

The EHP sequence and the Goodwillie tower

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problem: $\pi_k(S^n)$

problem:

$$\pi_* (S^n)$$



EHP Sequence

Goodwillie tower

problem: $\pi_* (S^n)$

EHP Sequence

Goodwillie tower

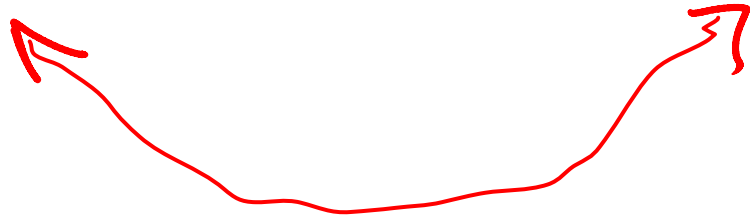


interrelated in a
complex but beautiful way

problem: $\pi_{k*}(S^n)$

EHP Sequence

Goodwillie tower



interrelated in a
complex but beautiful way

Today: everything is localized at $p=2$

Goodwillie Tower

Goodwillie Tower

$$F: \text{Top}_* \longrightarrow \text{Top}_*$$

Preserves

- w.e.
- filtered localim
- $F(a) = \rightarrow$

Goodwillie Tower

$$F: \text{Top}_* \longrightarrow \text{Top}_*$$

$$\begin{array}{c} \downarrow \\ P_3(F) \\ \downarrow \\ P_2(F) \\ \downarrow \\ P_1(F) \end{array}$$

F \nearrow $P_3(F)$
 F \nearrow $P_2(F)$
 F \longrightarrow $P_1(F)$

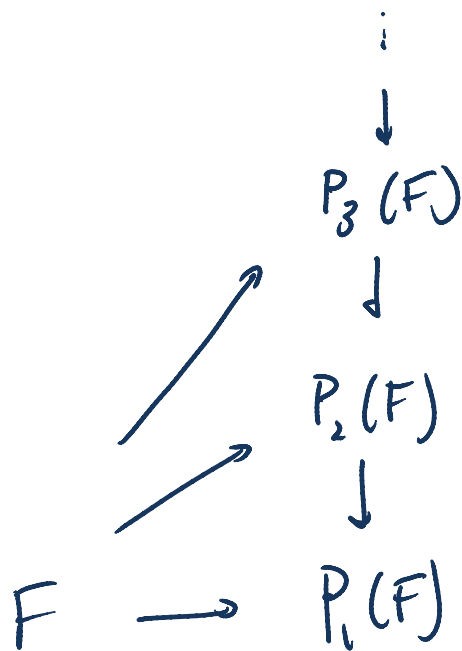
$$P_i(F) = i\text{-excisive}$$

Preserves

- w.e.
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Goodwillie Tower

$$F: \text{Top}_* \longrightarrow \text{Top}_*$$



$$P_i(F) = i\text{-excisive}$$

Preserves

- w.e.
- filtered hocolim
- $F(x) \Rightarrow$

$$F(x) \xrightarrow{\cong} \varprojlim P_i(F)(x)$$

if X is
sufficiently highly
connected

Goodwillie Tower

$$F : \text{Top}_* \longrightarrow \text{Top}_*$$

$$\begin{array}{c} \downarrow \\ P_3(F) \longleftarrow D_3(F) \\ \downarrow \\ P_2(F) \longleftarrow D_2(F) \\ \downarrow \\ P_1(F) \equiv D_1(F) \end{array}$$

$$P_i(F) = i\text{-excisive}$$

$$D_i(F)(X) = \Omega^{\heartsuit} D_i(F)(X)$$

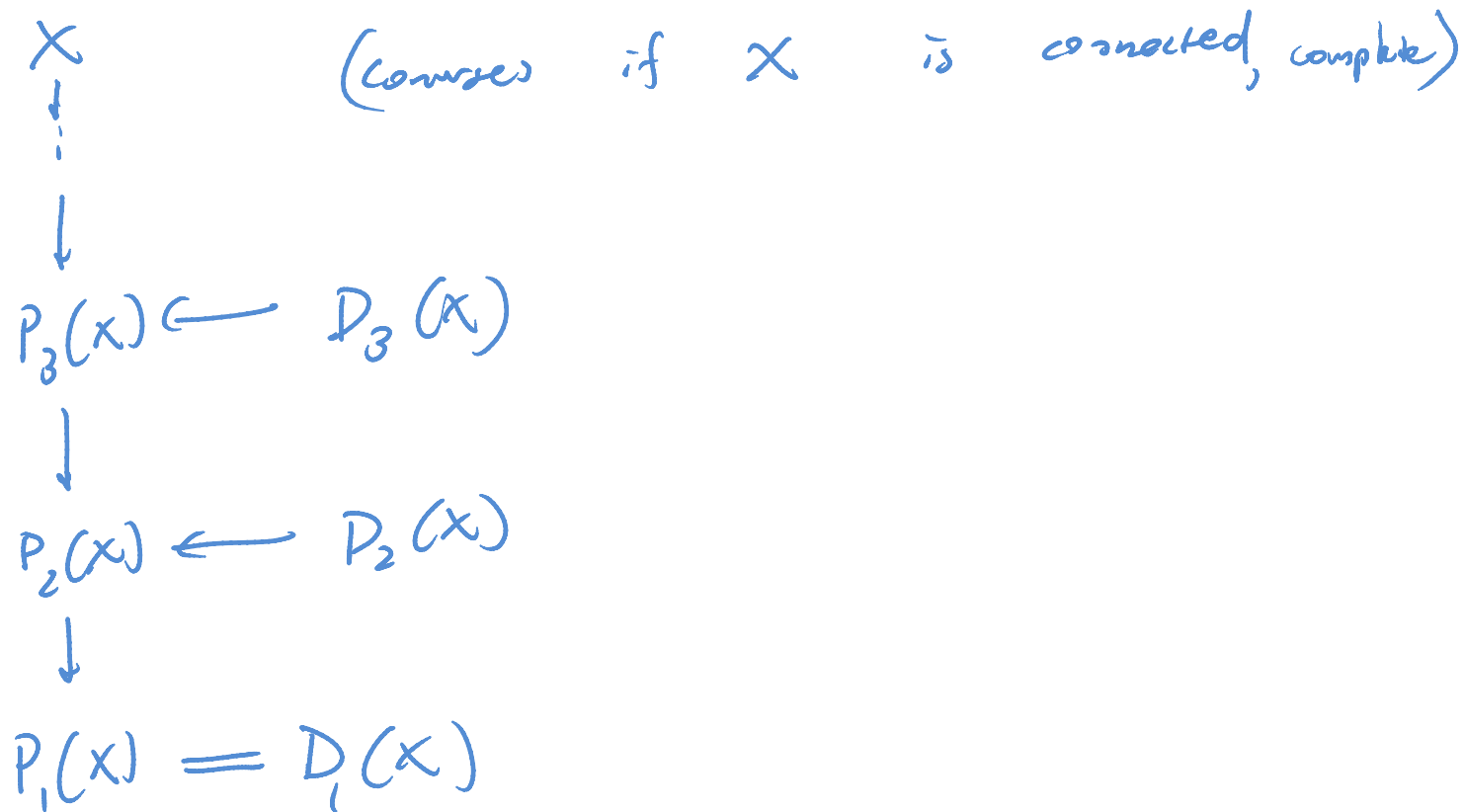
$$D_i F : \text{Top}_* \longrightarrow \text{Sp}$$

homogeneous deg i

Goodwillie tower of Id

$$\text{Id}: \text{Top}_* \longrightarrow \text{Top}_*$$

$$P_i(\text{Id})(X) =: P_i(X)$$



Goodwillie tower of Id

$$\text{Id}: \text{Top}_* \rightarrow \text{Top}_*$$

$$P_i(\text{Id})(X) =: P_i(X)$$

$$\begin{array}{c} X \\ \vdots \\ \downarrow \\ P_3(X) \leftarrow D_3(X) \\ \downarrow \\ P_2(X) \leftarrow P_2(X) \\ \downarrow \\ P_1(X) = D_1(X) \end{array}$$

(converse) if X is connected, complete)

Get a spectral sequence (SS)

$$E_1 = \pi_t D_i(X) \Rightarrow \pi_t X$$

[compute unstable homotopy groups from stable hty] sfs

GSS for S^n

Arone-Mahowald, Arone-Dwyer:

Thm:

$$\mathbb{D}_i(S^n) \cong \begin{cases} *, & i \neq 2^k \\ \sum^{n-k} L(k)_n, & i = 2^k \end{cases}$$

$$L(0)_n = \underline{S}, \quad L(1)_n = P_n^\infty = \sum_{i=0}^{\infty} \mathbb{R}P^i / \mathbb{R}P^{n-1}$$

GSS for S^n

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$L(k)_n$ are "well-known spectra"

studied by Kuhn, Mitchell, Priddy, ...

$$L(k)_n = e_{st} \sum^{\infty} (B\mathbb{F}_2^k)^{n\bar{p}}$$

GSS for S^n

The GSS takes the form:

$$\pi_t L(k)_n \implies \pi_{n+t-k}(S^n)$$

GSS for S^n

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Problems:

- (1) compute $\pi_s L(k)_n$ [stable hyp]
- (2) compute diff'ls [Mysterious]

EHP Sequence

Consider the sequence of functors

$$\text{Id} \xrightarrow[E]{} \Omega\Sigma \xrightarrow[H]{} \Omega\Sigma S_g$$

$$S_g(x) = X \wedge X$$

$$\sum \Omega\Sigma X \cong \sum_{n \geq 1} \bigvee X^{\wedge n} \xrightarrow{\quad} \sum X^{\wedge 2}$$

$\underbrace{\hspace{15em}}_{\sim H}$

EHP sequence

Consider the sequence of functors

$$\text{Id} \xrightarrow[E]{} \Omega \Sigma \xrightarrow[H]{} \Omega \Sigma S_q$$

$$\Omega^2 S^{2m+1} \xrightarrow[\downarrow]{} S^m \xrightarrow[E]{} \Omega S^{m+1} \xrightarrow[H]{} \Omega S^{2m+1}$$

Fiber sequence

EHP Sequence

Consider the sequence of functors

$$\text{Id} \xrightarrow[E]{} \Omega\Sigma \xrightarrow[H]{} \Omega\Sigma S_q$$

$$\Omega^{m+1} S^{2m+1} \xrightarrow[\downarrow P]{} \Omega^m S^m \xrightarrow[E]{} \Omega^{m+1} S^{m+1} \xrightarrow[H]{} \Omega^{m+1} S^{2m+1}$$

EHP Sequence

Consider the sequence of functors

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Writing: $\Omega S^1 \rightarrow \Omega^2 S^2 \rightarrow \dots \rightarrow \Omega S^0$

EHPSS $\bigoplus_{0 \leq m} \pi_{t+m+1} S^{2m+1} \Rightarrow \pi_{t} \underline{S}$

