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A Necessary and Sufficient Condition for Stability of Arbitrarily Switched Second-Order LTI System: Marginally Stable Case

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***Abstract*— In this paper, the stability analysis for a class of second order switched LTI systems with possible marginally stable subsystems is investigated. The main contribution here is the derivation of a necessary and sufficient condition for stability of such a switched system under arbitrary switching. The condition can be easily checked which is illustrated by an example.**

I. INTRODUCTION

The stability issues of switched systems, especially switched linear systems, have attracted considerable interest in the recent decade, see for example the survey papers [1],[2], the recent book [3] and the references cited therein. It is known that the stability of switched systems depends on not only the dynamics of the subsystems but also the properties of the switching signals. One of the basic problems for switched systems is to identify conditions that guarantee the stability of a switched system under all possible switching signals, or arbitrary switching. A popular way to deal with this problem is based on finding a common Lyapunov function. This approach is justified by the converse Lyapunov theorem proposed in [4] for arbitrary switching systems. However, most existing efforts, e.g. [5], [6], [7], are based on or imply the existence of a common quadratic Lyapunov function (CQLF), which is known to be sufficient only. Therefore, the study of non-quadratic Lyapunov functions has been attracting more and more attentions, e.g. [8], [9]. However, these non-quadratic Lyapunov functions are not easy to determine in general.

Another approach to derive both necessary and sufficient conditions for the stability of switched systems is based on the characterization of the worst case switching signal. The idea is very simple: if the switched system remains stable under the worst case switching signal, then the switched system must be stable for all possible switching signals. Similar idea has been used in [11] to derive a necessary and sufficient condition for absolute stability of second-order systems. However, the condition proposed in [11] is restricted to some special classes and not obvious to generalize. In addition, the checking of the condition in [11] could be computationally challenging. This motivates the development of an easily verifiable, necessary and sufficient condition for arbitrarily switched second order LTI systems with two asymptotically stable subsystems in [12]. In this paper, we further generalize the results in [12] to the case that the switched system may contain marginally stable subsystems.

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The switched systems considered here can be formulated as

$$S_{ij} : \dot{x} = \sigma x, \quad \sigma \in \{A_i, B_j\} \tag{1}$$

where A_i and B_j are both stable in sense of Lyapunov, and $i, j \in \{1, 2, 3\}$ denote the types of A and B respectively. A matrix $A \in \mathbb{R}^{2\times 2}$ was classified into three types according to its eigenvalue and eigenstructure. Type I: A has real eigenvalues and diagonalizable; Type II: A has real eigenvalues but undiagonalizable; Type III: A has two complex eigenvalues.

The problem studied here is under what condition the switched system (1) remains stability under all possible switching signals. Please note that the stability discussed in this paper is regarding the boundedness of the states. A switched system is said to be stable if its trajectory is bounded under all possible switching signals.

The rest of the paper is organized as follows. In Section II, some useful results presented in [12] are reviewed. In Section III, the main result, its proofs and examples are given. And the final section concludes the paper.

II. PRELIMINARIES

In this section, some results and definitions in [12] are briefly reviewed. Interested readers may refer to [12] for further details and proofs.

A. *Solution of single subsystem in polar coordinates*

Consider a second-order LTI system

$$\dot{x} = Ax = \begin{bmatrix} a & b \\ c & d \end{bmatrix} x \tag{2}$$

and define $x_1 = r cos\theta, \;\; x_2 = r sin\theta$, it follows that

$$\frac{dr}{dt} = r[a\cos^2\theta + d\sin^2\theta + (b+c)\sin\theta\cos\theta] \tag{3}$$

$$\frac{d\theta}{dt} = c\cos^2\theta - b\sin^2\theta + (d-a)\sin\theta\cos\theta \tag{4}$$

Lemma 2.1: The trajectories of system (2) in r-θ coordinates, except the ones lie on the eigenvectors[1], can be expressed as

$$r(t) = Cu(\theta(t)) \tag{5}$$

where C is a positive constant depending on initial state (r_0, θ_0) and $u(\theta)$ is positive.

[1] The case that the trajectory stays along an eigenvector corresponds to $\frac{d\theta}{dt} = 0$, which will be dealt with separately later.

