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Robust Stability of Discrete-Time Switched Delay Systems and Its Application to Network-Based Reliable Control

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Abstract— **This paper addresses the problem of robust stability of discrete-time switched systems with time-varying delays and polytopic uncertainties. It is supposed that the switched systems are composed of both stable and unstable subsystems. Delay-dependent sufficient conditions for robust exponential stability are proposed by employing parameter-dependent piecewise Lyapunov-like functional combined with the free-weighting matrix and average dwell time methods. These sufficient conditions are applied in the study of stability analysis for a network-based reliable control system with controller failures, and the effectiveness of the analysis procedure is demonstrated with numerical simulation on network-based reliable control of cart-pendulum.**

I. INTRODUCTION

A switched system is a hybrid system which consists of several subsystems described by differential or difference equations and a switching signal that orchestrates the switching among them [1], [2], [3], [4]. Switched systems with delay is called switched delay systems, where delay may be contained in the system state, control input or switching signal [5], [6]. Delay is a common phenomenon in practical control systems [7], which can usually degrade system's performance and even cause system instability. Stability of switched delay systems is often a natural requirement in practice. Consequently, increasing attention has been paid to the study of stability of switched delay systems, e.g., in continuous-time [5] and discrete-time [6], [8], [9], [10]. Recently, switched Lyapunov function method was extensively used to assess asymptotic stability of discrete-time switched delay systems [6], [8], [9], where arbitrary switching signal was considered and thus every subsystem in switched systems was required to be stable. In [10], the average dwell time technique [11] was utilized to design a stabilizing switching signal in the case where all subsystems were stable, and arbitrary switching signal was included as a special case. However, unstable subsystems could be involved in switched systems.

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It is well-known that switched systems composed of unstable subsystems may generate converge trajectories under class of switching signals [1], [3]. Average dwell time technique [1], [11], [12] is an effective tool for designing such stabilizing switching signals. As shown in [13], if the average dwell time was chosen sufficiently large and the total activation time of unstable subsystems was relatively small compared with that of stable subsystems, then the stability of the switched system can be guaranteed. So far, to the best of our knowledge, few result has appeared to construct stabilizing switching signals for discrete-time switched delay systems which might be composed of unstable subsystems [14]. In [14], exponential stability was proposed for delay switched systems with unstable subsystems, where the delay was assumed to be constant. However, in real-world models, the delay could be time varying, e.g., transmission delays in communication networks [15].

On the other hand, when designing a control system, it is often desirable to obtain robust stability against uncertainties on the physical parameters of the system. Polytopic uncertainty is an important class of uncertainties with explicit physical meaning, and many uncertain control systems can be modeled as systems with polytopic uncertainty. For example, a system affected by network-induced delay [16], or effected by quantization in networks [17], and or suffered by actuator failures [18] can be represented as a polytopic uncertain system. A traditional approach of robust control against parametric uncertainties is usually based on the notion of quadratic stability [19], [20], in which parameter-independent Lyapunov functions are used to drive quadratic stability conditions. However, the use of parameter-independent Lyapunov functions usually leads to more conservative results than those resulting from parameter-dependent Lyapunov functions, as shown in [21], [16], especially in the case where polytopic uncertainty is time-invariant or slowly time-varying.

Motivated by the idea of average dwell time [13] and the method of parameter-dependent Lyapunov functions, in this paper, robust stability is studied for discrete-time switched systems with time-varying delays and polytopic uncertainties, in which both stable and unstable subsystems are included. Compared with the existing results, the main contributions of the paper are that: i) parameter-dependent quadratic Lyapunov-like functionals are used to estimate the convergence or divergence rate of individual subsystem with time-varying delays and polytopic uncertainties; and ii) delay-dependent sufficient conditions for robustly exponential stability are presented for the switched systems.

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II. Problem statement

Consider a discrete-time uncertain linear delay system described by

$$\begin{aligned} x(k+1) &= A(\theta)x(k) + A_d(\theta)x(k-d(k)) \\ &\quad + B(\theta)u(k-d(k)) \\ x(k) &= \phi(k), \qquad -d_2, -d_2+1, \ldots, 0 \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the system state, $u(k) \in \mathbb{R}^p$ is the control input, and $\phi(k)$ is the initial condition sequence. $d(k)$ is a time-varying delay and satisfies

$$d_1 \leq d(k) \leq d_2 \tag{2}$$

where d_1 and d_2 are constant positive scalars representing the lower and upper delays, respectively. The system matrices $A(\theta)$, $A_d(\theta)$, and $B(\theta)$ are not precisely known, but belong to the following convex polytopic set

$$\begin{aligned} \Omega_1 := \{[A(\theta)\ A_d(\theta)\ B(\theta)] : [A(\theta)\ A_d(\theta)\ B(\theta)] = \\ \sum_{q=1}^{m} \theta_q [A_q\ A_{dq}\ B_q],\ \theta_q \geq 0,\ \sum_{q=1}^{m} \theta_q = 1\} \end{aligned} \tag{3}$$

where $\theta = [\theta_1\ \theta_2\ \cdots\ \theta_m]^T \in \mathbb{R}^m$ is a vector of uncertain real parameters, A_q, A_{dq}, B_q, $q \in \mathcal{M} = \{1, 2, \ldots, m\}$ are constant matrices denoting the extreme points in the uncertainties, and m is the number of the extreme points.

Assumption 1: The autonomous system (1), i.e., $u(k) = 0$, might be unstable; a state feedback controller $u(k) = Kx(k)$ has been designed, which results in the closed-loop system being robustly stable.