EHP Sequence

Consider the sequence of functors

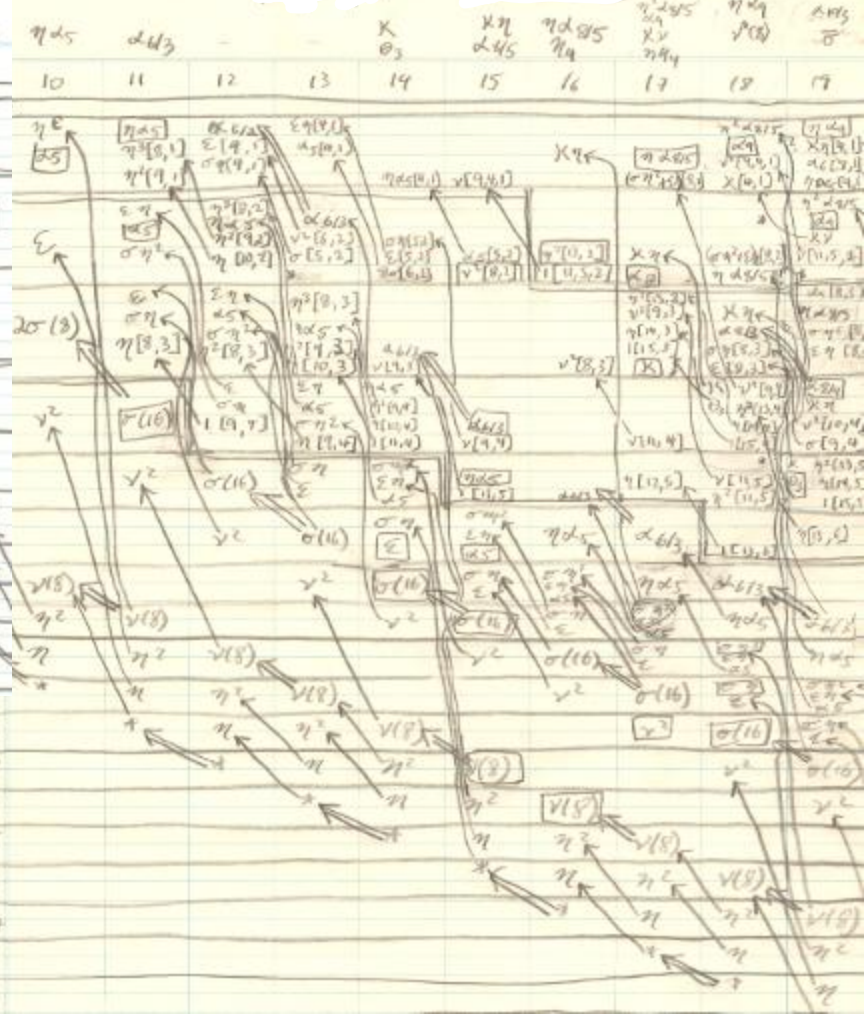
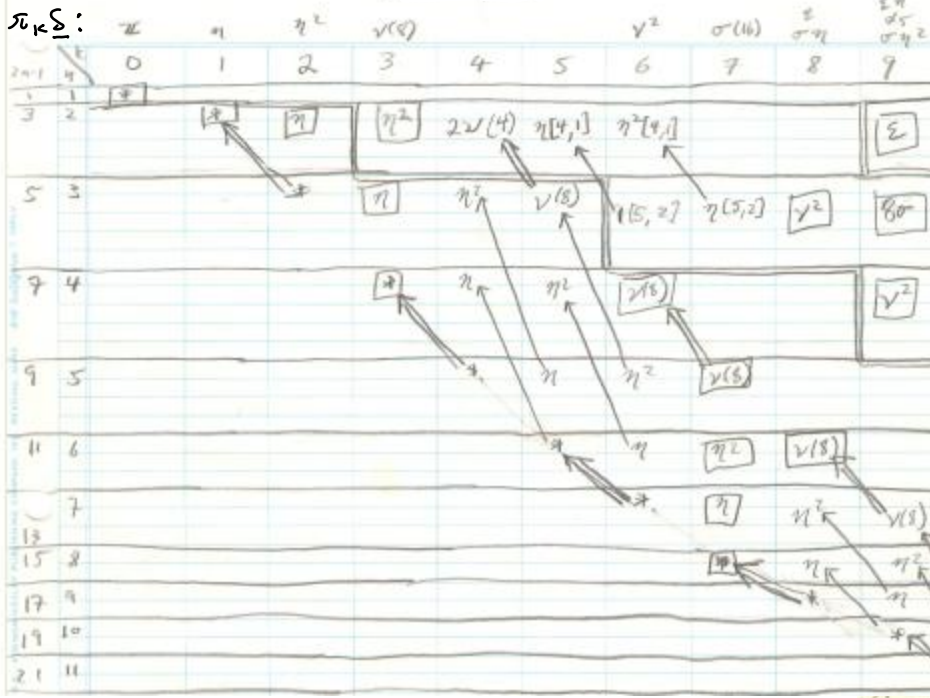
$$\text{Id} \xrightarrow{E} \Omega\Sigma \xrightarrow{H} \Omega\Sigma S_q$$

$$\Omega^{m+2} S^{2m+1} \xrightarrow{P} \Omega^m S^m \xrightarrow{E} \Omega^{m+1} S^{m+1} \xrightarrow{H} \Omega^{m+1} S^{2m+1}$$

Writing: $\Omega S^1 \rightarrow \Omega^2 S^2 \rightarrow \dots \rightarrow \Omega^n S^n \rightarrow QS^0$

EHPSS $\bigoplus_{0 \leq m} \pi_{t+m+1} S^{2m+1} \Rightarrow \pi_{t+} \underline{S}$

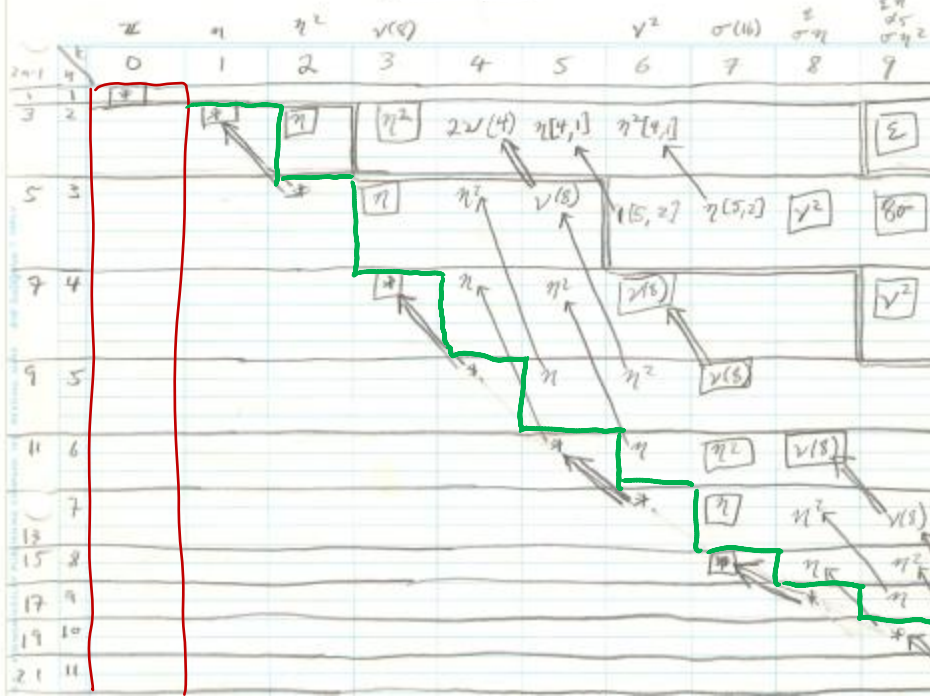
$$\bigoplus_{0 \leq m < n} \pi_{t+m+1} S^{2m+1} \Rightarrow \pi_{t+n} S^n \quad 1 \leq m < n$$



FHPSS

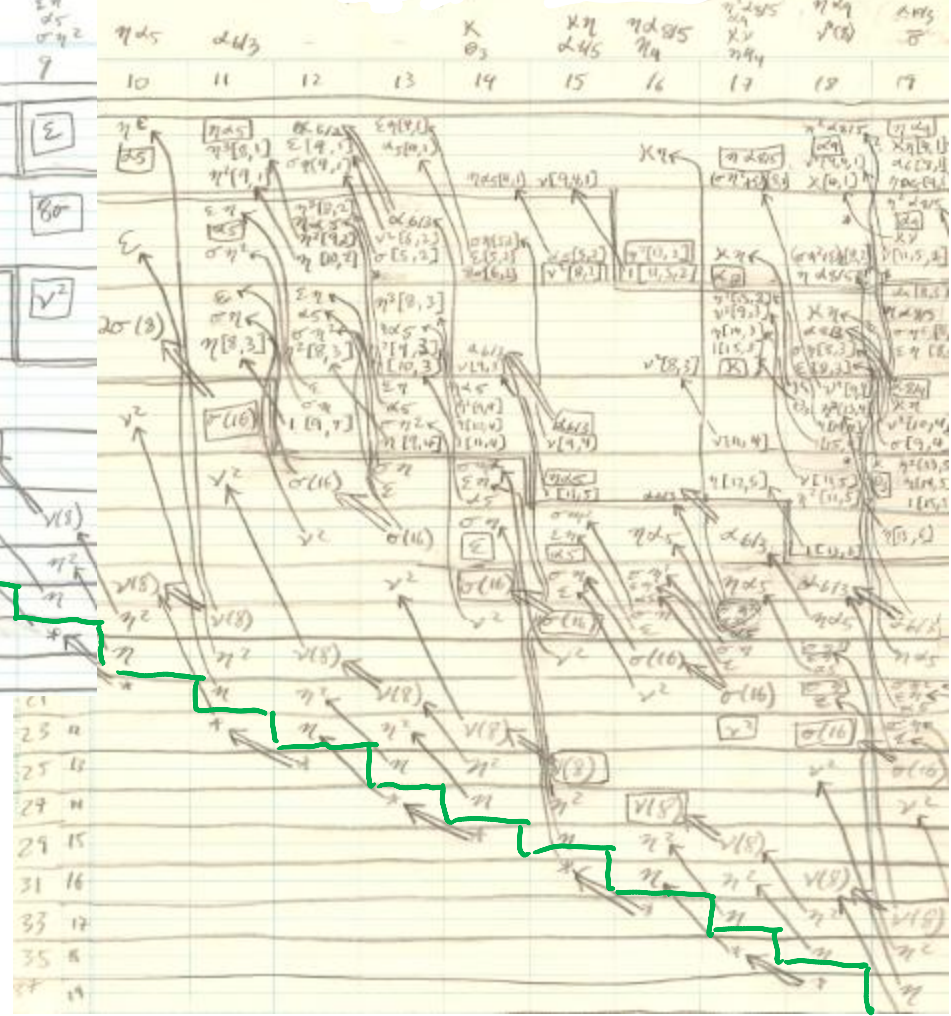
$$\pi_{\star} S^{2m+1} \Rightarrow \pi_{\star} \underline{S}$$

