T P x$ along the trajectory of A_{c2} as

$$\begin{aligned} \dot{V}(x) &= x^T(t) P (A_{c2} x(t) + B_{c2} w(t)) + (A_{c2} x(t) \\ &\quad + B_{c2} w(t))^T P x(t) \\ &= \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} A_{c2}^T P + P A_{c2} & P B_{c2} \\ B_{c2}^T P & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \\ &\leq - \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} \frac{1}{\gamma} C_{c2}^T C_{c2} & \frac{1}{\gamma} C_{c2}^T D_{c2} \\ \frac{1}{\gamma} D_{c2}^T C_{c2} & \frac{1}{\gamma} D_{c2}^T D_{c2} - \gamma I \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \\ &= -\frac{1}{\gamma} \Gamma(t), \end{aligned} \tag{12}$$

where $\Gamma(t) \triangleq z^T(t) z(t) - \gamma^2 w^T(t) w(t)$ and (10) was used to obtain the inequality. Integrating the above inequality from t_2 to t results in

$$\int_{t_2}^{t} \Gamma(s) ds \leq \gamma (V(x(t_2)) - V(x(t))). \tag{13}$$

During the time interval $[t_1, t_2 = t_1 + m_1\tau)$, A_{d1} is supposed to be activated. We compute the difference of the Lyapunov function $V(x) = x^T P x$ along the trajectory of A_{d1} to obtain

$$\begin{aligned} & V(x(t_1 + \tau)) - V(x(t_1)) \\ &= (A_{d1} x(t_1) + B_{d1} w(t_1))^T P (A_{d1} x(t_1) + B_{d1} w(t_1)) \\ &\quad - x^T(t_1) P x(t_1) \\ &= \begin{bmatrix} x(t_1) \\ w(t_1) \end{bmatrix}^T \begin{bmatrix} A_{d1}^T P A_{d1} - P & A_{d1}^T P B_{d1} \\ B_{d1}^T P A_{d1} & B_{d1}^T P B_{d1} \end{bmatrix} \begin{bmatrix} x(t_1) \\ w(t_1) \end{bmatrix} \\ &\leq - \begin{bmatrix} x(t_1) \\ w(t_1) \end{bmatrix}^T \begin{bmatrix} \frac{1}{\gamma} C_{d1}^T C_{d1} & \frac{1}{\gamma} C_{d1}^T D_{d1} \\ \frac{1}{\gamma} D_{d1}^T C_{d1} & \frac{1}{\gamma} D_{d1}^T D_{d1} - \gamma I \end{bmatrix} \begin{bmatrix} x(t_1) \\ w(t_1) \end{bmatrix} \\ &= -\frac{1}{\gamma} \Gamma(t_1), \end{aligned} \tag{14}$$

where (11) was used to obtain the inequality. Similarly, we obtain

$$\left. \begin{aligned} V(x(t_1 + 2\tau)) - V(x(t_1 + \tau)) &\leq -\frac{1}{\gamma} \Gamma(t_1 + \tau) \\ \vdots \quad \vdots \quad &\vdots \\ V(x(t_1 + m\tau)) - V(x(t_1 + (m-1)\tau)) &\leq -\frac{1}{\gamma} \Gamma(t_1 + (m-1)\tau), \end{aligned} \right\} \tag{15}$$

and thus

$$\sum_{j=0}^{m-1} \Gamma(t_1 + j\tau) \leq \gamma (V(x(t_1)) - V(x(t_2))). \tag{16}$$

Adding (13) and (16) with the description in Definition 1, we obtain

$$\int_{t_1}^{t} \Gamma(s) ds \leq \gamma (V(x(t_1)) - V(x(t))). \tag{17}$$

Same as for the interval $[t_2, t)$, we obtain for the time interval $[t_0, t_1)$ that

$$\int_{t_0=0}^{t_1} \Gamma(s) ds \leq \gamma (V(x(t_0)) - V(x(t_1))) \tag{18}$$

and thus

$$\int_{0}^{t} \Gamma(s) ds \leq \gamma (V(x(0)) - V(x(t))). \tag{19}$$

Using $V(x(t)) \geq 0$ and $x_0 = 0$ in the above leads to

$$\int_0^t z^T(s)z(s)ds \leq \gamma^2 \int_0^t w^T(s)w(s)ds. \tag{20}$$

This completes the proof.