In practical applications, it might be desirable to make controllers work intermittently. For example, controllers are suspended from time to time for an economic or system life under consideration [22], [23]. Take another example from network-based control systems (NCSs) [24], [25], a group of plants are distributively controlled over a shared communication network. Because of the limitation of communication capacity, only a limited number of controllers can access the shared network to communication with their plants while others must wait. Thus, a single control system switches on its open-loop and closed-loop status, e.g., [24], [25]. On the other hand, controller failures often occur in applications since control signals are not transmitted perfectly; the controllers are not available sometimes; or the controllers themselves are under complete outage [22], [23]. When the control system (1) is open/closed, it can be described by the following switched system:

$$x(k+1) = \bar{A}_{\sigma(k)}(\theta)x(k) + \bar{A}_{d\sigma(k)}(\theta)x(k-d(k)) \tag{4}$$

where the switching signal $\sigma(k) : \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \to \mathcal{N} = \{1, 2\}$ is a piecewise constant function of time k. Let subsystem "1" denote the case of the closed-loop status, and subsystem "2" describe the control loop being open, i.e.,

$$\bar{A}_1(\theta) = A(\theta), \quad \bar{A}_{d1}(\theta) = \bar{A}_d(\theta) + B(\theta)K, \tag{5}$$

$$\bar{A}_2(\theta) = A(\theta), \quad \bar{A}_{d2}(\theta) = \bar{A}_d(\theta) \tag{6}$$

Under Assumption 1, in the paper, we are interested in solving the problem: How often and how long system (1) should be closed its control loop such that the stability of the system is guaranteed?

III. Robust stability

This section aims to derive delay-dependent stability conditions for switched system (4) with subsystems (5) and (6).

Definition 1: The uncertain switched delay system (4) is said to be robustly exponentially stable with a decay rate ρ under switching signal $\sigma(k)$ if the equilibrium $x^*(k) = 0$ is robustly exponentially stable with a decay rate ρ for all admissible uncertain parameters θ, i.e., there exist constants $c > 0$ and $0 < \rho < 1$ such that the solution $x(k)$ of system (4) satisfies

$$\| x(k) \| \leq c\rho^{k-k_0} \| x(k_0) \|_\delta, \quad \forall k \geq k_0 \tag{7}$$

where $\| \cdot \|$ denotes the Euclidean norm and $\|x(k_0)\|_\delta = \sup_{-d_2 \leq \delta \leq 0}\{\|x(k_0+\delta)\|\}$, and k_0 is the initial time step.

Definition 2: [25] For any $k > 0$, let $\alpha_c(k)$ denote the total time interval of the plant being closed-loop (attended by the controller) during $[0\ k)$, and the ratio $\frac{\alpha_c(k)}{k}$ is said to be the attention rate of the plant, and let $N(k)$ denote the total number of the plant being closed-loop, which is said to be the attention frequency.

Consider the subsystem of the switched delay system (4) given by

$$x(k+1) = \bar{A}_i(\theta)x(k) + \bar{A}_{di}(\theta)x(k-d(k)),\ i \in \mathcal{N} \tag{8}$$

For this system, choose the following positive definite quadratic functional:

$$\begin{aligned} V_i(k) &= V_{1i}(k) + V_{2i}(k) + V_{3i}(k) + V_{4i}(k) + V_{5i}(k) \\ V_{1i}(k) &= x^T(k)P_i(\theta)x(k) \\ V_{2i}(k) &= \sum_{l=k-d(k)}^{k-1} x^T(l)\lambda_i^{k-1-l}Q_i(\theta)x(l) \\ V_{3i}(k) &= \sum_{l=k-d_2}^{k-1} x^T(l)\lambda_i^{k-1-l}R_i(\theta)x(l) \\ V_{4i}(k) &= \sum_{\delta=-d_2+1}^{-d_1} \sum_{l=k+\delta}^{k-1} x^T(l)\lambda_i^{k-1-l}Q_i(\theta)x(l) \\ V_{5i}(k) &= \sum_{\delta=-d_2}^{-1} \sum_{l=k+\delta}^{k-1} y^T(l)\lambda_i^{k-1-l}(Z_{1i}(\theta) + Z_{2i}(\theta))y(l) \\ y(l) &= x(l+1) - x(l), \end{aligned} \tag{9}$$

where $P_i(\theta)\ Q_i(\theta)\ R_i(\theta)\ Z_{1i}(\theta)\ Z_{2i}(\theta)$ have the following form:

$$\begin{aligned} \Omega_2 := \{&[P_i(\theta)\ Q_i(\theta)\ R_i(\theta)\ Z_{1i}(\theta)\ Z_{2i}(\theta)] \\ &: [P_i(\theta)\ Q_i(\theta)\ R_i(\theta)\ Z_{1i}(\theta)\ Z_{2i}(\theta)] \\ &= \sum_{q=1}^{m} \theta_q [P_{iq}\ Q_{iq}\ R_{iq}\ Z_{1iq}\ Z_{2iq}], \\ &\theta_q \geq 0,\ \sum_{q=1}^{m} \theta_q = 1\}, \end{aligned} \tag{10}$$

with $P_{iq} > 0$, $Q_{iq} \geq 0$, $R_{iq} \geq 0$, $Z_{1iq} > 0$, $Z_{2iq} > 0$, $i \in \mathcal{N}$, $q \in \mathcal{M}$.