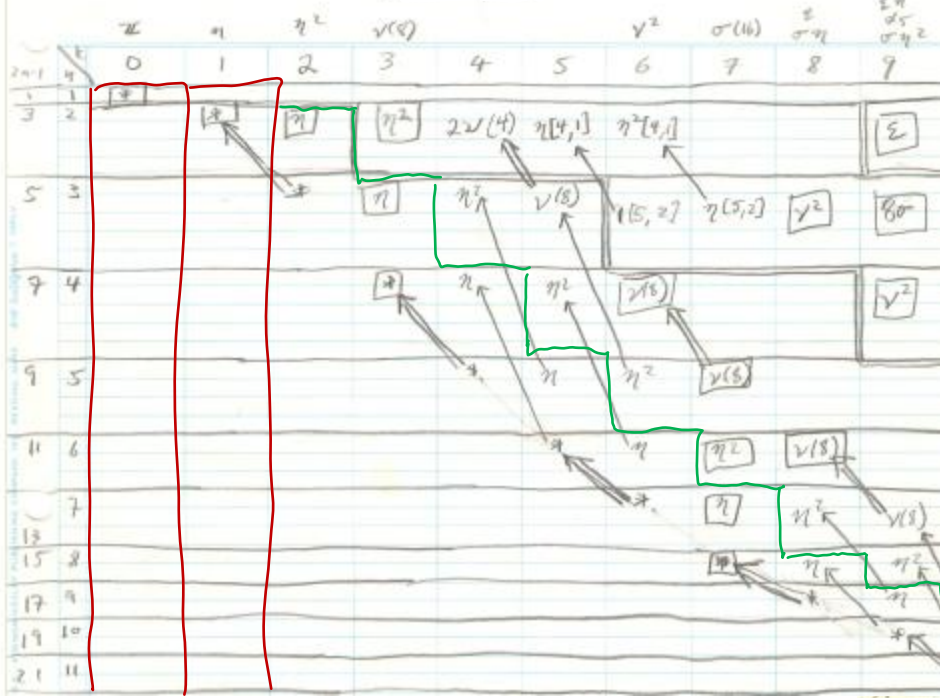
$$(* =: \mathbb{Z})$$



0-stem: $\pi_n S^n$

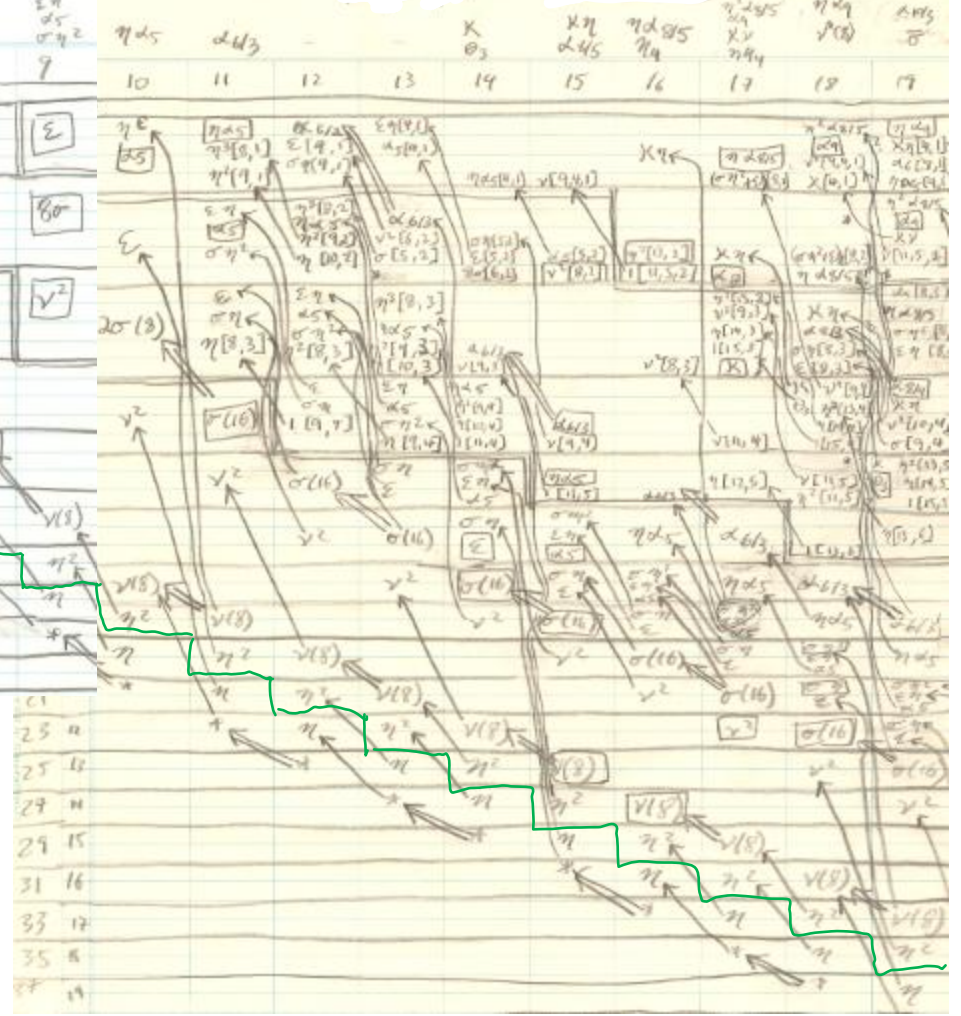
Curtis Algorithm

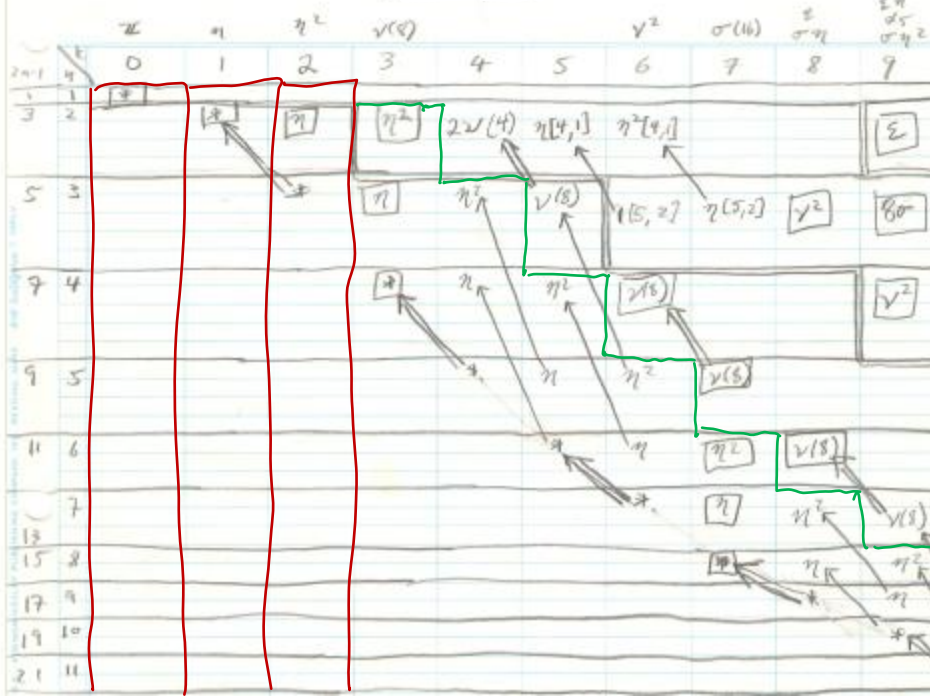




$$\pi_{n+1} S^n$$

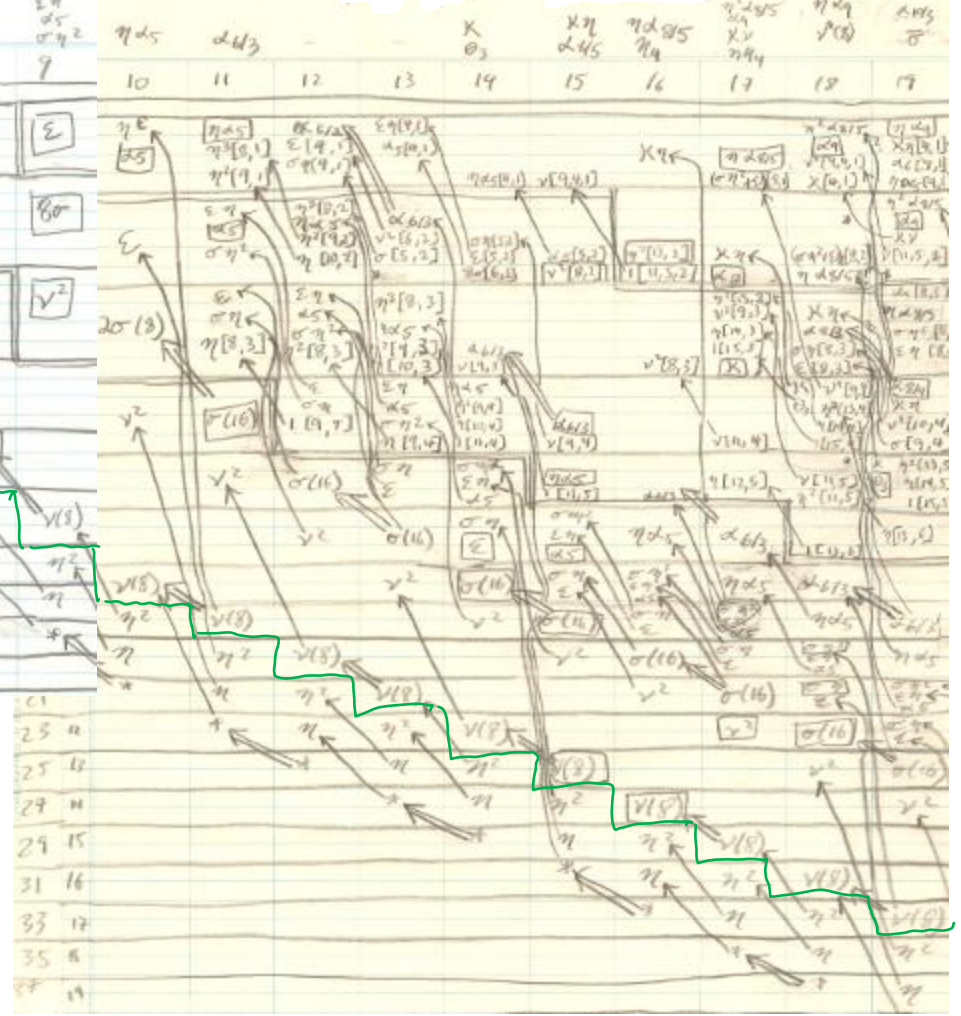
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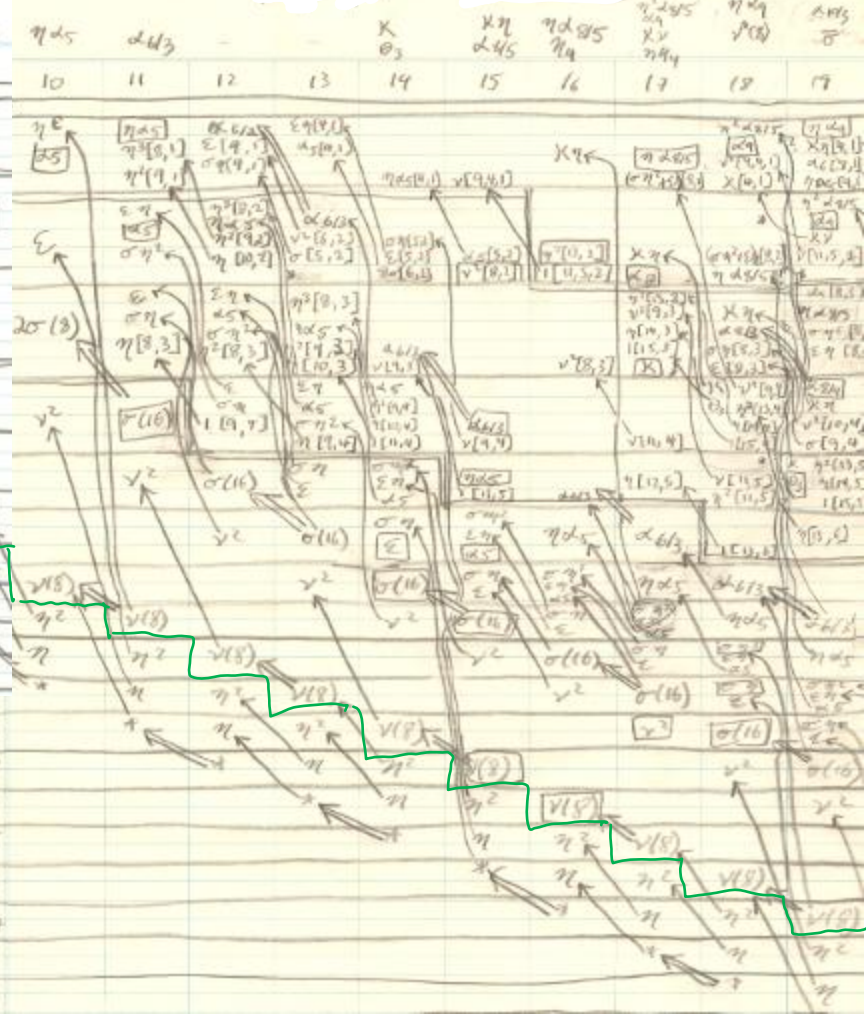
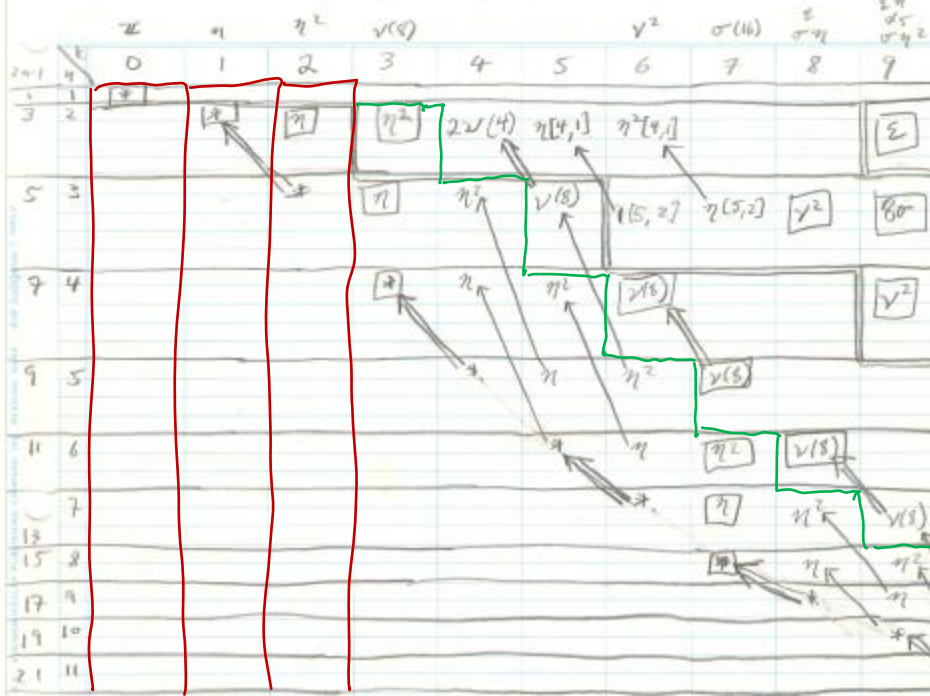




$\tau_{n+2} S^n$

Curtis Algorithm





$\tau_{intz} S^n$

Curtis Algorithm

Main problem for EMPSS:
 Compute diff'ls!