One may ask whether the situation in Theorem 1 really exists in real systems. In fact, it has been pointed out in Tan and Grigoriadis (2001a, b), Zhai *et al.* (2002a) that symmetric systems have good properties. Such symmetric systems appear quite often in many engineering disciplines (for example, RC or RL electrical networks, viscoelastic materials and chemical reactions) (Willems 1976), and thus belong to an important class in control engineering.

Lemma 3 (Tan and Grigoriadis 2001a)**:** *Assume that the continuous-time system*

$$\begin{cases} \dot{x}(t) = Ax(t) + Bw(t) \\ z(t) = Cx(t) + Dw(t) \end{cases} \tag{21}$$

is symmetric in the sense of $A = A^T$, $B = C^T$, $D = D^T$. *Then, the system* (21) *is Hurwitz stable and has* $\mathcal{L}_2$ *gain less than* γ *if and only if*

$$\begin{bmatrix} 2A & B & C^T \\ B^T & -\gamma I & D \\ C & D & -\gamma I \end{bmatrix} < 0. \tag{22}$$

Lemma 4 (Tan and Grigoriadis 2001b, Zhai *et al.* 2002a)**:** *Assume that the discrete-time system*

$$\begin{cases} x(k+1) = Ax(k) + Bw(k) \\ z(k) = Cx(k) + Dw(k) \end{cases} \tag{23}$$

is symmetric in the sense of $A = A^T$, $B = C^T$, $D = D^T$. *Then, the system* (21) *is Schur stable and has* $\mathcal{L}_2$ *gain less than* γ *if and only if*

$$\begin{bmatrix} -I & A & B & 0 \\ A & -I & 0 & C^T \\ B^T & 0 & -\gamma I & D^T \\ 0 & C & D & -\gamma I \end{bmatrix} < 0. \tag{24}$$

It is obvious that (22) and (24) mean that a common solution $P = I$ satisfies the matrix inequalities (5) and (8) respectively, for all the subsystems. Then, based on Theorem 1 and Lemmas 3 and 4, the following result is straightforward.

Corollary 1: *If all the subsystems are symmetric, stable and have* $\mathcal{L}_2$ *gain less than* γ, *then the switched system is stable and has* $\mathcal{L}_2$ *gain less than* γ *under arbitrary switching.*

Another class of switched systems for which there exist common Lyapunov functions in $\mathcal{L}_2$ sense has been established in our recent paper (Zhai *et al.* 2005), where we show that if for each subsystem an expanded matrix, including the subsystem's coefficient matrices, is normal and Schur stable, then $V(x) = x^T x$ serves as a common Lyapunov function in $\mathcal{L}_2$ sense for all subsystems.

The next section will discuss how to deal with the switched systems where there does not exist a common Lyapunov function in $\mathcal{L}_2$ sense for the subsystems.

4. Analysis using piecewise Lyapunov function

In this section, we loosen the requirement in the previous section that a common Lyapunov function should exist in $\mathcal{L}_2$ sense, and consider the case where all the subsystems are stable and have the $\mathcal{L}_2$ gain less than the same γ, but the Lyapunov matrices do not have to be same. Then, according to Lemmas 1 and 2, there exist a set of positive definite matrices P_{ci}'s, $i = 1, \ldots, N_c$, satisfying

$$\begin{bmatrix} A_{ci}^T P_{ci} + P_{ci} A_{ci} + \frac{1}{\gamma} C_{ci}^T C_{ci} & P_{ci} B_{ci} + \frac{1}{\gamma} C_{ci}^T D_{ci} \\ B_{ci}^T P_{ci} + \frac{1}{\gamma} D_{ci}^T C_{ci} & -\gamma I + \frac{1}{\gamma} D_{ci}^T D_{ci} \end{bmatrix} < 0 \tag{25}$$