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B. Solution for switched LTI systems in polar coordinates

Consider the switched LTI system (1), and denote the two subsystems as:

$$\Sigma_A : \dot{x} = Ax = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} x \tag{6}$$

$$\Sigma_B : \dot{x} = Bx = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} x \tag{7}$$

It follows from Lemma 2.1 that $r = C_A u_A(\theta)$, $r = C_B u_B(\theta)$, then the solution of the switched system is obtained by a combination of the solutions of two subsystems.

$$r = h_A(\theta) u_A(\theta) \tag{8}$$

where

$$h_A(\theta) = \begin{cases} C_A, & \sigma = A \\ C_B \frac{u_B(\theta)}{u_A(\theta)}, & \sigma = B \end{cases} \tag{9}$$

or similarly

$$r = h_B(\theta) u_B(\theta) \tag{10}$$

where

$$h_B(\theta) = \begin{cases} C_A \frac{u_A(\theta)}{u_B(\theta)}, & \sigma = A \\ C_B, & \sigma = B \end{cases} \tag{11}$$

With reference to (8), $u(\theta)$ is bounded for stable A and h_A is constant when $\sigma = A$, so the interesting part is the variation of h_A, which is described by $\frac{dh_A}{dt}$ when $\sigma = B$.

For convenience, we denote

$$H_A(\theta(t)) \triangleq \left.\frac{dh_A}{dt}\right|_{\sigma=B}, H_B(\theta(t)) \triangleq \left.\frac{dh_B}{dt}\right|_{\sigma=A} \tag{12}$$

$$Q_A(\theta(t)) \triangleq \left.\frac{d\theta}{dt}\right|_{\sigma=A}, Q_B(\theta(t)) \triangleq \left.\frac{d\theta}{dt}\right|_{\sigma=B} \tag{13}$$

To find the worst case switching signal for a given switched system (1), we need to know which subsystem is the worst for every θ and how θ varies with time t. The former one is determined by the signs of $H_A(\theta)$ and $H_B(\theta)$ while the latter one depends on the signs of $Q_A(\theta)$ and $Q_B(\theta)$. The geometrically meaning of positive $H_A(\theta)$ is that the vector field of subsystem Σ_B points outwards relative to Σ_A. And a positive Q_A implies a counter clockwise trajectory of Σ_A in $x - y$ coordinates.

It has been shown in [12] that it is sufficient to study stability of a switched system (1) in an interval of $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2})$. Denote $k = tg\theta$, the functions of θ can be transformed to the functions of k. Straightforward algebraic manipulations yield

$$\operatorname{sgn}(H_A(k)) = \operatorname{sgn}\left(\frac{N(k)}{D_A(k)}\right) \tag{14}$$

$$\operatorname{sgn}(H_B(k)) = -\operatorname{sgn}\left(\frac{N(k)}{D_B(k)}\right) \tag{15}$$

$$\operatorname{sgn}(Q_A(k)) = -\operatorname{sgn}(D_A(k)) \tag{16}$$

$$\operatorname{sgn}(Q_B(k)) = -\operatorname{sgn}(D_B(k)) \tag{17}$$

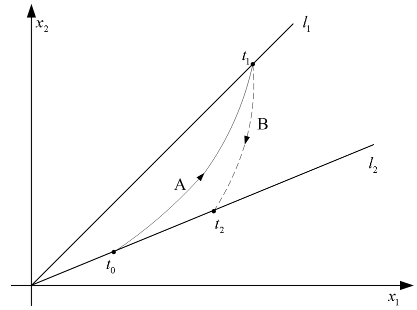

Fig. 1. The region where both H_A and H_B are positive

where $D_A(k) = b_1k^2 + (a_1 - d_1)k - c_1$, $D_B(k) = b_2k^2 + (a_2 - d_2)k - c_2$ and

$$\begin{aligned} N(k) = {} & (b_1d_2 - b_2d_1)k^2 + (b_1c_2 + a_1d_2 - b_2c_1 - a_2d_1)k \\ & + a_1c_2 - a_2c_1 \triangleq p_2k^2 + p_1k + p_0. \end{aligned} \tag{18}$$

Denote two distinct real solutions of $N(k) = 0$, if exist, by k_1 and k_2, and assume $k_2 < k_1$. Define a continuous interval of k, in which the signs of (14)-(17) preserve, as a region of k. Some useful properties are obtained from (14)-(17)

1) If the signs of Q_A and Q_B are the same (opposite), then the signs of H_A and H_B are opposite (same).
2) The boundaries of regions of k, if exist, are real eigenvectors of subsystems ($D_A(k) = 0$ or $D_B(k) = 0$) and the vectors where $\frac{dr}{d\theta}\big|_{\sigma=A} = \frac{dr}{d\theta}\big|_{\sigma=B}$ (N(k)=0).
3) When trajectories cross boundaries of k_1 or k_2, the trajectory directions will not change, but the signs of $H_A(k)$ and $H_B(k)$ change simultaneously.
4) For two neighbor regions whose common boundary are an eigenvector of Σ_A, the trajectory along A can not cross the boundary and the directions are opposite in these two region. Moreover, the signs of $H_A(k)$ changes when trajectory of Σ_B across the boundary.