Back to $L(k)_n$

EHP sequence:
(exact on spheres)

$$\Omega^1 \Sigma^n \rightarrow \Omega^{n+1} \Sigma^{n+1} \rightarrow \Omega^{n+1} \Sigma^{n+1} S_q$$

Back to $L(k)_n$

EHP sequence:
(exact on spheres)

$$\Omega^n \Sigma^n \rightarrow \Omega^{n+1} \Sigma^{n+1} \rightarrow \Omega^{n+1} \Sigma^{n+1} S_q$$

Apply Goodwillie tower:
(exact on spheres)

$$\mathbb{D}_{2^k}(\Omega^n \Sigma^n) \rightarrow \mathbb{D}_{2^k}(\Omega^{n+1} \Sigma^{n+1}) \rightarrow \mathbb{D}_{2^k}(\Omega^{n+1} \Sigma^{n+1} S_q)$$

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$$\Sigma^{-n} \circ \mathbb{D}_{2^k} \circ \Sigma^n \quad \Sigma^{-n-1} \circ \mathbb{D}_{2^k} \circ \Sigma^{n+1} \quad \Sigma^{-n-1} \circ \mathbb{D}_{2^{k-1}} \circ \Sigma^{n+1} \circ S_q$$

$$\begin{aligned} f(x) &= \sum a_k x^{2^k} \\ \Rightarrow f(x^2) &= \sum a_{k-1} x^{2^k} \end{aligned}$$

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\downarrow \downarrow \downarrow

Cofiber sequence

$$\Sigma^{-n} \circ \mathbb{D}_{2^k} \circ \Sigma^n \quad \quad \quad \Sigma^{-n-1} \circ \mathbb{D}_{2^k} \circ \Sigma^{n+1} \quad \quad \quad \Sigma^{-n-1} \circ \mathbb{D}_{2^{k-1}} \circ \Sigma^{n+1} \circ S_q$$

$$\Sigma^n L(k-1)_{2n+1} \longrightarrow L(k)_n \longrightarrow L(k)_{n+1} \longrightarrow \Sigma^{n+1} L(k-1)_{2n+1}$$

[originally due to Kuhn and Takasaku]

Back to $L(k)_n$

EHP sequence:
(exact on spheres)

$$\Omega^n \Sigma^n \longrightarrow \Omega^{n+1} \Sigma^{n+1} \longrightarrow \Omega^{n+1} \Sigma^{n+1} S_q$$

Apply Goodwillie tower:
(exact on spheres)

$$\mathbb{D}_{2^k}(\Omega^n \Sigma^n) \longrightarrow \mathbb{D}_{2^k}(\Omega^{n+1} \Sigma^{n+1}) \longrightarrow \mathbb{D}_{2^k}(\Omega^{n+1} \Sigma^{n+1} S_q)$$

\Downarrow \Downarrow \Downarrow

Cofiber sequence

$$\Sigma^{-n} \circ \mathbb{D}_{2^k} \circ \Sigma^n \longrightarrow \Sigma^{-n-1} \circ \mathbb{D}_{2^k} \circ \Sigma^{n+1} \longrightarrow \Sigma^{-n-1} \circ \mathbb{D}_{2^{k-1}} \circ \Sigma^{n+1} \circ S_q$$

$$\Sigma^n L(k-1)_{2n+1} \longrightarrow L(k)_n \longrightarrow L(k)_{n+1} \longrightarrow \Sigma^{n+1} L(k-1)_{2n+1}$$

eg.
 $k=1$

$$S^n \longrightarrow P_n^\infty \longrightarrow P_{n+1}^\infty \longrightarrow S^{n+1}$$

Homology of $L(k)_n$

$$H_* = H_*(-; \mathbb{F}_2)$$

$$H_* L(k)_n \cong \mathbb{F}_2 \left\{ Q^{\mathbf{I}} \mid \mathbf{I} = (i_1, \dots, i_k) \text{ c.u.}, i_k \geq n \right\}$$

$$Q^{\mathbf{I}} = Q^{i_1} \cdots Q^{i_k} \quad \text{sequence of D-L operations}$$

c.u. = "completely inadmissible"

$$i_j \geq 2i_{j+1}$$

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$$i_j \geq 2i_{j+1}$$

Dual action of A : Nishida Relations

Homology of $L(k)_n$

$$H_* L(k)_n \cong \mathbb{F}_2 \{ Q^{\mathbb{I}} \mid \mathbb{I} = (i_1, \dots, i_k) \text{ c.u.}, i_k \geq n \}$$

$$0 \rightarrow H_* \Sigma^n L(k-1)_{2n+1} \rightarrow H_* L(k)_n \rightarrow H_* L(k)_{n+1} \rightarrow 0$$

$$\sigma^n Q^{i_1} \dots Q^{i_{k-1}} \mapsto Q^{i_1} \dots Q^{i_{k-1}} Q^n$$

$$Q^{i_1} \dots Q^{i_k} \mapsto Q^{i_1} \dots Q^{i_k}$$

$i_k > n$

Computing $\pi_* L(k)_n$: "AHSS's"

$$L(k)_n^m \longrightarrow L(k)_n \longrightarrow L(k)_{m+1}$$

$$\lim_{\substack{\longrightarrow \\ m}} L(k)_n^m = L(k)_n$$

Computing $\pi_* L(k)_n$: "AHSS's"

$$L(k)_n^m \longrightarrow L(k)_n \longrightarrow L(k)_{m+1}$$

$$\lim_{\substack{\longrightarrow \\ m}} L(k)_n^m = L(k)_n$$

AHSS:

$$E_1 = \bigoplus_{m \geq n} \pi_* \sum^m L(k-1)_{2m+1} \implies \pi_* L(k)_n$$

Computing $\pi_* L(k)_n$: "AHSS's"

Iterated AHSS:

$$\bigoplus \pi_* (\Sigma^{|\mathbf{I}|}) \Rightarrow \underbrace{\dots}_{k \text{ spectral sequences}} \Rightarrow \pi_* L(k)_n$$

$$\mathbf{I} = (i_1, \dots, i_k) \\ i_k \geq n$$

k spectral sequences

AHSS:

$$E_1 = \bigoplus_{m \geq n} \pi_* \Sigma^m L(k-1)_{2m+1} \Rightarrow \pi_* L(k)_n$$

Computing $\pi_* L(k)_n$: "AHSS's"

Iterated AHSS:

$$\alpha \Rightarrow \dots \Rightarrow \alpha [i_1, \dots, i_k]$$

$\underbrace{\hspace{10em}}_m$

$$\bigoplus \pi_* (\Sigma^{|\mathbf{I}|}) \Rightarrow \dots \Rightarrow \pi_* L(k)_n$$

$$\mathbf{I} = (i_1, \dots, i_k)$$

$$i_k \geq n$$



k spectral sequences

AHSS:

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Computing $\pi_* L(k)_n$: "AHSS's"

Iterated AHSS:

$$\alpha \Rightarrow \dots \Rightarrow \alpha [i_1, \dots, i_k]$$

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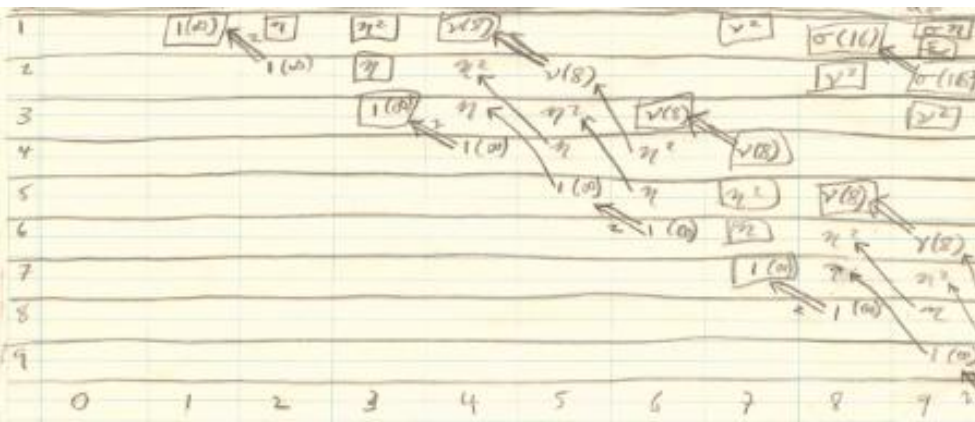
k spectral sequences

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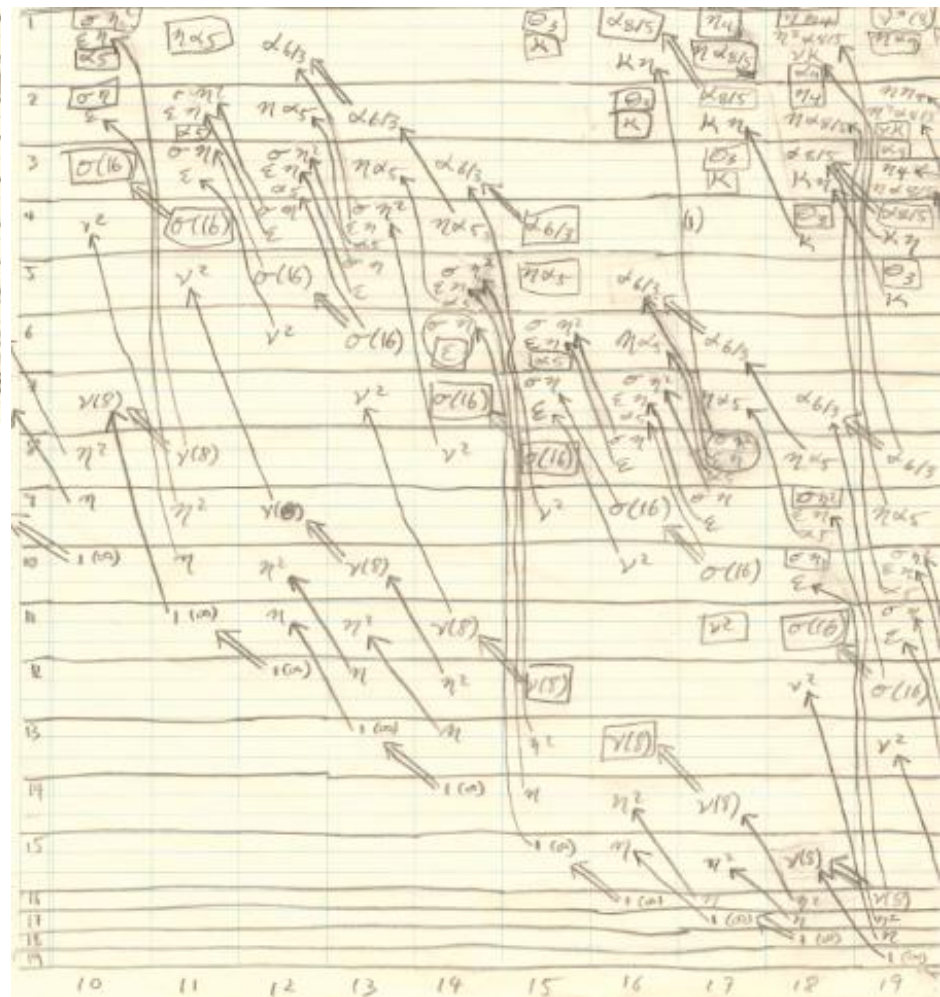
Diff's: attaching maps between "cells" Q^{i_1}, \dots, Q^{i_k} [largely determined by Steenrod action]