It should be mentioned that the positive definite quadratic functional (9) is similar to the form in [26], i.e., when $\lambda_i = 1$, and $P_{iq} = P_i$, $Q_{iq} = Q_i$, $R_{iq} = R_i$, $Z_{1iq} = Z_{1i}$, $Z_{2iq} = Z_{2i}$, $q \in \mathcal{M}$, functional (9) is that employed in [26] where it is used to derive asymptotic stability conditions for certain non-switched delay system. The positive matrices $P_i(\theta)$ $Q_i(\theta)$ $R_i(\theta)$ $Z_{1i}(\theta)$ $Z_{2i}(\theta)$ in (10) depend on the uncertain parameters θ. When $P_{iq} = P_i$, $Q_{iq} = Q_i$, $R_{iq} = R_i$, $Z_{1iq} = Z_{1i}$, $Z_{2iq} = Z_{2i}$, $q \in \mathcal{M}$, $V_i(k)$ in (9) shrinks the parameter-independent functional. Under consideration that parameter-independent Lyapunov function usually renders conservative stability conditions as shown in [16], [21], parameter-dependent positive function (9) is used to estimate the convergence or divergence rate of individual subsystem (4).

Along any state trajectory of system (8), an exponential decay or increase estimation of $V_i(k)$ in (9) is presented in the following lemma.

Lemma 1: Consider system (8) with the time-varying delay (2) and the uncertainty domain (3). For given scalars $\lambda_i > 0$ and $d_2 \geq d_1 \geq 0$, if there exist matrices $P_{iq} > 0$, $Q_{iq} \geq 0$, $R_{iq} \geq 0$, $Z_{1iq} > 0$, $Z_{2iq} > 0$, $T_i = [T_{1i}^T\ T_{2i}^T]^T$, $V_{iq} = [V_{1iq}^T\ V_{2iq}^T\ V_{3iq}^T\ V_{4iq}^T]^T$, $W_{iq} = [W_{1iq}^T\ W_{2iq}^T\ W_{3iq}^T\ W_{4iq}^T]^T$, $S_{iq} = [S_{1iq}^T\ S_{2iq}^T\ S_{3iq}^T\ S_{4iq}^T]^T$, $i \in \mathcal{N}$, $q \in \mathcal{M}$, such that the following inequalities

$$\begin{bmatrix} \Phi_{iq} & \Psi_{iq} \\ \Psi_{iq}^T & \Gamma_{iq} \end{bmatrix} < 0, \quad i \in \mathcal{N},\ q \in \mathcal{M} \tag{11}$$

hold, then along any state trajectory of system (8), the following inequalities satisfy

$$V_i(k+1) \leq \lambda_i V_i(k) \tag{12}$$

where

$$\Phi_{iq} = \begin{bmatrix} \Phi_{11iq} & \Phi_{12iq} & \Phi_{13iq} & \Phi_{14iq} \\ \Phi_{12iq}^T & \Phi_{22iq} & \Phi_{23iq} & \Phi_{24iq} \\ \Phi_{13iq}^T & \Phi_{23iq}^T & \Phi_{33iq} & \Phi_{34iq} \\ \Phi_{14iq}^T & \Phi_{24iq}^T & \Phi_{34iq}^T & \Phi_{44iq} \end{bmatrix}$$

$$\Psi_{iq} = \begin{bmatrix} c_{1i}V_{1iq} & c_{2i}W_{1iq} & c_{1i}S_{1iq} \\ c_{1i}V_{2iq} & c_{2i}W_{2iq} & c_{1i}S_{2iq} \\ c_{1i}V_{3iq} & c_{2i}W_{3iq} & c_{1i}S_{3iq} \\ c_{1i}V_{4iq} & c_{2i}W_{4iq} & c_{1i}S_{4iq} \end{bmatrix}$$

$$\Gamma_{iq} = \mathrm{diag}\{-c_{1i}Z_{1iq},\ -c_{2i}Z_{1iq},\ -c_{1i}Z_{2iq}\}$$

$$\begin{aligned}
\Phi_{11iq} &= -\lambda_i P_{iq} + T_{1i}\bar{A}_{iq} + \bar{A}_{iq}^T T_{1i}^T + R_{iq} + d_2(Z_{1iq} + Z_{2iq}) \\
&\quad + (d_2 - d_1 + 1)Q_{iq} + V_{1iq} + V_{1iq}^T + S_{1iq} + S_{1iq}^T \\
\Phi_{12iq} &= -T_{1i} + \bar{A}_{iq}^T T_{2i}^T - d_2(Z_{1iq} + Z_{2iq}) + V_{2iq}^T + S_{2iq}^T \\
\Phi_{13iq} &= T_{1i}\bar{A}_{diq} + V_{3iq}^T - V_{1iq} + W_{1iq} + S_{3iq}^T \\
\Phi_{14iq} &= V_{4iq}^T - W_{1iq} + S_{4iq}^T - S_{1iq} \\
\Phi_{22iq} &= P_{iq} - T_{2i} - T_{2i}^T + d_2(Z_{1iq} + Z_{2iq}) \\
\Phi_{23iq} &= T_{2i}\bar{A}_{diq} - V_{2iq} + W_{2iq} \\
\Phi_{24iq} &= -W_{2iq} - S_{2iq} \\
\Phi_{33iq} &= -\lambda_i^{d_2} Q_{iq} - V_{3iq} - V_{3iq}^T + W_{3iq} + W_{3iq}^T
\end{aligned}$$

(for $0 < \lambda_i \leq 1$), or

$$\Phi_{33iq} = -\lambda_i^{d_1} Q_{iq} - V_{3iq} - V_{3iq}^T + W_{3iq} + W_{3iq}^T \ (\text{for } \lambda_i \geq 1)$$

$$\begin{aligned}
\Phi_{34iq} &= -V_{4iq}^T + W_{4iq}^T - W_{3iq} - S_{3iq} \\
\Phi_{44iq} &= -\lambda_i^{d_2} R_{iq} - W_{4iq} - W_{4iq}^T - S_{4iq} - S_{4iq}^T \\
c_{1i} &= \lambda_i^{-(d_2+1)d_2/2},\ \ c_{2i} = \lambda_i^{-(d_2+d_1+1)(d_2-d_1)/2}
\end{aligned}$$

Proof: See Appendix A. ■

By iterative substitutions, inequality (12) yields

$$V_i \leq \lambda_i^{(k-k_0)} V(k_0) \tag{13}$$

which implies the functional $V_i(k)$ in (9) along any state trajectory of system (8) has an exponential decay rate λ_i ($0 < \lambda_i < 1$) or increase rate λ_i ($\lambda_i > 1$). When $\lambda_i = 1$, inequality (11) gives delay-dependent sufficient conditions for asymptotical stability of system (8).