C. Criteria of the worst case switching signal (WCSS)

1) Both H_A and H_B are positive:

Lemma 2.2: The switched system (1) is not stable under arbitrary switching if there exists a region of k, where both H_A and H_B are positive.

Fig.1 shows that the trajectories of the switched system will go to infinity by keeping switching in the region bounded by l_1 and l_2.

2) H_A is positive and H_B is negative: The worse subsystem is Σ_B. Fig.2 shows that the trajectories of Σ_B always have a larger magnitude than the corresponding ones of Σ_A for all θ in this region.

3) H_A is negative and H_B is positive: Similarly, the worse subsystem is Σ_A.

4) Both H_A and H_B are negative:

Lemma 2.3: The switched system (1) is stable under arbitrary swiching *if* one of $H_A(k)$ and $H_B(k)$ is non-positive for all k.

Based on Lemma 2.3, if the trajectory stays in this region, the switched system is stable. Hence, the worst case

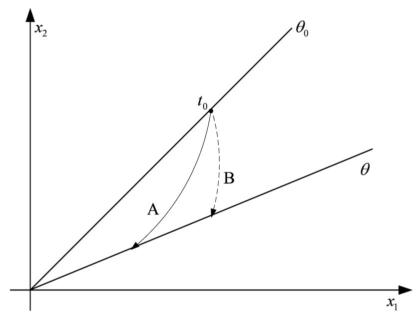

Fig. 2. The region where H_A is positive and H_B is negative

switching signal is the subsystem whose trajectory is able to go out of this region.

If both trajectories can go out, we need to consider two different cases: If one of the boundaries is k_1 or k_2, it follows from (14) and (15) that there is an unstable region near this region. The switched system is not stable under arbitrary switching; Otherwise, the boundaries are eigenvectors of subsystems, both trajectories can go out and can not come back. Hence no matter which subsystem is chosen, trajectory will leave this region and the stability of the system is not affected.

5) Both H_A and H_B are zero: In this case, we can choose either one as the worse case switching signal.

6) On real eigenvectors: It can be readily shown that the worse subsystem is Σ_A if trajectory is on the eigenvector of B, and vice versa.

III. Main result

Without loss of generality, we define the standard form J_i for different types of second order matrix.

$$J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},\ J_2 = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix},\ J_3 = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix} \tag{19}$$

where

$$\lambda_2 \leq \lambda_1 \leq 0, \quad \lambda < 0, \quad \mu \leq 0, \omega > 0 \tag{20}$$

for stable matrix. The trivial case when $\lambda_2 = \lambda_1 = 0$ is excluded in general proofs.

It is assumed that one of subsystems of (1) is in its standard form, ie., $A_i = J_i$. Then, the other one can be expressed as $B_j = Q_j J_j Q_j^{-1}$ with $i \leq j$. If A_i and B_j do not share a real eigenvector (these cases will be proved separately), then transformation matrix Q_j can be obtained in the following structures based on the eigenvector of B_j.

$$Q_1 = \begin{bmatrix} 1 & 1 \\ \alpha & \beta \end{bmatrix}, Q_2 = \begin{bmatrix} 0 & 1 \\ \beta & \alpha \end{bmatrix}, Q_3 = \begin{bmatrix} 0 & 1 \\ \beta & \alpha \end{bmatrix} \tag{21}$$

Additional assumptions are required for individual combinations.

1) If $S_{ij} = S_{11}$, $\beta < 0$
2) If $S_{ij} = S_{12}$, $\alpha < 0$
3) If $S_{ij} = S_{13}$, $k_1, k_2 \leq 0$ (if exist)
4) If $S_{ij} = S_{33}$, $p_2 \neq 0$ (if $A \neq B$)
5) If $S_{ij} = S_{33}$, $p_2 < 0$ (if $N(k)$ has two distinct real roots)

Any given switched linear systems (1) can be transformed to satisfy these assumptions by coordinates transformation while stability properties of the switched system preserve. Assumptions 1-3 can be satisfied by the transformation $\bar{x}_1 = -x_1$ and assumption 4-5 can be satisfied by similarity transformation with a unitary matrix $P = \begin{bmatrix} \gamma & -\eta \\ \eta & \gamma \end{bmatrix}$ when necessary, where $\det(P) = \sqrt{\gamma^2 + \eta^2} = 1$.