AMSS : $\pi_n \subseteq \Rightarrow \pi_n L(1)$,

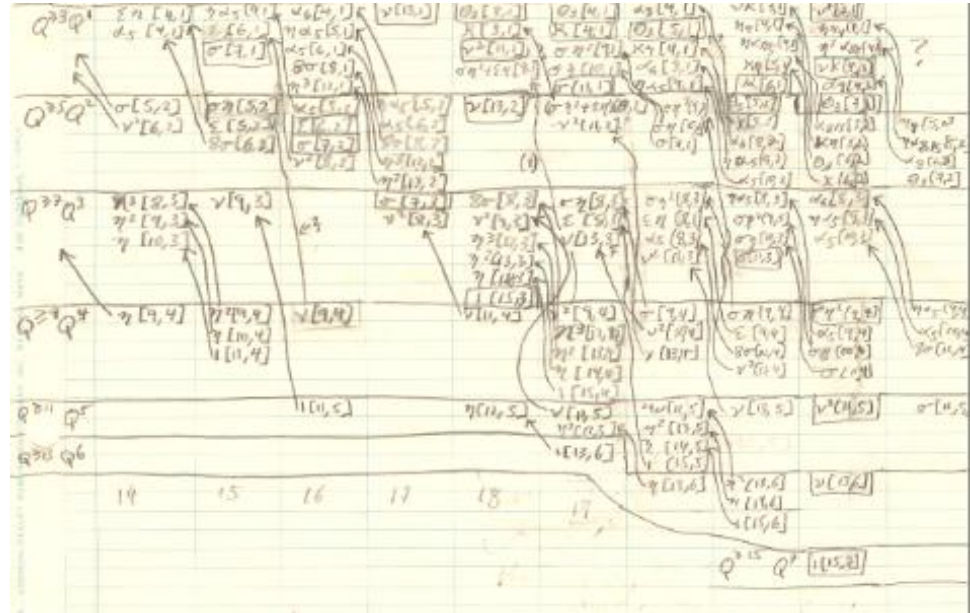
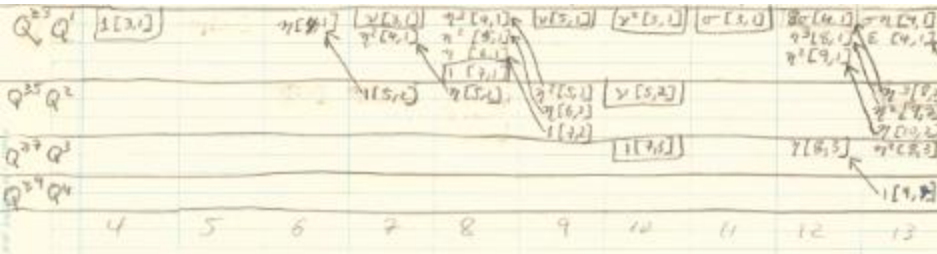


$4[1]$ $n[1]$ $n[1]$
 $n[2]$
 $1[3]$

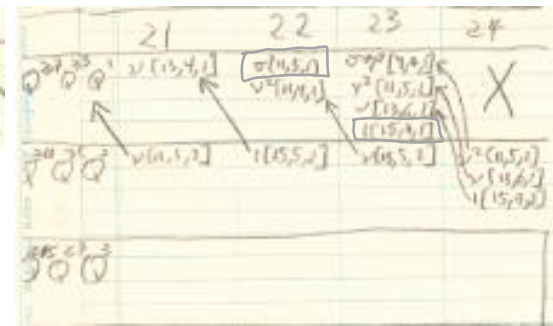
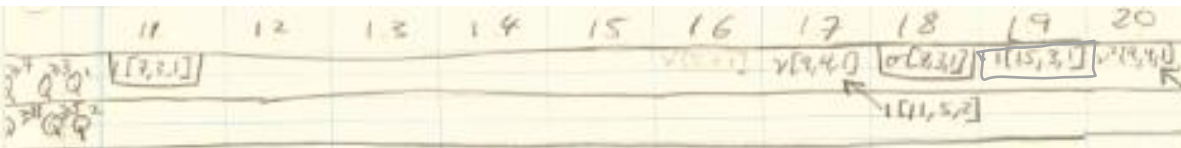
$\pi_1 P^\infty$ $\pi_2 P^\infty$ $\pi_3 P^\infty$



$$\pi_x L(1)_{2m+1} \Rightarrow \pi_b L(2)$$



$$\pi_{\star} L(2)_{2n+1} \Rightarrow \pi_{\star} L(3)$$



Summary so far:

We "understand" $E_i^{GSS} = \bigoplus_k \pi_k L(k)_n$

Need to understand diff'ls!

Summary so far:

We "understand" $E_i^{GSS} = \bigoplus_k \pi_* L(k)_n$

Need to understand diff'ls!

First: Understand meaning of "Goodwillie filtration"

[Note: Biedermann-Dwyer have a different perspective]

Understanding Goodwillie Filtration

Thm: Consider sequence of spectral sequences

$$\begin{array}{c}
 \bigoplus_k \bigoplus_{(i_1, \dots, i_k)} \pi_* (S^{|I|}) \implies \dots \implies \bigoplus_k \pi_* L(k)_n \implies \pi_* S^n \\
 \begin{array}{c} i_k \geq n \end{array} \quad \text{Iterated} \quad \text{ARSS} \quad \text{GSS}
 \end{array}$$

$$\begin{array}{c}
 \psi \\
 \alpha \implies \dots \implies \alpha [i_1, \dots, i_k] \implies \beta \\
 \psi
 \end{array}$$

Then:

$$\beta = E^{-l_k} P_{i_k} E^{-l_{k-1}} P_{i_{k-1}} \dots E^{-l_1} P_{i_1} \tilde{\alpha}$$

where $l_j = i_j - (2i_{j+1} + 1)$, $l_k = i_k - n$, $\tilde{\alpha}$ stabilizes to α

Slogan:

$$\left(\begin{array}{l} B \text{ has goodwill} \\ \text{filtration} \geq 2^k \end{array} \right) \iff \left(B \in \text{"Im } P^k \text{"} \right)$$

Sketch of proof

Define fibers:

$$R_{2^k}(S^n) \longrightarrow S^n \longrightarrow P_{2^{k-1}}(S^n)$$

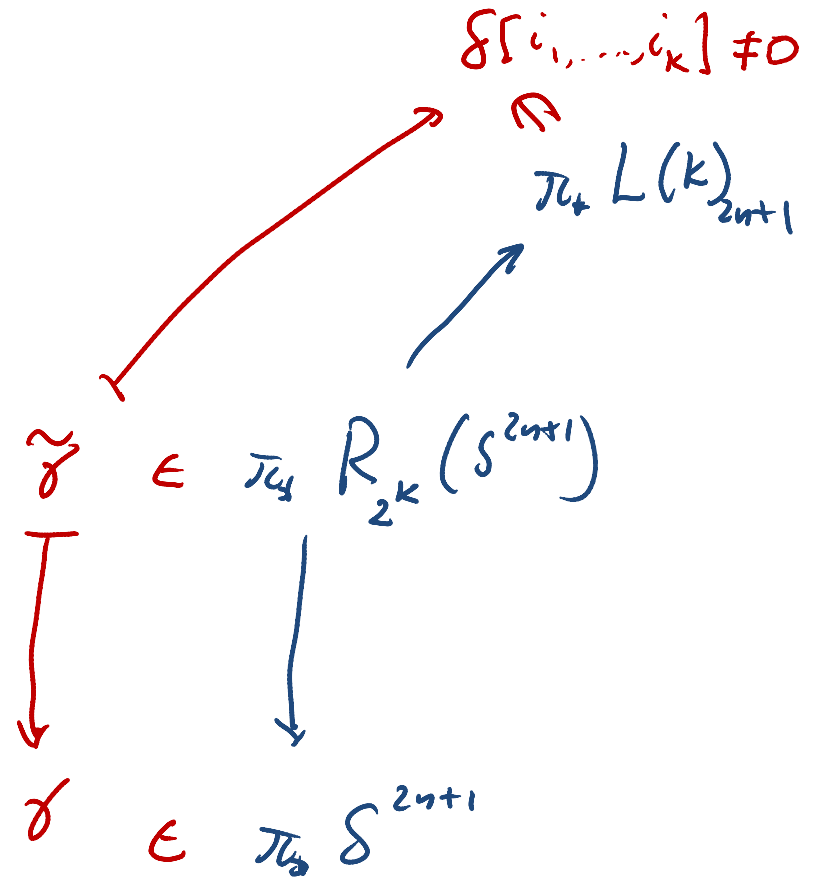
γ in Goodwillie
filtration 2^k

\Rightarrow

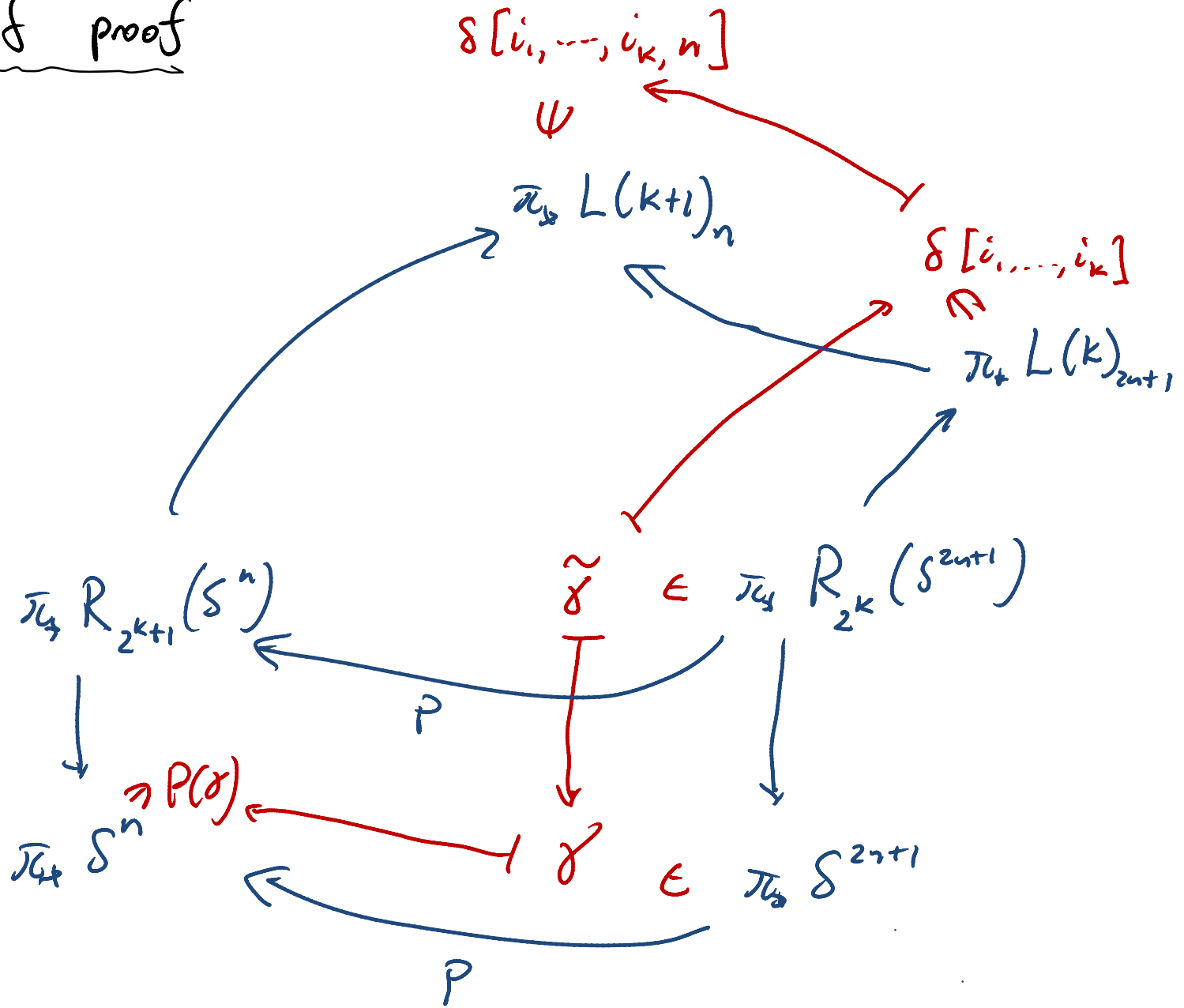
$$\begin{array}{ccc} \tilde{\gamma} & \in & \pi_{2^k} R_{2^k}(S^{2n+1}) \\ \downarrow & & \downarrow \\ \gamma & \in & \pi_{2^k} S^{2n+1} \end{array}$$