and there exist a set of positive definite matrices P_{dj}'s, $j = 1, \ldots, N_d$, satisfying

$$\begin{bmatrix} A_{dj}^T P_{dj} A_{dj} - P_{dj} + \frac{1}{\gamma} C_{dj}^T C_{dj} & A_{dj}^T P_{dj} B_{dj} + \frac{1}{\gamma} C_{dj}^T D_{dj} \\ B_{dj}^T P_{dj} A_{dj} + \frac{1}{\gamma} D_{dj}^T C_{dj} & \begin{Bmatrix} -\gamma I + \frac{1}{\gamma} D_{dj}^T D_{dj} \\ + B_{dj}^T P_{dj} B_{dj} \end{Bmatrix} \end{bmatrix} < 0. \tag{26}$$

Noting that (25) and (26) are LMIs with respect to $P_{ci} > 0$ and $P_{dj} > 0$, respectively, and thus are readily solved by the existing LMI softwares.

Using the solution P_{ci}'s of (25) and P_{dj}'s of (26), we define the following piecewise Lyapunov function candidate

$$V_\sigma(x) = x^T P_\sigma x \tag{27}$$

for the switched system, where P_σ is switched among the solution P_{ci}'s and P_{dj}'s in accordance with the piecewise constant switching signal. Then, the piecewise Lyapunov function (27) has the following properties:

(i) When subsystem A_{ci} is activated, $V_{ci} = x^T P_{ci} x$ in (27) is continuous and its derivative along the trajectories of A_{ci} satisfies

$$\begin{aligned}\dot{V}_{ci} &= x^T(t)P_{ci}(A_{ci}x(t) + B_{ci}w(t)) \\ &\quad + (A_{ci}x(t) + B_{ci}w(t))^T P_{ci}x(t) \\ &= x^T(t)(P_{ci}A_{ci} + A_{ci}^T P_{ci})x(t) \\ &\quad + x^T(t)P_{ci}B_{ci}w(t) + w^T(t)B_{ci}^T P_{ci}x(t) \\ &\le -\frac{1}{\gamma}(z^T(t)z(t) - \gamma^2 w^T(t)w(t)),\end{aligned} \tag{28}$$

where the inequality is obtained using the matrix inequality (25), as done in the previous section.

(ii) When subsystem A_{dj} is activated, $V_{dj} = x^T P_{dj} x$ in (27) is continuous and its difference along the trajectories of A_{dj} satisfies

$$\begin{aligned}&V_{dj}(x(k+1)) - V_{dj}(x(k)) \\ &\quad = (A_{dj}x(k) + B_{dj}w(k))^T P_{dj}(A_{dj}x(k) + B_{dj}w(k)) \\ &\quad\quad - x^T(k)P_{dj}x(k) \\ &\quad = x^T(k)(A_{dj}^T P_{dj}A_{dj} - P_{dj})x(k) \\ &\quad\quad + x^T(k)A_{dj}^T P_{dj}B_{dj}w(k) + w^T(k)B_{dj}^T P_{dj}A_{dj}x(k) \\ &\quad \le -\frac{1}{\gamma}(z^T(k)z(k) - \gamma^2 w^T(k)w(k)),\end{aligned} \tag{29}$$

where the inequality is obtained using the matrix inequality (26).

(iii) There exist constant scalars $\alpha_1 > 0$, $\alpha_2 > 0$ such that

$$\alpha_1 \|x\|^2 \le \{V_{ci}(x), V_{dj}(x)\} \le \alpha_2 \|x\|^2 \tag{30}$$

holds for all $x \in \mathcal{R}^n$ and all i,j. It is easy to see that (30) is true if we choose $\alpha_1 = \inf_{i,j}\{\lambda_m(P_{ci}), \lambda_m(P_{dj})\}$, $\alpha_2 = \sup_{i,j}\{\lambda_M(P_{ci}), \lambda_M(P_{dj})\}$, where $\lambda_M(P)$ ($\lambda_m(P)$) denotes the largest (smallest) eigenvalue of a symmetric matrix P.