Theorem 3.1: Switched system (1) is not stable under arbitrary switching signals *if and only if* $N(k)$ (18) has two distinct real roots, $k_2 < k_1$, satisfying

$$\begin{cases} N \leq k_2 < k_1 \leq M & if\, \det(Q_j) < 0 \\ \left\| e^{B_j T_B} e^{A_i T_A} x(0) \right\|_2 > \|x(0)\|_2 & if\, \det(Q_j) > 0 \end{cases} \tag{22}$$

where

$$\begin{cases} N = \alpha, M = 0 & S_{ij} = S_{11} \\ N \to -\infty, M \to +\infty & otherwise \end{cases} \tag{23}$$

$$T_A = \int_{\theta_2}^{\theta_1} \frac{1}{c_1 \cos^2\theta - b_1 \sin^2\theta + (d_1 - a_1)\sin\theta\cos\theta} d\theta \tag{24}$$

$$T_B = \int_{\theta_1}^{\theta_2+\pi} \frac{1}{c_2 \cos^2\theta - b_2 \sin^2\theta + (d_2 - a_2)\sin\theta\cos\theta} d\theta \tag{25}$$

where $\theta_1 = tg^{-1}k_1$, $\theta_2 = tg^{-1}k_2$ and $x(0) = [1, k_2]^T$.

A. Proof of the special cases

If A and B share a common real eigenvector, then A and B can be transformed to lower-triangular matrix $\bar{A}$ and $\bar{B}$ simultaneously by a nonsingular matrix whose second column is the common real eigenvector. The switched system after transformation is $\dot{x} = \begin{bmatrix} a_{\sigma(t)} & 0 \\ c_{\sigma(t)} & d_{\sigma(t)} \end{bmatrix} x$, and the solution of the switched system can be obtained by studying the two states $x_1(t)$, $x_2(t)$ separately

$$x_1(t) = x_1(0) e^{\int_0^t a_{\sigma(\tau)} d\tau} \tag{26}$$

$$x_2(t) = e^{\int_0^t d_{\sigma(\tau)} d\tau} \left[\int_0^t c_{\sigma(\tau)} x_1(\tau) e^{-\int_0^\tau d_{\sigma(\xi)} d\xi} d\tau + x_2(0) \right] \tag{27}$$

It can be readily shown that both states $x_1(t)$ and $x_2(t)$ are bounded due to non-positive $a_{\sigma(t)}$, $d_{\sigma(t)}$ and stable subsystems.

The result can also be obtained by using Theorem 3.1. Since $b_1 = b_2 = 0$ violate the condition $N(k) = 0$ has two distinct eigenvalues, the switched system is stable.

Comment: If one subsystem is of type I and has identical eigenvalues, then all real vectors in the phase plane are the eigenvectors of this subsystem. If the other subsystem is of type I or type II, which happens in S_{11} and S_{12}, then these two subsystems share a common real eigenvectors and the switched system is stable.

Now, we proceed to prove general cases of S_{ij} with marginally stable subsystems.

B. Proof for $S_{ij} = S_{11}$

$$A_1 = \begin{bmatrix} \lambda_{1a} & 0 \\ 0 & \lambda_{2a} \end{bmatrix} \tag{28}$$

$$B_1 = \frac{1}{\beta - \alpha} \begin{bmatrix} \beta\lambda_{1b} - \alpha\lambda_{2b} & \lambda_{2b} - \lambda_{1b} \\ \alpha\beta(\lambda_{1b} - \lambda_{2b}) & \beta\lambda_{2b} - \alpha\lambda_{1b} \end{bmatrix} \tag{29}$$

Define $\lambda_{1a} = k_A\lambda_{2a}$, $\lambda_{1b} = k_A\lambda_{2b}$. Note that $k_A = 1$,$k_B = 1$ or $\alpha = 0$ correspond to special cases, then we have $\lambda_{2a}, \lambda_{2b} < 0$, $k_A, k_B \in [0, 1)$, $\alpha \neq 0$. In addition, we have $k_A k_B = 0$ for at least one marginally stable subsystem and $\beta < 0$ by assumption. Substitute (28) and (29) to (14)-(18), it follows that

$$N(k) = \lambda_{2a}\lambda_{2b}(1 - k_A)\frac{\bar{N}(k)}{\alpha - \beta} \tag{30}$$

where $\bar{N}(k)$ is a monic polynomial with the same solutions as $N(k)$.