Sketch of proof

γ in Goodwillie
filtration 2^k



Sketch of proof



□

Goodwillie diff's

Hopf invariant

EMPS:

$$\bigoplus_n \pi_n \Omega^{m+1} \Sigma^{2m+1} \implies \pi_n \underline{S}$$

\Downarrow
 $\alpha \implies \beta$

$$\alpha = HI(\beta)$$

Goodwillie diff's

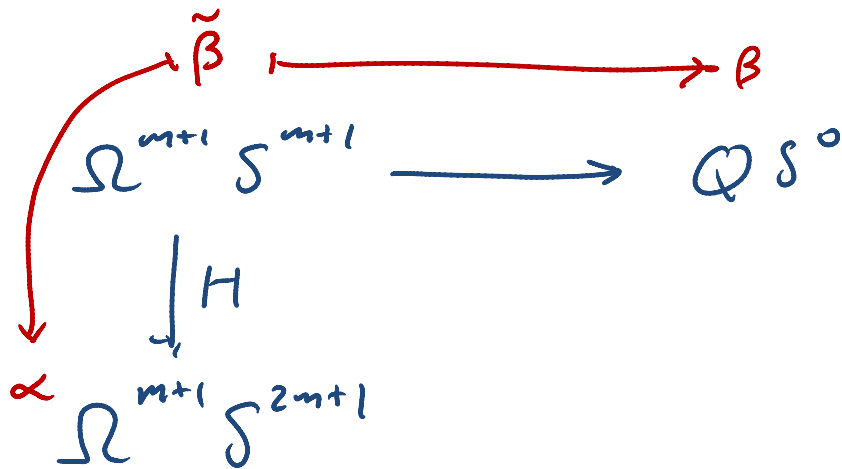
Hopf invariant

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$$\bigoplus_n \pi_n \Omega^{m+1} \Sigma^{2m+1} \implies \pi_n \underline{S}$$

$$\downarrow \alpha \qquad \implies \qquad \downarrow \beta$$

$$\alpha = HI(\beta)$$



Goodwillie diff's

$$\mathcal{Q}S^0 \xrightarrow{J-H} \mathcal{Q}\mathbb{R}P^\infty$$

adjoint to

$$\Sigma^\infty \mathcal{Q}S^0 \rightarrow \Sigma^\infty \underline{S}_{h\Sigma_2} \simeq \Sigma^\infty \mathbb{R}P^\infty$$

Goodwillie diff'ls

$$\mathcal{Q}S^0 \xrightarrow{J-H} \mathcal{Q}\mathbb{R}P^\infty$$

adjoint to

$$\Sigma^\infty \mathcal{Q}S^0 \rightarrow \Sigma^\infty \underline{S}_{h\Sigma_2} \cong \Sigma^\infty \mathbb{R}P^\infty$$

Stable Hopf invariant!

$$\begin{array}{ccc}
 \bigoplus_m \pi_n S^m & \xrightarrow{\text{AHSS}} & \pi_n P^\infty \\
 \downarrow \alpha & & \downarrow J-H \\
 & & J-H(\beta)
 \end{array}$$

$\pi_n \underline{S} \ni \beta$
 $\downarrow J-H$
 \downarrow

$$\alpha = SHI(\beta)$$

Goodwillie diff's

Fundamental diagram

$$\begin{array}{ccccc} \Omega^m S^m & \xrightarrow{E} & \Omega^{m+1} S^{m+1} & \xrightarrow{H} & \Omega^{m+1} S^{2m+1} \\ \downarrow JH & & \downarrow JH & & \downarrow E^\infty \\ \mathbb{Q} \mathbb{R}P^{m-1} & \longrightarrow & \mathbb{Q} \mathbb{R}P^m & \longrightarrow & \mathbb{Q} S^m \end{array}$$

Deduce:

$$\text{If } E^\infty HI(B) \neq 0$$

$$\Rightarrow E^\infty HI(B) = SHI(B)$$

Goodwillie Diff's

Thus:

In the Goodwillie spectral sequence

$$d_1(\alpha [i_1, \dots, i_k]) = SHI(\alpha) [m, i_1, \dots, i_k] + \text{lower terms}$$

where $SHI(\alpha)$ is carried by
 m -cell of RP^∞

Idea of proof:

First show the following diagram commutes:

$$\begin{array}{ccccc} Q\mathcal{S} & \xrightarrow{J-H} & Q\mathbb{R}P^\infty & \longrightarrow & Q\mathbb{R}P_n^\infty \\ \parallel & & & & \parallel \\ \Omega^\infty L(0)_n & \xrightarrow{d_1} & & & \Omega^\infty L(1)_n \end{array}$$

(This establishes the chain for $k=0$)

Idea of proof

$$\begin{array}{ccc}
 \Omega D_{2^k} (S^{n+1}) & \xrightarrow{d_1} & D_{2^{k+1}} (S^{n+1}) \\
 \downarrow H & & \downarrow H \\
 D_{2^k} (\Omega \Sigma') (S^n) & \xrightarrow{d_1} & B D_{2^{k+1}} (\Omega \Sigma') (S^n) \\
 \downarrow H & & \downarrow H \\
 D_{2^k} (\Omega \Sigma' S_2) (S^n) & \xrightarrow{d_1} & B D_{2^{k+1}} (\Omega \Sigma' S_2) (S^n) \\
 \downarrow H & & \downarrow H \\
 \Omega D_{2^{k-1}} (S^{2n+1}) & \xrightarrow{d_1} & D_{2^k} (S^{2n+1})
 \end{array}$$

(relates k to $k-1$)

TABLE 5. The GSS for $\pi_{n+1}(S^1)$

n	$\pi_n(L(0))$	$\pi_{n-1}(L(1))$	$\pi_{n-2}(L(2))$	$\pi_{n-3}(L(3))$
0	$1(\infty)$	$1[1]$		
1	η	$\eta[1]$		
2	η^2	$\eta^2[1]$		
3	4ν 2ν ν	$\eta[2]$ $\eta[3]$ $\nu[1]$	$1[3, 1]$	
4				
5		$\nu[3]$	$\nu[3, 1]$	
6	ν^2	$\nu^2[1]$	$\nu[3, 1]$	
7	8σ 4σ 2σ σ	$\eta^3[4]$ $\eta^2[5]$ $\eta[6]$ $1[7]$ $\sigma[1]$	$1[7, 1]$	
8	ϵ $\sigma\eta$	$\nu^2[2]$ $\nu[5]$ $\sigma\eta[1]$	$\nu[5, 1]$	
9	$\epsilon\eta$ $\sigma\eta^2$ α_5	$\epsilon[1]$ $\nu^2[3]$ $\nu^2[3, 1]$ $\nu^2[3]$ $\nu[5, 2]$ $\nu[5, 2]$ $1[7, 3]$ $1[7, 3]$ $\sigma[3, 1]$	$1[7, 3, 1]$	
10	$\eta\alpha_5$	$\alpha_5[1]$		
11	α_6 $\alpha_{6/2}$ $\alpha_{6/3}$	$\eta\alpha_5[1]$ $\alpha_5[2]$ $8\sigma[4]$		
12				
13	κ θ_3	$\epsilon[6]$ $\sigma[7]$ $\theta_3[1]$	$\sigma[7, 1]$	

n	$\pi_n(L(0))$	$\pi_{n-1}(L(1))$	$\pi_{n-2}(L(2))$	$\pi_{n-3}(L(3))$
14	κ θ_3	$\epsilon[6]$ $\sigma[7]$ $\theta_3[1]$ $\kappa[1]$ $\alpha_6[4]$ $\eta\alpha_5[5]$ $\alpha_5[6]$ $8\sigma[8]$ $\eta^3[12]$	$\sigma[7, 1]$ $\sigma[7, 2]$ $\epsilon[6, 2]$	
15	$\eta\kappa$ α_8 $\alpha_{8/2}$ $\alpha_{8/3}$ $\alpha_{8/4}$ $\alpha_{8/5}$	$\theta_3[2]$ $\kappa[2]$ $\nu[13]$ $\alpha_{8/5}[1]$	$\sigma[7, 3]$ $\nu[13, 1]$ $\sigma[7, 3, 1]$	
16	η_4 $\eta\alpha_{8/5}$	$\theta_3[3]$ $\eta_4[1]$ $\nu^2[11]$ $\eta\alpha_{8/5}[1]$ $\kappa[3]$ $\alpha_8[2]$	$\theta_3[3, 1]$ $\nu^2[11, 1]$ $\nu[13, 2]$ $1[15, 3]$ $\theta_3[4, 1]$	$\sigma[7, 3, 1]$
17	$\eta\eta_4$ $\eta^2\alpha_{8/5}$ $\nu\kappa$ α_9	$\eta\eta_4[1]$ $\eta_4[2]$ $\theta_3[4]$ $\sigma\eta^2[9]$ $\sigma\eta[10]$ $\sigma[11]$ $\alpha_9[1]$	$\nu^2[11, 1]$ $\nu[13, 2]$ $1[15, 3]$ $\theta_3[4, 1]$ $\kappa[4, 1]$ $\sigma[11, 1]$	$1[15, 3, 1]$
18	$4\nu^*$ $2\nu^*$ ν^* $\eta\alpha_9$	$\nu\kappa[2]$ $\nu^*[1]$ $\theta_3[5]$ $\eta\alpha_9[1]$ $\alpha_9[2]$ $\alpha_8[4]$	$\kappa[4, 1]$ $\sigma[11, 1]$ $\theta_3[5, 1]$	
19	σ α_{10} $\alpha_{10/2}$ $\alpha_{10/3}$ 4κ 2κ κ	$\bar{\sigma}[1]$ $\kappa\nu[3]$ $\kappa\eta[5]$ $\kappa[6]$ $\bar{\sigma}[2]$ $\nu^*[3]$	$\kappa[6, 1]$ $\theta_3[5, 2]$ $\sigma[11, 3]$ $\nu^*[3, 1]$	$\sigma[11, 3, 1]$

only red diff's don't follow

Goodwillie - Whitehead conj

The GSS

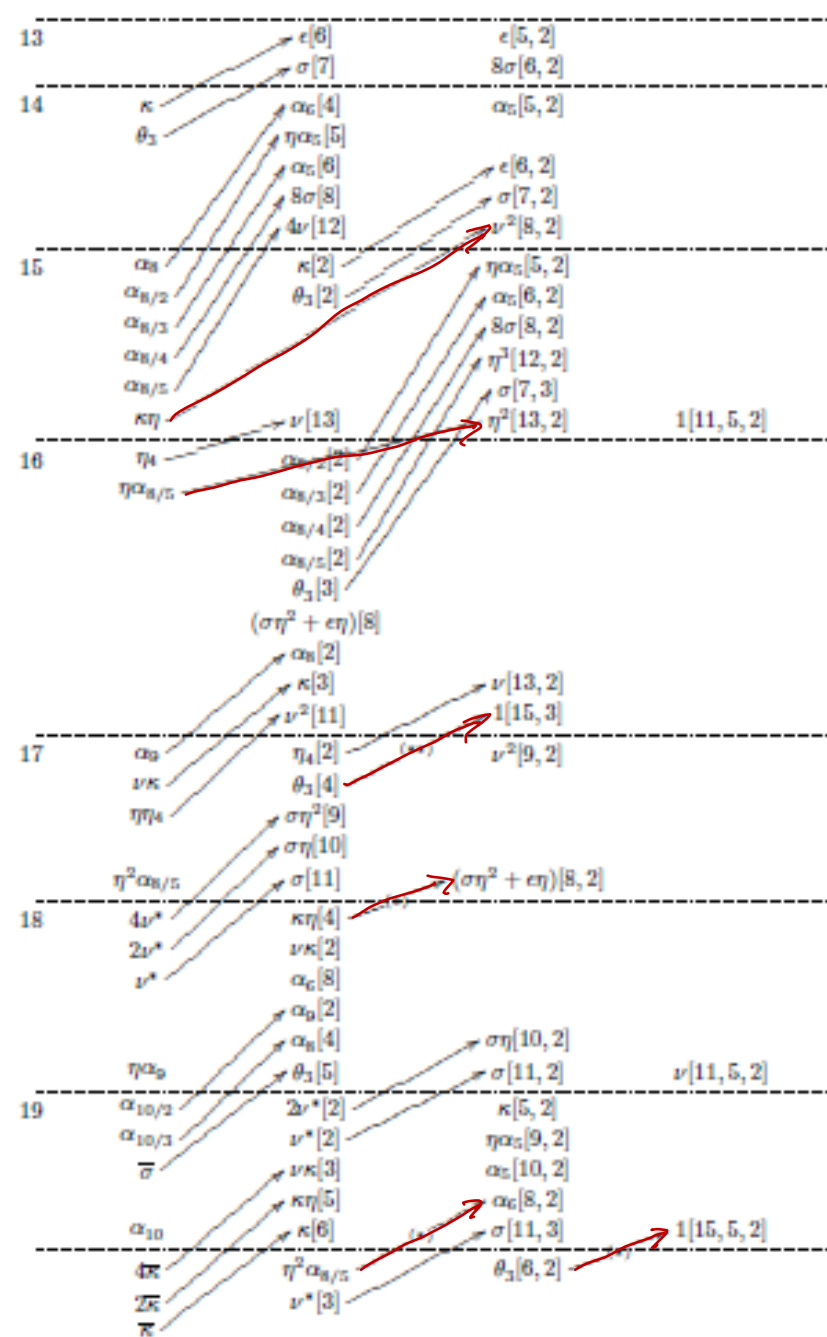
$$E_1 = \bigoplus_k \pi_n L(k) \implies \pi_n S^1$$

collapses at E_2

(Arone
Dwyer
Kuhn
Lesh
Mahowald)