Based on the above exponential decay or increase estimation of $V_i(k)$, the following result gives robustly exponential stability of system (4).

Theorem 1: Consider the switched delay system (4) with subsystem (5) and subsystem (6). For given scalars $0 < \lambda_1 < 1$, $\lambda_2 > \lambda_1$, $d_2 \geq d_1 \geq 0$, and $\mu \geq 1$, if there exist matrices $P_{iq} > 0$, $Q_{iq} \geq 0$, $R_{iq} \geq 0$, $Z_{1iq} > 0$, $Z_{2iq} > 0$, $T_i = [T_{1i}^T\ T_{2i}^T]^T$, $V_{iq} = [V_{1iq}^T\ V_{2iq}^T\ V_{3iq}^T\ V_{4iq}^T]^T$, $W_{iq} = [W_{1iq}^T\ W_{2iq}^T\ W_{3iq}^T\ W_{4iq}^T]^T$, $S_{iq} = [S_{1iq}^T\ S_{2iq}^T\ S_{3iq}^T\ S_{4iq}^T]^T$, $i \in \mathcal{N}$, $q \in \mathcal{M}$ such that inequalities (11) hold, then system (4) is robustly exponentially stable with decay rate $0 < \rho < 1$ under the switching signal $\sigma(k)$ with the following conditions:

i) The attention rate satisfies

$$\frac{\alpha_c(k)}{k} \geq \frac{\ln\lambda_2 - \ln\lambda^*}{\ln\lambda_2 - \ln\lambda_1} \tag{14}$$

ii) The attention frequency satisfies

$$\begin{aligned}
N(k) &\leq N_0 + k/T_a,\ N_0 = \frac{\ln c}{2\ln\mu + \ln\lambda_d}, \\
T_a &> T_a^* = \frac{2\ln\mu + \ln\lambda_d}{2\ln\rho - \ln\lambda^*}
\end{aligned} \tag{15}$$

where T_a and N_0 are said to be the average dwell time and the chatter bound [11], respectively, $\lambda_1 < \lambda^* < \rho^2 < 1$, $c > 0$, $\lambda_d = (\frac{\lambda_2}{\lambda_1})^{d_2-1}$ and μ satisfies the following inequalities

$$\begin{aligned}
&P_{\alpha q} \leq \mu P_{\beta q},\ Q_{\alpha q} \leq \mu Q_{\beta q},\ R_{\alpha q} \leq \mu R_{\beta q}, \\
&Z_{1\alpha q} \leq \mu Z_{1\beta q},\ Z_{2\alpha q} \leq \mu Z_{2\beta q},\ \alpha, \beta \in \mathcal{N}
\end{aligned} \tag{16}$$

Further, the state decay estimation is given by

$$\|x(k)\| \leq \sqrt{\frac{bc}{a}}\rho^k \|x(0)\|_\delta \tag{17}$$

where $a = \min_{\forall i \in \mathcal{N}, q \in \mathcal{M}}\{\lambda_{\min}(P_{iq})\}$, $\quad b = \max\{b_1, b_2\}$

$$\begin{aligned}
b_1 = &\max_{\forall q \in \mathcal{M}}\{\lambda_{\max}(P_{1q})\} + d_2 \max_{\forall q \in \mathcal{M}}\{\lambda_{\max}(Q_{1q}) \\
&+ \lambda_{\max}(R_{1q})\} + d_2^2 \max_{\forall q \in \mathcal{M}}\{\lambda_{\max}(Z_{11q} + Z_{21q})\} \\
&+ (d_2 - d_1)(d_2 - 1)\max_{\forall q \in \mathcal{M}}\{\lambda_{\max}(Q_{1q})\} \\
b_2 = &\max_{\forall q \in \mathcal{M}}\{\lambda_{\max}(P_{2q})\} + d_2\lambda_2^{d_2+1}\max_{\forall q \in \mathcal{M}}\{\lambda_{\max}(Q_{2q}) \\
&+ \lambda_{\max}(R_{2q})\} + d_2^2\lambda_2^{d_2-1}\max_{\forall q \in \mathcal{M}}\{\lambda_{\max}(Z_{11q} + Z_{21q})\} \\
&+ (d_2 - d_1)(d_2 - 1)\lambda_2^{d_2-2}\max_{\forall q \in \mathcal{M}}\{\lambda_{\max}(Q_{1q})\}
\end{aligned}$$

in which $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the minimum and maximum eigenvalues of a symmetric matrix, respectively,

Proof: See Appendix B. ■

Remark 1: Condition i) implies that the attention rate of the stable subsystem is required to be sufficiently large. The lower bound of attention rate depends on the estimation of the stable subsystem convergence rate and unstable subsystem divergence rate. While the attention frequency is small enough which is restricted with Condition ii). This implies low frequency switching is required.

Remark 2: Although only two subsystems is considered in the switched system (4), it is easy to extend Theorem 1 to the case of more than two subsystems. In this case, the subsystem "1" and "2" will denote, respectively, the sets of stable and unstable subsystems.