$$\bar{N}(k) = k^2 + \frac{(k_A - k_B)\beta + (1 - k_A k_B)\alpha}{k_B - 1}k + \alpha\beta k_A \tag{31}$$

$$\operatorname{sgn}(H_A(k)) = \operatorname{sgn}(\alpha - \beta)\operatorname{sgn}(\bar{N}(k))\operatorname{sgn}(k) \tag{32}$$

$$\operatorname{sgn}(H_B(k)) = -\operatorname{sgn}(\bar{N}(k))\operatorname{sgn}(k - \alpha)\operatorname{sgn}(k - \beta) \tag{33}$$

$$\operatorname{sgn}(Q_A(k)) = -\operatorname{sgn}(k) \tag{34}$$

$$\operatorname{sgn}(Q_B(k)) = -\operatorname{sgn}(\alpha - \beta)\operatorname{sgn}(k - \alpha)\operatorname{sgn}(k - \beta) \tag{35}$$

In order to determine the signs of above equations, we need to know the locations of k_1, k_2 relative to α, β and 0, which correspond to the eigenvectors of the two subsystems.

$$(\alpha - k_1)(\alpha - k_2) = \frac{k_B\alpha(1 - k_A)(\alpha - \beta)}{k_B - 1} \tag{36}$$

1) $\bar{N}(k)$ do not have two distinct real roots: $\operatorname{sgn}(\bar{N}(k)) \geq 0$, then what we care is the relative positions among α, β and 0. Note that (36) ≥ 0 is assured in this case. It follows from (31) and (36) ≥ 0 that $\beta < \alpha < 0$. Then the signs of (32)-(34) are determined.

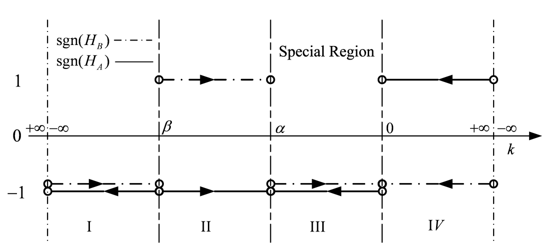

Fig. 3. The signs of $H_A(k)$ and $H_B(k)$ for S_{11} when $N(k)$ does not have two distinct real roots, the switched system is stable.

With reference to Fig.3, Region I, III are stable since both $H_A(k)$ and $H_B(k)$ are negative in these regions. Furthermore, none of the trajectories can go out of the region III.

- If the initial state is in region III, it can not go out of this region.
- If the initial state is in region II or IV, it will be brought into region III by the worst case switching signal, which is Σ_A in region II and Σ_B in region IV.
- If the initial state is in region I, it must be brought out because region I is stable. Then the trajectory will go to region II or region IV, and go to region III eventually.

Therefore, the switched system is stable since all the trajectories under the worst switching signal will go into a stable region and can not go out again.

2) $\bar{N}(k)$ has two distinct real roots and $\det(Q_1) < 0$*:* $\alpha > \beta$, with reference to (31) and (36), there are only three possibilities:

a) $\beta < \alpha \leq k_2 < k_1 \leq 0$: In this case, the switched system is not stable under arbitrary switching since $H_A(k)$ and $H_B(k)$ are both positive when $k \in (k_2, k_1)$. In Fig.4, $k_1 = 0$ only when $k_B = 0$ and $k_2 = \alpha$ only when $k_A = 0$. This result corresponds to the first inequality in Theorem 3.1.

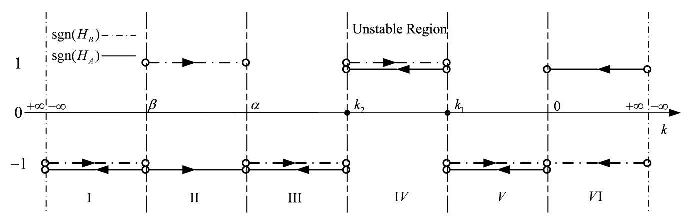

Fig. 4. The signs of $H_A(k)$ and $H_B(k)$ for S_{11} when $N(k)$ has two distinct real roots between α and 0, the switched system is not stable under arbitrary switching.

b) $\beta = k_2 < k_1 < \alpha < 0$: Similar process shows that all trajectories along WCSS will go into the region, bounded by α and 0, can not go out again. The switched system is stable since $H_A(k)$ and $H_B(k)$ are negative in this region.

c) $\beta < k_2 \leq 0 < \alpha \leq k_1$: It can be readily shown that the switched system is stable by similar process.