TABLE 6. The GSS for $\pi_{n+2}(S^2)$

n	$\pi_n(L(0)_2)$	$\pi_{n-1}(L(1)_2)$	$\pi_{n-2}(L(2)_2)$	$\pi_{n-3}(L(3)_2)$
0	$1(\infty)$			
1	η	$1(\infty)[2]$		
2	η^2	$\eta[2]$ $1[3]$		
3	$\frac{2\nu}{\nu}$ $\frac{\nu}{4\nu}$			
4		$2\nu[2]$ $\nu[2]$		
5		$\nu[3]$	$1[5, 2]$	
6	ν^2	$\eta^3[4]$ $\eta^2[5]$ $\eta[6]$ $1[7]$		
7	8σ 4σ 2σ σ		$\eta[5, 2]$	
8	ϵ $\sigma\eta$	$\nu^2[2]$ $\nu[5]$ $4\sigma[2]$ $2\sigma[2]$ $\sigma[2]$	$\eta^2[5, 2]$ $\eta[6, 2]$ $1[7, 2]$	
9	α_5 $\sigma\eta^2$ $\epsilon\eta$	$8\sigma[2]$ $\nu^2[3]$ $\sigma\eta[2]$ $\sigma[3]$	$\nu[5, 2]$ $1[7, 3]$	
10		$4\nu[8]$ $\alpha_5[2]$ $8\sigma[4]$		
11	$\eta\alpha_5$ $\alpha_{8/2}$ $\alpha_{8/3}$ α_8		$\eta^2[9, 2]$ $\eta[10, 2]$ $\eta^3[8, 2]$	
12		$\epsilon\eta[4]$ $\alpha_{8/2}[2]$ $\alpha_{8/3}[2]$	$\sigma[5, 2]$ $\nu^2[6, 2]$	



Longer diff's

Thm: Suppose

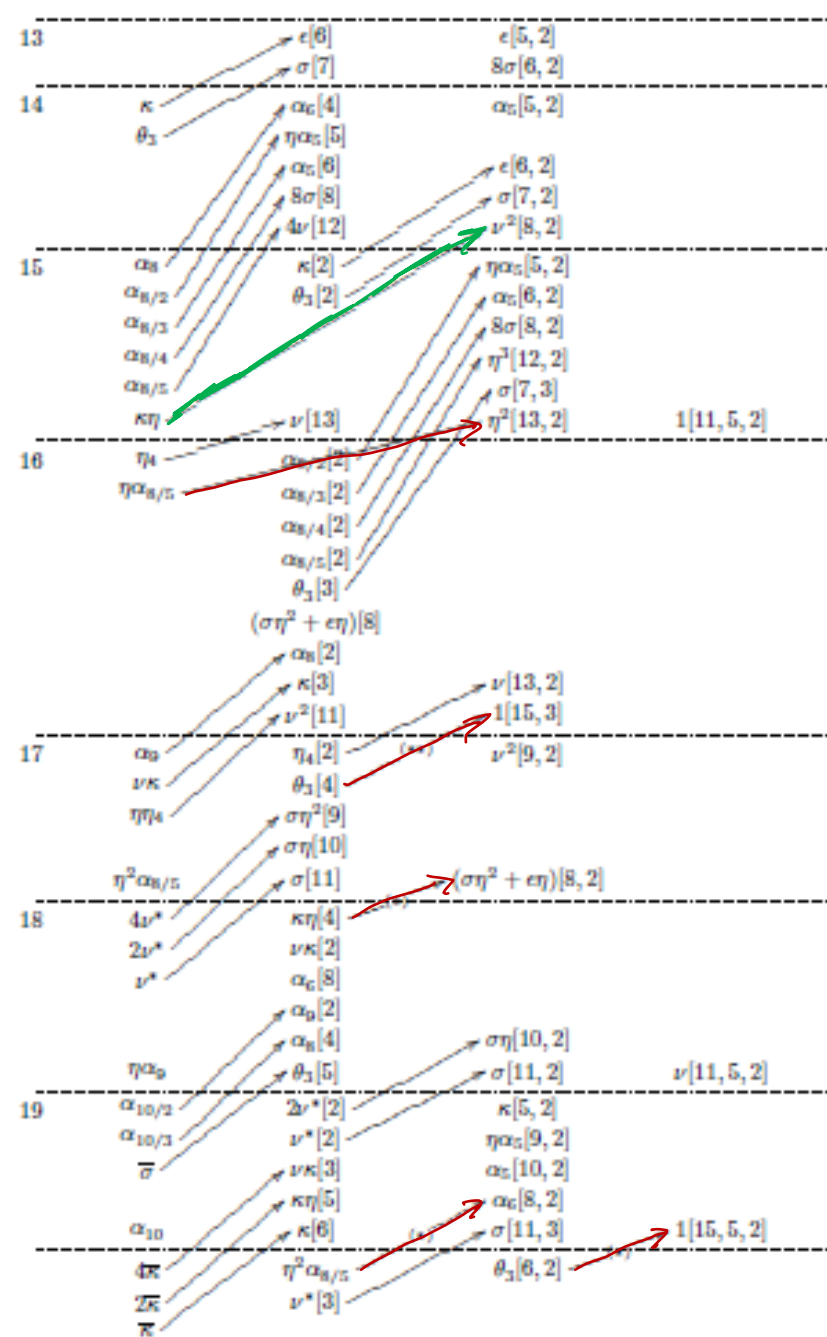
(1) $\alpha[i_1, \dots, i_k]$ persists to E_r^{GSS}

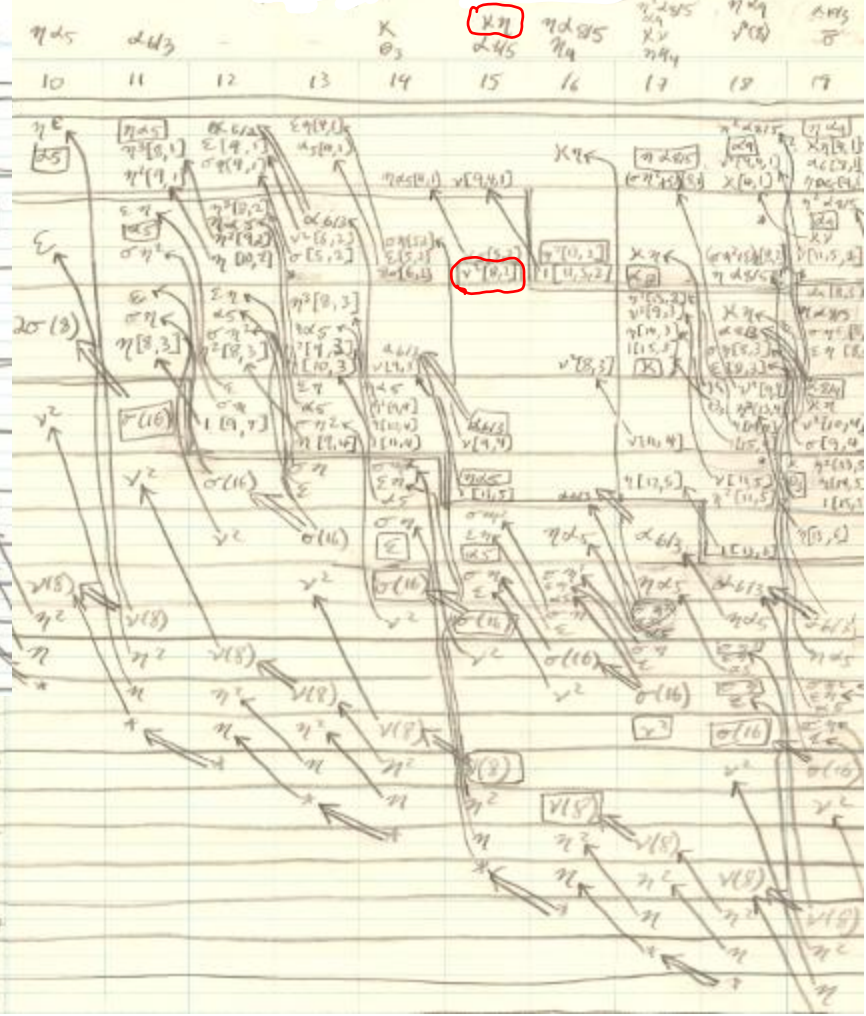
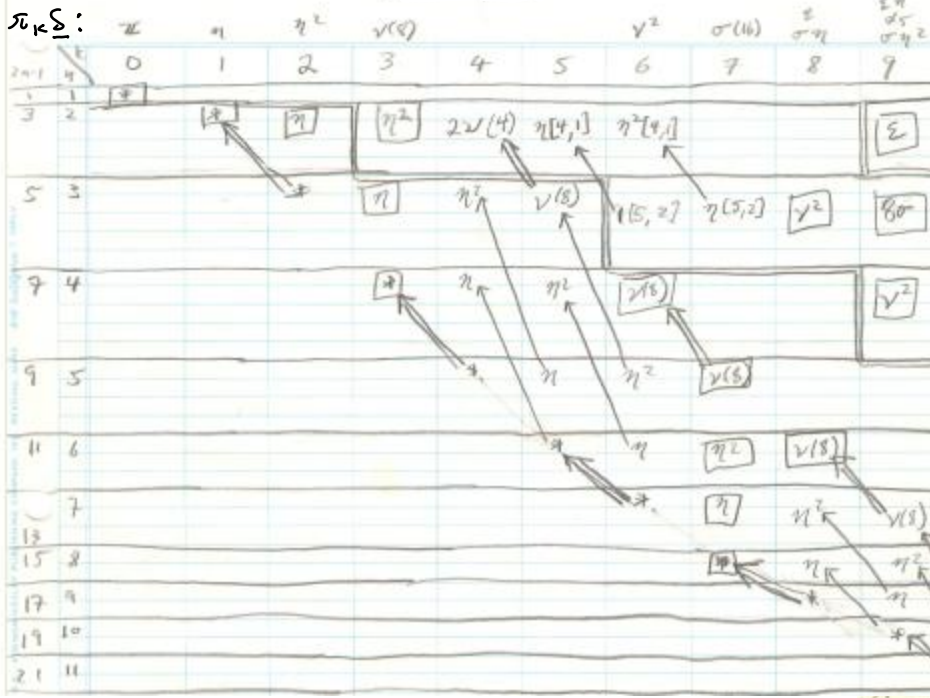
(2) $HI(\alpha) \in \pi, S^{2m+1}$ has Goodwillie filtration 2^{r-1}
detected by $\delta[j_1, \dots, j_{r-1}]$

Then: $d_r(\alpha[i_1, \dots, i_k]) = \delta[j_1, \dots, j_{r-1}, m, i_1, \dots, i_k]$
+ lower terms