IV. Applications to Network-Based Reliable Control

In this section, robust stability conditions proposed in Theorem 1 are applied to study the stability analysis for network-based reliable control system with controller failure. Here, controller failure means the complete outage of control signal i.e., $u(k) = 0$, and the controller can be recovered through a time interval [14], [22]. The control network setup in Fig. 1 is used for simulation studies, in which cart-pendulum is distributively controlled over a network link.

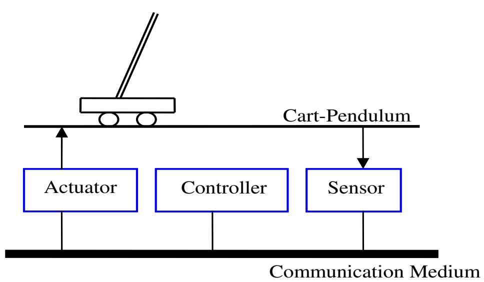

Fig. 1. Cart-pendulum is distributively controlled over a network link.

The simplified and discretized model of the cart-pendulum is described by

$$\xi(k+1) = A_d\xi(k) + B_d u(k) \tag{18}$$

where the system matrices are given as follows [26]:

$$A_d = \begin{bmatrix} 1.0078 & 0.0301 \\ 0.5202 & 1.0078 \end{bmatrix}, \quad B_d = \begin{bmatrix} -0.0001 \\ -0.0053 \end{bmatrix}$$

The eigenvalues of matrix A_d are $\text{eig}(A_d) = \{1.1329; 0.8827\}$, which means that A_d is an unstable matrix. Since unavoidable transmission delay in the communication network, it is assumed that a state-feedback delayed control law is described by

$$u(k) = K\xi(k - d(k)) \tag{19}$$

where $K = [348.7626\ 73.3999]$ and $0 \le d(k) \le 1$.

For the single input system (18), let us consider the reliability with respect to actuator failures. Denote u^F as the faulty actuator output. Then the following failure model is adopted [27]:

$$u^F(k) = \alpha u(k) \tag{20}$$

where $0 \le \underline{\alpha} \le \alpha \le \overline{\alpha} \le 1$. Here $\underline{\alpha}$ and $\overline{\alpha}$ represent the lower and upper bounds, respectively. Note that, when $\underline{\alpha} = \overline{\alpha} = 1$ then the model (20) represents the system without any failure. When $0 < \underline{\alpha} \le \alpha \le \overline{\alpha} < 1$, then this model represents a partial failure. Then, system (18) with controller failure $(u(k) = 0)$ and actuator failure (20) is expressed as the switched delay system (4) with

$$\bar{A}_1 = A_d, \quad \bar{A}_{d1}(\alpha) = A_d + \alpha BK,$$
$$\bar{A}_2 = A_d, \quad \bar{A}_{d2} = 0.$$

Set $0.5 \le \alpha \le 1$, which means that the actuator is tolerable to partial failure and its control effectiveness can vary from the normal case (100% effectiveness) to 50% effectiveness. Selecting the parameters $c = 0$, $\lambda_1 = 0.4$, $\lambda_2 = 1.3$, and $\mu = 1.5$, then LMIs (11) in Theorem 1 is feasible. Choosing the parameters $\rho = 0.84$ and $\lambda^* = 0.65$, then it follows that the attention rate $\frac{\alpha_c(k)}{k} \ge 0.5881$ and the average dwell time $T_a^* = 9.8802$ according to inequalities (14) and (15).

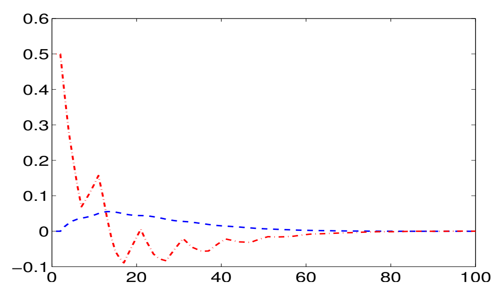

Fig. 2. State trajectories of cart-pendulum.

Let the initial states $x(0) = [0, 0.5]$ and $\alpha = 0.6$. According to the above attention rate and average dwell time, we thus can design such a periodic switching rule: Let networked controller (19) work for the duration of 6 time steps, then open the control loop for the duration of 4 time steps, then back to closing the control loop for the duration of 6 time steps again. Under the above period switchings, the state trajectories of system (18) are shown in Fig. 2, where the network-induce delays are generated randomly.

V. Conclusions

In this paper, delay-dependent sufficient conditions for robust exponential stability of switched delay systems composed of unstable systems have been proposed by employing parameter-dependent piecewise Lyapunov-like functional combined with free-weighting matrix and average dwell time methods. These sufficient conditions have been applied in the study of stability analysis for network-based reliable control system with controller failure, which have shown a control system's stability can be guaranteed if its control loop is opened with sufficiently low frequency and small duration.

Future works will be devoted to design state-feedback for network-based reliable control system, and take the effect of packet-dropout into account in the communication network.