3) $\bar{N}(k)$ has two distinct real roots and $\det(Q_1) > 0$*:* $\alpha < \beta$, it follows from (31) and (36) that $k_2 \leq \alpha < \beta < k_1 \leq 0$.

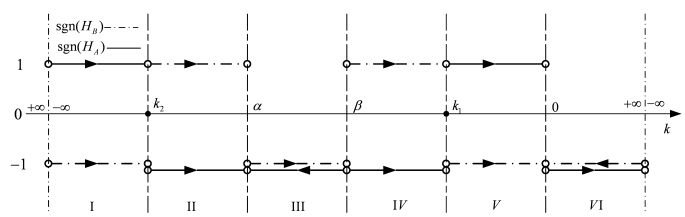

Fig. 5. The signs of $H_A(k)$ and $H_B(k)$ for S_{11} when trajectories can rotate towards the origin.

With reference to Fig.5, it is straightforward that the worst case switching signal is Σ_B in region I and V because H_A is positive and H_B are negative. Similarly, the WCSS is Σ_A in region II and IV because H_A is positive and H_B are negative. In region III, both of H_A and H_B are negative, but Σ_A is the only subsystem whose trajectory can go out of region III because the boundaries of region III are α and β that correspond to the eigenvectors of B. Similarly, the WCSS is Σ_B in region VI. On k_1 and k_2, without loss of generality, we choose Σ_B as the WCSS. Based on above analysis, it is concluded that the WCSS in the whole interval of k is

$$\begin{cases} \sigma = A & k_2 < k < k_1, \\ \sigma = B & \text{otherwise.} \end{cases} \quad (37)$$

The simplest way to determine stability of this system is to follow a trajectory under WCSS originating from a line l until it gets to l again and evaluate its expansion or contraction in the radial direction. Without loss of generality, let $x(0) = [1, k_2]$, the system is not stable under arbitrary switching if and only if $\|\exp(B_1 T_B)\exp(A_1 T_A)x(0)\|_2 > \|x(0)\|_2$, which follows the second inequality in Theorem 3.1.

C. *Proof of $S_{ij} = S_{12}$*

In this case, two subsystems are expressed as

$$A_1 = \begin{bmatrix} \lambda_{1a} & 0 \\ 0 & \lambda_{2a} \end{bmatrix}, B_2 = \frac{1}{\beta}\begin{bmatrix} \beta\lambda_b - \alpha & 1 \\ -\alpha^2 & \beta\lambda_b + \alpha \end{bmatrix}$$

where $\beta < 0$ by assumption, $\lambda_{1a} = 0$ for marginally stable cases and $\lambda_{2a}, \lambda_b < 0$. Denote $\lambda_{1a} = k_A\lambda_{2a}$, then $k_A = 0$. Substitute the entries of A_1 and B_2 into (14)-(17), it follows that

$$\text{sgn}(H_A(k) = \text{sgn}(\beta)\,\text{sgn}(\bar{N}(k))\,\text{sgn}(k) \quad (38)$$

$$\text{sgn}(H_B(k) = -\,\text{sgn}(\bar{N}(k)) \quad (39)$$

$$\text{sgn}(Q_A(k)) = -\,\text{sgn}(k) \quad (40)$$

$$\text{sgn}(Q_B(k)) = -\,\text{sgn}(\beta) \quad (41)$$

where

$$\bar{N}(k) = k^2 - [(k_A - 1)\beta\lambda_b + (k_A + 1)\alpha]k + k_A\alpha^2 \quad (42)$$

Similarly with S_{11}, we need to know the locations of k_1, k_2 relative to α, which is based on the sign of (43)

$$\text{sgn}((\alpha - k_1)(\alpha - k_2)) = \text{sgn}((1 - k_A)(\lambda_b\alpha)\beta) = \text{sgn}(\beta) \quad (43)$$

1) $\bar{N}(k)$ does not have two distinct real roots: (39) is non-positive for all k, the switched system is stable based on Lemma 2.3.

2) $\bar{N}(k)$ has two distinct real roots and $\det(Q_2) < 0$*:* $\beta > 0$, k_1 and k_2 are in the same side of α and $|k_1 k_2| = k_A\alpha^2 < \alpha^2$, so we have $\alpha < k_2 < k_1 = 0$. Both (38) and (39) are positive when $k \in (k_2, k_1)$. Therefore, the switched system is not stable under arbitrary switching based on Lemma 2.2.