(iv) There exists a constant scalar $\mu \ge 1$ such that

$$V_*(x) \le \mu V_{**}(x) \tag{31}$$

holds for all $x \in \mathcal{R}^n$, where "$_*$" and "$_{**}$" can be any subsystem index. It is easy to see that one choice of such μ is $(\sup_{i,j}\{\lambda_M(P_{ci}), \lambda_M(P_{dj})\})/(\inf_{i,j}\{\lambda_m(P_{ci}), \lambda_m(P_{dj})\})$. Since $\mu = 1$ is the case where all positive definite matrices are the same (and thus a common Lyapunov matrix exists as discussed in the previous section), we exclude such case and assume $\mu > 1$ here.

For simplicity, let us now consider the same switching signal as we used before: subsystem A_{c1} on $[0 = t_0, t_1)$, subsystem A_{d1} on $[t_1, t_2 = t_1 + m_1\tau)$, and subsystem A_{c2} on $[t_2, t)$. Then, we obtain from (28) and (29) that

$$\begin{aligned}\int_0^{t_1} \Gamma(s)ds &\le \gamma(V_{c1}(0) - V_{c1}(x(t_1))) \\ \sum_{j=0}^{m-1} \Gamma(t_1 + j\tau) &\le \gamma(V_{d1}(x(t_1)) - V_{d1}(x(t_2))) \\ \int_{t_2}^{t} \Gamma(s)ds &\le \gamma(V_{c2}(x(t_2)) - V_{c2}(x(t))).\end{aligned} \tag{32}$$

Using $V_{c2}(x(t_2)) \le \mu V_{d1}(x(t_2))$, $V_{d1}(x(t_1)) \le \mu V_{c1}(x(t_1))$, and $x(0) = 0$, $V_{c2}(x(t)) \ge 0$ to add the above three inequalities, we obtain

$$\int_0^{t_1} \mu^2 \Gamma(s)ds + \sum_{j=0}^{m-1} \mu\Gamma(t_1 + j\tau) + \int_{t_2}^{t} \Gamma(s)ds \le 0. \tag{33}$$

If we denote by $N(t)$ the number of switchings that occurred during $[0, t)$, then the above inequality can be rewriten as

$$\int_0^t \mu^{N(t)-N(s)} \Gamma(s)ds \le 0, \tag{34}$$

or equivalently

$$\int_0^t \mu^{-N(s)} \Gamma(s)ds \le 0. \tag{35}$$

It is not difficult to confirm that the above inequality holds for any other switching signal.

If the switching signal satisfies

$$c_1 e^{-\lambda_1 s} \le \mu^{-N(s)} \le c_2 e^{-\lambda_2 s} \tag{36}$$

with positive scalars $c_1, c_2, \lambda_1, \lambda_2$, we obtain from (35) that

$$c_1 \int_0^t e^{-\lambda_1 s} z^T(s)z(s)ds \le c_2 \gamma^2 \int_0^t e^{-\lambda_2 s} w^T(s)w(s)ds. \tag{37}$$

Integrating (37) from $t = 0$ to $t = \infty$ (by rearranging the double-integral area) leads to

$$\int_0^\infty z^T(s)z(s)ds \le \frac{\lambda_1 c_2}{\lambda_2 c_1}\gamma^2 \int_0^\infty w^T(s)w(s)ds, \tag{38}$$

which implies that an ultimate $\mathcal{L}_2$ gain $\sqrt{(\lambda_1 c_2/\lambda_2 c_1)}\gamma$ is achieved.

We observe that the inequality (36) is exactly an average dwell time scheme since it can be rewriten as

$$a_2 + \frac{s}{\tau_{a2}} \leq N(s) \leq a_1 + \frac{s}{\tau_{a1}} \tag{39}$$

where

$$\begin{aligned} a_1 &= -\frac{\ln c_1}{\ln \mu}, \quad \tau_{a1} = \frac{\ln \mu}{\lambda_1} \\ a_2 &= -\frac{\ln c_2}{\ln \mu}, \quad \tau_{a2} = \frac{\ln \mu}{\lambda_2}. \end{aligned} \tag{40}$$

We summarize the above discussion in the following theorem.