Appendix A: Proof of Lemma 1

Proof: Define $\Delta V_i(k) = V_i(k+1) - \lambda_i V_i(k)$. From (9), it follows that

$$\Delta V_{1i}(k) = x^T(k+1)P_i(\theta)x(k+1) - x^T(k)\lambda_i P_i(\theta)x(k) \tag{21}$$

In addition, we have

$$\begin{aligned}\Delta V_{2i}(k) \le\ & x^T(k)Q_i(\theta)x(k) \\ & -x^T(k-d(k))\lambda_i^{d(k)}Q_i(\theta)x(k-d(k)) \\ & + \sum_{l=k+1-d_2}^{k-d_1} x^T(l)\lambda_i^{k-l}Q_i(\theta)x(l) \end{aligned} \tag{22}$$

$$\begin{aligned}\Delta V_{3i}(k) =\ & x^T(k)R_i(\theta)x(k) \\ & -x^T(k-d_2)\lambda_i^{d_2}R_i(\theta)x(k-d_2)\end{aligned} \tag{23}$$

$$\begin{aligned}\Delta V_{4i}(k) =\ & (d_2-d_1)x^T(k)Q_i(\theta)x(k) \\ & - \sum_{l=k+1-d_2}^{k-d_1} x^T(l)\lambda_i^{k-l}Q_i(\theta)x(l)\end{aligned} \tag{24}$$

$$\begin{aligned}\Delta V_{5i}(k) =\ & d_2 y^T(k)(Z_{1i}(\theta)+Z_{2i}(\theta))y(k) - \\ & \sum_{l=k-d_2}^{k-1} y^T(l)\lambda_i^{k-l}(Z_{1i}(\theta)+Z_{2i}(\theta))y(l) \\ =\ & d_2 x^T(k)(Z_{1i}(\theta)+Z_{2i}(\theta))x(k) \\ & +d_2 x^T(k+1)(Z_{1i}(\theta)+Z_{2i}(\theta))x(k+1) \\ & -2d_2 x^T(k)(Z_{1i}(\theta)+Z_{2i}(\theta))x(k+1) \\ & - \sum_{l=k-d(k)}^{k-1} y^T(l)\lambda_i^{k-l}Z_{1i}(\theta)y(l) \\ & - \sum_{l=k-d_2}^{k-d(k)-1} y^T(l)\lambda_i^{k-l}Z_{1i}(\theta)y(l) \\ & - \sum_{l=k-d_2}^{k-1} y^T(l)\lambda_i^{k-l}Z_{2i}(\theta)y(l)\end{aligned} \tag{25}$$

Here, the improved free-weighting matrix technique [26] will be employed. Note that $y(l) = x(l+1) - x(l)$ provides

$$\begin{aligned}\upsilon_i(k) :=\ & 2\eta^T(k)V_i(\theta)[x(k) - x(k-d(k)) \\ & - \sum_{l=k-d(k)}^{k-1} y(l)] = 0, \\ \varpi_i(k) :=\ & 2\eta^T(k)W_i(\theta)[x(k-d(k)) - x(k-d_2) \\ & - \sum_{l=k-d_2}^{k-d(k)-1} y(l)] = 0, \\ \varphi_i(k) :=\ & 2\eta^T(k)S_i(\theta)[x(k) - x(k-d_2) \\ & - \sum_{l=k-d_2}^{k-1} y(l)] = 0,\end{aligned} \tag{26}$$

for appropriate dimension matrices $V_i(\theta) = \sum_{q=1}^m \theta_q V_{iq}$, $W_i(\theta) = \sum_{q=1}^m \theta_q W_{iq}$, $S_i(\theta) = \sum_{q=1}^m \theta_q S_{iq}$, where $\eta(k) = [x^T(k)\ \ x^T(k+1)\ \ x^T(k-d(k))\ \ x^T(k-d_2)]^T$. On the other hand, in order to decouple the product entries of matrices $\bar{A}_i(\theta(k))$ with $P_i(\theta(k))$, the following equation is introduced:

$$\begin{aligned}\mu_i(k) :=\ & 2\varsigma^T(k)T_i[-x(k+1) + \bar{A}_i(\theta)x(k) \\ & +\bar{A}_{di}(\theta)x(k-d(k))] = 0,\end{aligned} \tag{27}$$

according to the state equation in (8) with appropriately dimensioned matrices T_i, where $\varsigma(k) = [x^T(k)\ x^T(k+1)]^T$. From (21)-(27), it follows that

$$\begin{aligned}\Delta V_i(k) =\ & \Delta V_{1i}(k) + \Delta V_{2i}(k) + \Delta V_{3i}(k) + \Delta V_{4i}(k) \\ & +\Delta V_{5i}(k) + \upsilon_i(k) + \varpi_i(k) + \varphi_i(k) + \mu_i(k) \\ \le\ & \eta^T(k)[\Phi_i(\theta) + c_{1i}V_i(\theta)Z_{1i}^{-1}(\theta)V_i^T(\theta) \\ & +c_{2i}W_i(\theta)Z_{1i}^{-1}(\theta)W_i^T(\theta) \\ & +c_{1i}S_i(\theta)Z_{2i}^{-1}(\theta)S_i^T(\theta)]\eta(k) \\ & - \sum_{l=k-d(k)}^{k-1}[\eta^T(k)V_i(\theta) + \lambda_i^{k-l}y^T(l)Z_{1i}(\theta)] \\ & \times[\lambda_i^{k-l}Z_{1i}(\theta)]^{-1}[V_i^T(\theta)\eta(k) + \lambda_i^{k-l}Z_{1i}(\theta)y(l)] \\ & - \sum_{l=k-d_2}^{k-d(k)-1}[\eta^T(k)W_i(\theta) + \lambda_i^{k-l}y^T(l)Z_{1i}(\theta)] \\ & \times[\lambda_i^{k-l}Z_{1i}(\theta)]^{-1}[W_i^T(\theta)\eta(k) + \lambda_i^{k-l}Z_{1i}(\theta)y(l)] \\ & - \sum_{l=k-d_2}^{k-1}[\eta^T(k)S_i(\theta) + \lambda_i^{k-l}y^T(l)Z_{2i}(\theta)] \\ & \times[\lambda_i^{k-l}Z_{2i}(\theta)]^{-1}[S_i^T(\theta)\eta(k) + \lambda_i^{k-l}Z_{2i}(\theta)y(l)]\end{aligned} \tag{28}$$