3) $\bar{N}(k)$ has two distinct real roots and $\det(Q_2) > 0$*:* $\beta < 0$, with reference to (42) and (43), we have $k_2 < \alpha < k_1 = 0$. Since both k_1 and k_2 are non-positive, the worst case switching signal is obtained which is the same as (37) in case S_{11}.

D. *Proof of $S_{ij} = S_{13}$*

$$A_1 = \begin{bmatrix} \lambda_{1a} & 0 \\ 0 & \lambda_{2a} \end{bmatrix}, B_3 = \frac{\omega}{\beta}\begin{bmatrix} \beta\xi - \alpha & 1 \\ -(\alpha^2 + \beta^2) & \beta\xi + \alpha \end{bmatrix}$$

where $\mu \le 0$, $\omega > 0$, and $\xi = \frac{\mu}{\omega} \le 0$. Substitute the entries of A_1 and B_3 into (14)-(17), it follows that

$$\text{sgn}(H_A(k)) = \text{sgn}(\beta)\,\text{sgn}(\bar{N}(k))\,\text{sgn}(k) \quad (44)$$

$$\text{sgn}(H_B(k)) = -\,\text{sgn}(\bar{N}(k)) \quad (45)$$

$$\text{sgn}(Q_A(k)) = -\,\text{sgn}(k) \quad (46)$$

$$\text{sgn}(Q_B(k)) = -\,\text{sgn}(\beta) \quad (47)$$

where

$$\bar{N}(k) = k^2 - [(k_A - 1)\beta\xi + (k_A + 1)\alpha]k + k_A(\alpha^2 + \beta^2) \quad (48)$$

1) $\bar{N}(k)$ does not have two distinct real roots: (45) is non-positive for all k, the switched system is stable based on Lemma 2.3.

2) $\bar{N}(k)$ has two distinct real roots and $\det(Q_3) < 0$*:* We have $\beta > 0$ and $k_1, k_2 \le 0$ by assumption. It follows that $H_A(k)$ and $H_B(k)$ are positive when $k \in (k_2 \quad k_1)$, thus the switched system is not stable under arbitrary switching.

3) $\bar{N}(k)$ has two distinct real roots and $\det(Q_3) > 0$*:* $\beta < 0$, Similarly, we obtain the WCSS as (37).

E. *Proof of $S_{ij} = S_{22}$*

In this case, there is no marginally stable subsystem, which was considered in [12].

F. *Proof of $S_{ij} = S_{23}$*

$$A = \begin{bmatrix} \lambda_a & 0 \\ 1 & \lambda_a \end{bmatrix}, B = \frac{\omega}{\beta}\begin{bmatrix} \beta\xi - \alpha & 1 \\ -(\alpha^2 + \beta^2) & \beta\xi + \alpha \end{bmatrix}$$

where $\mu = 0$, $\omega > 0$, and $\xi = \frac{\mu}{\omega} = 0$.

$$\text{sgn}(H_A(k)) = -\,\text{sgn}(\beta)\,\text{sgn}(\bar{N}(k)) \quad (49)$$

$$\text{sgn}(H_B(k)) = -\,\text{sgn}(\bar{N}(k)) \quad (50)$$

$$\text{sgn}(Q_A(k)) = 1 \quad (51)$$

$$\text{sgn}(Q_B(k)) = -\,\text{sgn}(\beta) \quad (52)$$

where

$$\bar{N}(k) = k^2 - \frac{2\alpha\lambda_a - 1}{\lambda_a}k + \frac{\lambda_a(\alpha^2 + \beta^2) + (\beta\xi - \alpha)}{\lambda_a} \quad (53)$$

1) $\bar{N}(k)$ does not have two distinct real roots: (50) is non-positive for all k, the switched system is stable based on Lemma 2.3.

2) $\bar{N}(k)$ has two distinct real roots and $\det(Q_3) < 0$*:* $\beta > 0$, both of (49) and (50) are positive when $k \in (k_2, k_1)$, thus the switched system is not stable under arbitrary switching as long as the two roots $k_2 < k_1$ exist.

3) $\bar{N}(k)$ has two distinct real roots and $\det(Q_3) > 0$*:* $\beta < 0$, Similarly, we obtain the WCSS as (37).