Theorem 2: *Assume that all the subsystems are Hurwitz/Schur stable and have the $\mathcal{L}_2$ gain less than γ. Then, the switched system under the average dwell time scheme* (39) *achieves the ultimate $\mathcal{L}_2$ gain less than $\sqrt{(\lambda_1 c_2)/(\lambda_2 c_1)}\gamma$.*

This theorem only gives a kind of "worst" estimation of $\mathcal{L}_2$ gain property for the switched systems under a wide class of switching law. A more practical problem is to design the class of switching signal so that the switched system can achieve the $\mathcal{L}_2$ gain close to the original level. This is an interesting problem in our future research.

We observe that the inequality (39) gives an upper bound together with a lower bound for the average dwell time and thus the number of switchings. In many applications, it is not desirable to set a lower bound for $N(s)$. Thus, we consider the switching signal satisfying

$$e^{-\lambda s} \leq \mu^{-N(s)} \tag{41}$$

which is rewritten as

$$N(s) \leq \frac{s}{\tau_a}, \quad \tau_a = \frac{\ln \mu}{\lambda}, \tag{42}$$

which is also an average dwell time scheme, specifying the lower bound of the dwell time averagely between the subsystems ($\tau_a \leq s/N(s)$). Then, we obtain from (35) that

$$\int_0^t e^{-\lambda s} z^T(s) z(s) ds \leq \gamma^2 \int_0^t w^T(s) w(s) ds. \tag{43}$$

Due to the existence of the term $e^{-\lambda s}$, we see that the switched system achieves a weighted $\mathcal{L}_2$ gain γ under the average dwell time scheme (47).

Theorem 3: *Assume that all the subsystems are Hurwitz/Schur stable and have the $\mathcal{L}_2$ gain less than γ. Then, the switched system under the average dwell time scheme* (42) *achieves a weighted $\mathcal{L}_2$ gain γ in the sense of* (43).

It is important to point out that if λ is chosen close enough to zero, which means the average dwell time in (42) is chosen sufficiently large, then the inequality (43) approaches the normal $\mathcal{L}_2$ gain definition. This observation is consistent with the results in the case of switched continuous-time systems (Zhai *et al.* 2001a) and the case of switched discrete-time systems (Zhai *et al.* 2002b).

Finally we note that using the approach in this paper together with Zhai *et al.* (2001a, 2002b), we can also analyse $\mathcal{L}_2$ gain properties for the case where unstable subsystems exist and the case where perturbations exist in all subsystems.

5. Conclusion

In this paper, we have studied $\mathcal{L}_2$ gain property for a class of switched systems which are composed of both continuous-time subsystems and discrete-time subsystems. Under the assumption that all subsystems are Hurwitz/Schur stable and have the $\mathcal{L}_2$ gain less than γ, we have discussed the $\mathcal{L}_2$ gain that the switched system could achieve. We have shown that when a common Lyapunov function exists for all subsystems in $\mathcal{L}_2$ sense, the switched system has the $\mathcal{L}_2$ gain less than the same level γ under arbitrary switching. As an example, we have established a common Lyapunov function in $\mathcal{L}_2$ sense for switched symmetric systems. In the case where no common Lyapunov function exists in $\mathcal{L}_2$ sense, we have proposed a piecewise Lyapunov function to show that the switched system achieves an ultimate (or weighted) $\mathcal{L}_2$ gain under an average dwell time scheme.

We observe that when there does not exist a common Lyapunov function in $\mathcal{L}_2$ sense, the results involving the average dwell time scheme in this paper are still conservative. One possible solution to this problem may be the use of the controller realization strategy proposed in Hespanha and Morse (2002), although the extension from stabilization to $\mathcal{L}_2$ gain analysis and design is a difficult task. We also note that the results in this paper can be applied to the multi-controller design problem in digital control systems and furthermore networked control systems.

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