where $\Phi_i(\theta) = \sum_{q=1}^m \theta_q \Phi_{iq}$ and Φ_{iq} is defined in Lemma 1. On the other hand, multiplying inequality (11) by θ_q, and then summing them, and using Schur complement formula [19], it follows

$$\begin{aligned}& c_{1i}V_i(\theta)Z_{1i}^{-1}(\theta)V_i^T(\theta) + c_{2i}W_i(\theta)Z_{1i}^{-1}(\theta)W_i^T(\theta) \\ & +\widetilde{\Phi}_i(\theta) + c_{1i}S_i(\theta)Z_{2i}^{-1}(\theta)S_i^T(\theta) < 0\end{aligned} \tag{29}$$

It can be seen that the last three parts in (28) are all less than 0, since $Z_{1i}(\theta) > 0$ and $Z_{2i}(\theta) > 0$. Thus, we can conclude that $\Delta V_i(k) \le 0$, which means that $V_i(k+1) \le \lambda_i V_i(k)$. This completes the proof. ∎

Appendix B: Proof of Theorem 1

Proof: Without loss of generality, we assume that the controller works during $[k_{2j}\ \ k_{2j+1})$, and the plant is open-loop during $[k_{2j+1}\ \ k_{2j+2})$, $j = 0, 1, \cdots$, where $k_0 = 0$. Choose the following piecewise Lyapunov-like function candidate as

$$V(k) = \begin{cases} V_1(k), & \text{if closed-loop} \\ V_2(k), & \text{if open-loop.}\end{cases} \tag{30}$$

For any $k > 0$, it holds from (13) that

$$V(k) \le \begin{cases} \lambda_1^{k-k_{2j}}V_1(k_{2j}), & \text{if } k_{2j} \le k < k_{2j+1} \\ \lambda_2^{k-k_{2j+1}}V_2(k_{2j+1}), & \text{if } k_{2j+1} \le k < k_{2j+2}\end{cases} \tag{31}$$

where $0 < \lambda_1 < 1$ and $\lambda_2 > \lambda_1$. Considering functional (9) and inequalities (16), it is easy to verify that

$$\begin{aligned} V_1(k) &\le \mu V_2(k) \\ V_2(k) &\le \mu d_\lambda V_1(k)\end{aligned} \tag{32}$$

where $d_\lambda = (\frac{\lambda_2}{\lambda_1})^{d_2-1}$. Therefore, if $k \in [k_{2j+1}\ k_{2j+2})$, it follows from inequalities (31) and (32), and Definition 2 that

$$\begin{aligned} V(k) &\le \lambda_2^{k-k_{2j+1}} V_2(k_{2j+1}) \\ &\le \mu d_\lambda \lambda_2^{k-k_{2j+1}} V_1(k_{2j+1}^-) \\ &\le \mu^2 d_\lambda \lambda_2^{k-k_{2j+1}} \lambda_1^{k_{2j+1}-k_{2j}} V_2(k_{2j}^-) \\ &\le \mu^3 d_\lambda^2 \lambda_2^{k-k_{2j+1}+k_{2j}-k_{2j-1}} \lambda_1^{k_{2j+1}-k_{2j}} V_1(k_{2j-1}^-) \\ &\le \cdots \\ &\le \mu^{2j+1} d_\lambda^{j+1} \lambda_1^{\alpha_c(k)} \lambda_2^{(k-\alpha_c(k))} V_1(0) \end{aligned} \tag{33}$$

where $\alpha_c(k)$ is defined in Definition 2, and k_{2j}^- denotes the time instant that is immediately before k_{2j}.

Similarly, we have

$$V(k) \le \mu^{2j} d_\lambda^j \lambda_1^{\alpha_c(k)} \lambda_2^{(k-\alpha_c(k))} V_1(0) \tag{34}$$

for $k \in [k_{2j}\ k_{2j+1})$. According to the Definition 2, it is clear that $N(k) = j+1$ for $k \in [k_{2j+1}\ k_{2j+2})$ and $N(k) = j$ for $k \in [k_{2j}\ k_{2j+1})$. Therefore, it follows from inequalities (33) and (34) that

$$V(k) \le \mu^{2N(k)} d_\lambda^{N(k)} \lambda_1^{\alpha_c(k)} \lambda_2^{(k-\alpha_c(k))} V(0). \tag{35}$$

From (14), we have

$$(\ln\lambda_1 - \ln\lambda_2)\alpha_c(k) \ge (\ln\lambda_1 - \ln\lambda^*)k$$

which is equivalent to

$$\lambda_1^{\alpha_c(k)} \lambda_2^{k-\alpha_c(k)} \le (\lambda^*)^k. \tag{36}$$

From (15), we have

$$\begin{aligned} \mu^{2N(k)} d_\lambda^{N(k)} &= e^{N(k)(2\ln\mu + \ln\lambda_d)} \\ &\le e^{N_0(2\ln\mu+\ln\lambda_d)} e^{(k/T_a)(2\ln\mu+\ln\lambda_d)} \\ &\le c e^{k(2\ln\rho + \ln\lambda^*)} = c(\frac{\rho^2}{\lambda^*})^k. \end{aligned} \tag{37}$$

Combining (35), (36) and (37) yields

$$V(k) \le c\rho^{2k} V(0). \tag{38}$$

From the piecewise Lyapunov-like function (30) and inequalities (38), it follows that

$$a\|x(k)\|^2 \le V(k), \quad V(0) \le b\|x(0)\|_\delta^2$$

which leads to inequality (17) by (38). It means that system (4) is robustly exponentially stable with decay rate $0 < \rho < 1$ according to Definition 1. This completes the proof. ■

References

[1] D. Liberzon, *Switching in Systems and Control*. MA: Birkhäuser: Boston, 2003.