G. Proof of $S_{ij} = S_{33}$

$$A_3 = \begin{bmatrix} \mu_a & -1 \\ 1 & \mu_a \end{bmatrix}, B_3 = \frac{\omega_b}{\beta} \begin{bmatrix} \beta\xi - \alpha & 1 \\ -(\alpha^2 + \beta^2) & \beta\xi + \alpha \end{bmatrix}$$

where $\mu_a, \mu_b \leq 0$, $\omega_b > 0$ and $\xi = \frac{\mu_b}{\omega_b} \leq 0$. Similarly, we have

$$\mathrm{sgn}(H_A(k)) = -\,\mathrm{sgn}(N(k)) \tag{54}$$

$$\mathrm{sgn}(H_B(k)) = -\,\mathrm{sgn}(\beta)\,\mathrm{sgn}(N(k)) \tag{55}$$

$$\mathrm{sgn}(Q_A(k)) = 1 \tag{56}$$

$$\mathrm{sgn}(Q_B(k)) = -\,\mathrm{sgn}(\beta) \tag{57}$$

1) $N(k)$ does not have two distinct real roots:

a) $\beta < 0$: At least one of (54) and (55) is negative for all k regardless of the sign of leading coefficient of $N(k)$.

b) $\beta > 0$ and the leading coefficient of $N(k)$ is positive: In this case, (54) is negative for all k. The switched system is stable under arbitrary switching based on Lemma 2.3.

c) $\beta > 0$ and the leading coefficient of $N(k)$ is negative.: It can be proved by contradiction that leading coefficient of $N(k)$ is not possible to be negative if $N(k)$ has no distinct real roots and $\beta > 0$.

2) $N(k)$ has two distinct real roots and $\det(Q_3) < 0$: Note that the sign of $N(k)$ is positive when $k \in (k_2, k_1)$ because p_2, the leading coefficient of N(k), was assumed to be negative. In this case, $\beta > 0$, both of (54) and (55) are positive when $k \in (k_2, k_1)$, thus the switched system is not stable under arbitrary switching as long as the two roots $k_2 < k_1$ exist.

3) $N(k)$ has two distinct real roots and $\det(Q_3) > 0$: $\beta < 0$, Similarly, we obtain the WCSS as (37).

H. An example

Consider a switched linear system with two subsystems

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 \\ -5 & 1 \end{bmatrix} \tag{58}$$

1) Step 1: Simple checking yields that A has two eigenvalues $\pm i$ and B has two eigenvalues $\pm 2i$. So it is the case S_{33} with two marginally stable cases. And $J_3 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$ is obtained based on the eigenvalues of B.

2) Step 2: It is noticed that A is already in its standard form and the Assumption 4 and 5 for S_{33} are satisfied since $p_2 = b_1 d_2 - b_2 d_1 = -1$. Thus no further transformations are needed. Since B and the standard form of B are known, we can obtain $Q_3 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ according to (21).

3) Step 3: Substitute entries of A and B into (18), we have $k_2 = -0.236, k_1 = 4.236$ and $\det(Q) = -2 < 0$. It follows from Theorem 3.1 that the switched system (58) is not stable for arbitrary switching. It is shown in Fig.6 that there exists a switching signal $(x(t_0) \to x(t_1) \to x(t_2)...)$ that makes the system unstable.

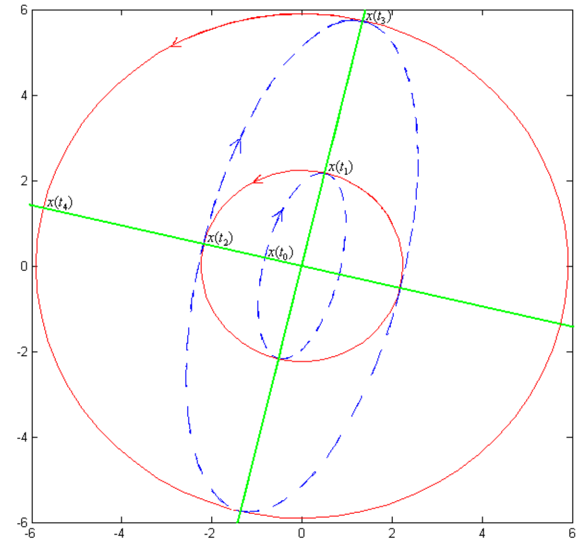

Fig. 6. A switched system with two marginally stable subsystems is not stable under arbitrary switching. Solid curves belong to Σ_A and dashed curves belong to Σ_B

IV. Conclusion

In this paper, a necessary and sufficient condition for stability of arbitrarily switched second order LTI systems with marginally stable subsystem was derived. It turns out that the condition for the marginally stable case is similar with the one for asymptotically stable except boundary conditions are included.

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