[2] Z. Sun and S. S. Ge, "Analysis and synthesis of switched linear control systems," *Automatica*, vol. 41, no. 2, pp. 181–195, 2005.

[3] ——, *Switched Linear Systems–Control and Design*. New York: Springer-Verlag, 2005.

[4] H. Lin and P. J. Antsaklis, "Stability and stabilizability of switched linear systems: A survey of recent results," *IEEE Transactions on Automatic Control*, vol. 54, no. 2, pp. 308–322, 2009.

[5] X.-M. Sun, J. Zhao, and D. J. Hill, "Stability and L_2-gain analysis for switched delay systems: A delay-dependent method," *Automatica*, vol. 42, no. 10, pp. 1769–1774, 2006.

[6] Y. G. Sun, L. Wang, and G. Xie, "Delay-dependent robust stability and H_∞ control for uncertain discrete-time switched systems with mode-dependent time delays," *Applied Mathematics and Computation*, vol. 187, pp. 1228–1237, 2007.

[7] K. Gu, V. L. Kharitonov, and J. Chen, *Stability of Time-Telay Systems*. MA: Birkhäuser: Boston, 2003.

[8] D. Du, B. Jiang, P. Shi, and S. Zhou, "H_∞ filtering of discrete-time switched systems with state delays via switched Lyapunov function approach," *IEEE Transactions on Automatic Control*, vol. 52, no. 8, pp. 1520–1525, 2007.

[9] L. Zhang, P. Shi, and E. H. Boukas, "H_∞ output-feedback control for switched linear discrete-timesystems with time-varying delays," *International Journal of Control*, vol. 80, no. 8, pp. 1354–1365, 2006.

[10] R. Wang and J. Zhao, "Exponential stability analysis for discrete-time swithced linear systems with time-delay," *International Journal of Innovative Computing, Information and Control*, vol. 3, no. 6, pp. 1557–1564, 2007.

[11] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in *Proceedings of the 38th IEEE Conference on Decision and Control*, Phoenix, Arizona USA, 1999, pp. 2655–2660.

[12] Z. Sun, "Combined stabilizing strategies for switched linear systems," *IEEE Transactions on Automatic Control*, vol. 51, no. 4, pp. 666–674, 2006.

[13] G. Zhai, B. Hou, K. Yasuda, and A. M. Michel, "Stability analysis of switched systems with stable and unstable subsystems: An average dwell time approach," in *Proceedings of the 2000 American Control Conference*, Chicago, Illinois, 2000, pp. 200–204.

[14] X.-M. Sun, J. Zhao, and W. Wang, "State feedback control for discrete delay systems with controller failures based on average dwell-time method," *IET Control Theory and Applications*, vol. 2, no. 2, pp. 126–132, 2008.

[15] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 138–162, 2007.

[16] L. Hetel, J. Daafouz, and C. Iung, "Stabilization of arbitrary switched linear systems with unknown time-varying delays," *IEEE Transactions on Automatic Control*, vol. 51, no. 10, pp. 1668–1674, 2006.

[17] H. Lin and P. J. Antsaklis, "Robust regulation of polytopic uncertain linear hybrid systems with networked control system applications," in *Stability and Control of Dynamical Systems with Applications*, Birkhauser: Basel, 2003, pp. 83–108.

[18] H.-N. Wu and H.-Y. Zhang, "Reliable H_∞ fuzzy control for continuous-time nonlinear systems with actuator failures," *IEEE Transactions on Fuzzy Systems*, vol. 14, no. 5, pp. 609–618, 2006.

[19] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia: SIAM, 1994.

[20] G. Zhai, H. Lin, and P. J. Antsaklis, "Quadratic stabilizability of switched linear systems with polytopic uncertainties," *International Journal of Control*, vol. 76, no. 7, pp. 747–753, 2003.

[21] P. Gahinet, P. Apkarian, and M. Chilali, "Affine parameter-dependent Lyapunov functions and real parametric uncertainty," *IEEE Transactions on Automatic Control*, vol. 41, no. 3, pp. 436–442, 1996.

[22] G. Zhai and H. Lin, "Controller failure time analysis for symmetric H_∞ control systems," *International Journal of Control*, vol. 77, no. 6, pp. 598–605, 2004.

[23] X.-M. Sun, G.-P. Liu, D. Rees, and W. Wang, "Stability of systems with controller failure and time-varying delay," *IEEE Transactions on Automatic Control*, vol. 53, no. 10, pp. 2391–2396, 2008.

[24] D. Hristu-Varsakelis, "Feedback control systems as users of a shared network: communication sequences that guarantee stability," in *Proceedings of the 40th IEEE Conference on Decision and Control*, Orlando, Florida USA, 2001, pp. 3631–3636.

[25] H. Lin, G. Zhai, L. Fang, and P. J. Antsaklis, "Stability and H_∞ performace preserving scheduling policy for networked control systems," in *Proceedings of the 16th IFAC World Congress*, Prague, Czech Republic, 2005.

[26] H. Gao and T. Chen, "New results on stability of discrete-time systems with time-varying state delay," *IEEE Transactions on Automatic Control*, vol. 52, no. 2, pp. 328–334, 2007.

[27] G.-H. Yang, J. L. Wang, and Y. C. Soh, "Reliable H_∞ controller design for linear system," *Automatica*, vol. 37, no. 3, pp. 717–725, 2001.