TABLE 6. The GSS for $\pi_{n+2}(S^2)$

n	$\pi_n(L(0)_2)$	$\pi_{n-1}(L(1)_2)$	$\pi_{n-2}(L(2)_2)$	$\pi_{n-3}(L(3)_2)$
0	$1(\infty)$			
1	η	$1(\infty)[2]$		
2	η^2	$\eta[2]$ $1[3]$		
3	$\frac{2\nu}{\nu}$ $\frac{\nu}{4\nu}$			
4		$2\nu[2]$ $\nu[2]$		
5		$\nu[3]$	$1[5, 2]$	
6	ν^2	$\eta^3[4]$ $\eta^2[5]$ $\eta[6]$ $1[7]$		$\eta[5, 2]$
7	8σ 4σ 2σ σ	$\nu^2[2]$ $\nu[5]$ $\sigma[2]$	$\eta^2[5, 2]$ $\eta[6, 2]$ $1[7, 2]$	
8	ϵ $\sigma\eta$	$4\sigma[2]$ $2\sigma[2]$ $8\sigma[2]$ $\nu^2[3]$ $\sigma\eta[2]$ $\sigma[3]$	$\nu[5, 2]$ $1[7, 3]$	
9	α_5 $\sigma\eta^2$ $\epsilon\eta$	$4\nu[8]$ $\alpha_5[2]$ $8\sigma[4]$		
10	$\eta\alpha_5$			
11	$\alpha_{8/2}$ $\alpha_{8/3}$ α_8		$\eta^2[9, 2]$ $\eta[10, 2]$ $\eta^3[8, 2]$	
12		$\epsilon\eta[4]$ $\alpha_{8/2}[2]$ $\alpha_{8/3}[2]$	$\sigma[5, 2]$ $\nu^2[6, 2]$	





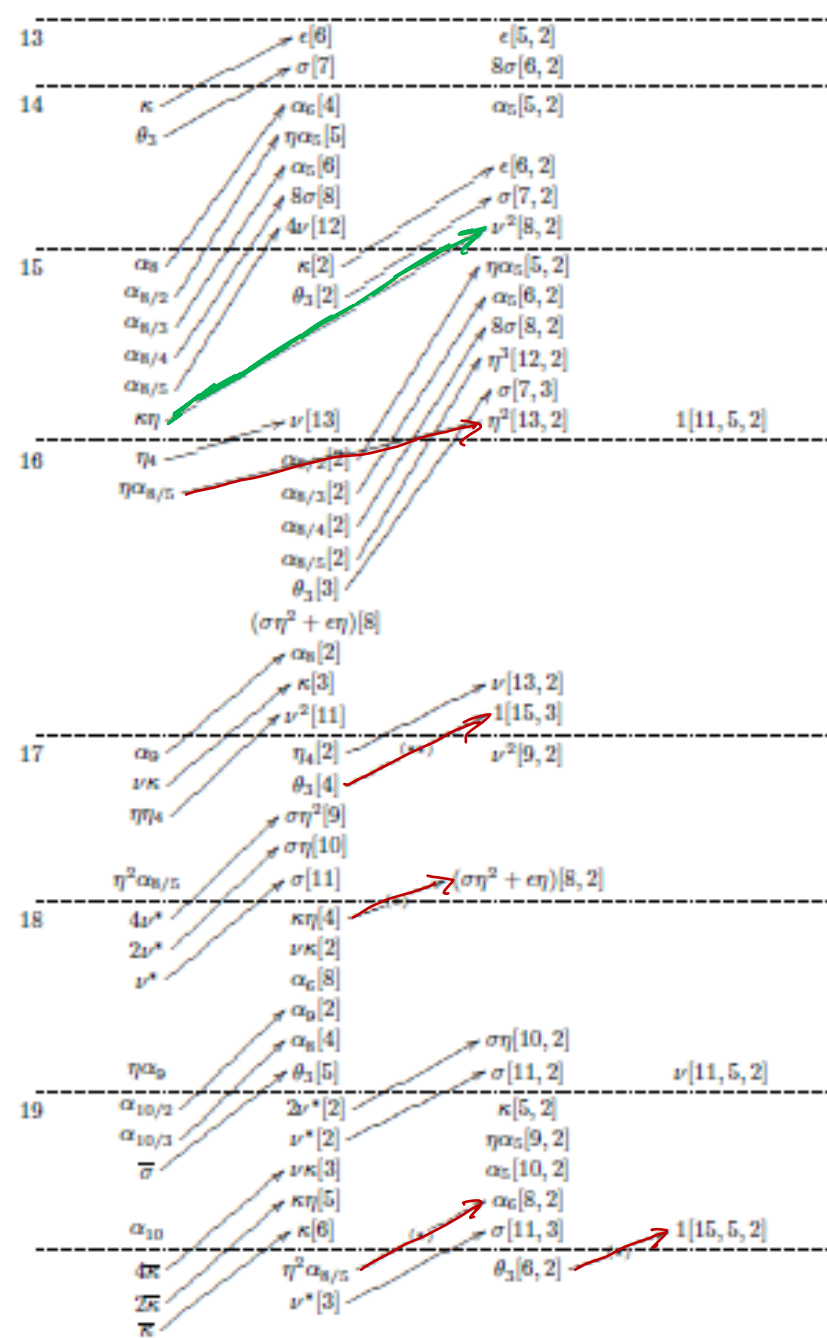
EHPSS

$$\pi_{\star} S^{2m+1} \Rightarrow \pi_{\star} \underline{S}$$

$$(* =: \pi)$$

TABLE 6. The GSS for $\pi_{n+2}(S^2)$

n	$\pi_n(L(0)_2)$	$\pi_{n-1}(L(1)_2)$	$\pi_{n-2}(L(2)_2)$	$\pi_{n-3}(L(3)_2)$
0	$1(\infty)$			
1	η	$1(\infty)[2]$		
2	η^2	$\eta[2]$ $1[3]$		
3	$\frac{2\nu}{\nu}$ $\frac{\nu}{4\nu}$			
4		$2\nu[2]$ $\nu[2]$		
5		$\nu[3]$	$1[5, 2]$	
6	ν^2	$\eta^3[4]$ $\eta^2[5]$ $\eta[6]$ $1[7]$		
7	8σ 4σ 2σ σ		$\eta[5, 2]$	
8	ϵ $\sigma\eta$	$\nu^2[2]$ $\nu[5]$ $4\sigma[2]$ $2\sigma[2]$ $\sigma[2]$	$\eta^2[5, 2]$ $\eta[6, 2]$ $1[7, 2]$	
9	α_5 $\sigma\eta^2$ $\epsilon\eta$	$8\sigma[2]$ $\nu^2[3]$ $\sigma\eta[2]$ $\sigma[3]$	$\nu[5, 2]$ $1[7, 3]$	
10		$4\nu[8]$ $\alpha_5[2]$ $8\sigma[4]$		
11	$\eta\alpha_5$ $\alpha_{8/2}$ $\alpha_{8/3}$ α_8		$\eta^2[9, 2]$ $\eta[10, 2]$ $\eta^3[8, 2]$	
12		$\epsilon\eta[4]$ $\alpha_{8/2}[2]$ $\alpha_{8/3}[2]$	$\sigma[5, 2]$ $\nu^2[6, 2]$	



Summary:

EHP sequence: E_1 term = potential Hopf invariants
diff's = "Whitehead products"

GSS : E_1 - term = potential "Whitehead products"
diff's = Hopf invariants

Computing diff'ls in EMPSS

$$\begin{array}{ccccc} \Omega^m \mathcal{S}^m & \longrightarrow & \Omega^{m+1} \mathcal{S}^{m+1} & \longrightarrow & \Omega^{m+1} \mathcal{S}^{2m+1} \\ \downarrow \mathcal{JH} & & \downarrow \mathcal{JH} & & \downarrow E^\infty \\ \mathbb{Q} \mathbb{R}P^{m-1} & \longrightarrow & \mathbb{Q} \mathbb{R}P^m & \longrightarrow & \mathbb{Q} \mathcal{S}^m \end{array}$$

Consequence! Can "lift" diff'ls from
the AHSS for $\mathbb{R}P^\infty$
to the EMPSS

$\pi_{\star} \underline{S}$:

	\mathbb{Z}	η	η^2	$v(8)$	v^2	$\sigma(16)$	σ^2	σ^4
201	0	1	2	3	4	5	6	7
21	1	*						
3	2		η	η^2	$2v(4)$	$\eta[4,1]$	$\eta^2[4,1]$	Σ
5	3			η	η^2	$v(8)$	$v[8,2]$	v^2
7	4				η	η^2	$v(8)$	v^2
9	5					η	η^2	$v(8)$
11	6						η	$v(8)$
13	7							$v(8)$
15	8							$v(8)$
17	9							$v(8)$
19	10							$v(8)$
21	11							$v(8)$

η_{d5}	d_{d3}			K_{θ_3}	η_{d5}	η_{d5}	η_{d5}
10	11	12	13	14	15	16	17
18	19	20	21	22	23	24	25
26	27	28	29	30	31	32	33
34	35	36	37	38	39	40	41
42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57
58	59	60	61	62	63	64	65
66	67	68	69	70	71	72	73
74	75	76	77	78	79	80	81
82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97
98	99	100	101	102	103	104	105
106	107	108	109	110	111	112	113
114	115	116	117	118	119	120	121
122	123	124	125	126	127	128	129
130	131	132	133	134	135	136	137
138	139	140	141	142	143	144	145
146	147	148	149	150	151	152	153
154	155	156	157	158	159	160	161
162	163	164	165	166	167	168	169
170	171	172	173	174	175	176	177
178	179	180	181	182	183	184	185
186	187	188	189	190	191	192	193
194	195	196	197	198	199	200	201
202	203	204	205	206	207	208	209
210	211	212	213	214	215	216	217
218	219	220	221	222	223	224	225
226	227	228	229	230	231	232	233
234	235	236	237	238	239	240	241
242	243	244	245	246	247	248	249
250	251	252	253	254	255	256	257
258	259	260	261	262	263	264	265
266	267	268	269	270	271	272	273
274	275	276	277	278	279	280	281
282	283	284	285	286	287	288	289
290	291	292	293	294	295	296	297
298	299	300	301	302	303	304	305
306	307	308	309	310	311	312	313
314	315	316	317	318	319	320	321
322	323	324	325	326	327	328	329
330	331	332	333	334	335	336	337
338	339	340	341	342	343	344	345
346	347	348	349	350	351	352	353
354	355	356	357	358	359	360	361
362	363	364	365	366	367	368	369
370	371	372	373	374	375	376	377
378	379	380	381	382	383	384	385
386	387	388	389	390	391	392	393
394	395	396	397	398	399	400	401
402	403	404	405	406	407	408	409
410	411	412	413	414	415	416	417
418	419	420	421	422	423	424	425
426	427	428	429	430	431	432	433
434	435	436	437	438	439	440	441
442	443	444	445	446	447	448	449
450	451	452	453	454	455	456	457
458	459	460	461	462	463	464	465
466	467	468	469	470	471	472	473
474	475	476	477	478	479	480	481
482	483	484	485	486	487	488	489
490	491	492	493	494	495	496	497
498	499	500	501	502	503	504	505
506	507	508	509	510	511	512	513
514	515	516	517	518	519	520	521
522	523	524	525	526	527	528	529
530	531	532	533	534	535	536	537
538	539	540	541	542	543	544	545
546	547	548	549	550	551	552	553
554	555	556	557	558	559	560	561
562	563	564	565	566	567	568	569
570	571	572	573	574	575	576	577
578	579	580	581	582	583	584	585
586	587	588	589	590	591	592	593
594	595	596	597	598	599	600	601
602	603	604	605	606	607	608	609
610	611	612	613	614	615	616	617
618	619	620	621	622	623	624	625
626	627	628	629	630	631	632	633
634	635	636	637	638	639	640	641
642	643	644	645	646	647	648	649
650	651	652	653	654	655	656	657
658	659	660	661	662	663	664	665
666	667	668	669	670	671	672	673
674	675	676	677	678	679	680	681
682	683	684	685	686	687	688	689
690	691	692	693	694	695	696	697
698	699	700	701	702	703	704	705
706	707	708	709	710	711	712	713
714	715	716	717	718	719	720	721
722	723	724	725	726	727	728	729
730	731	732	733	734	735	736	737
738	739	740	741	742	743	744	745
746	747	748	749	750	751	752	753
754	755	756	757	758	759	760	761
762	763	764	765	766	767	768	769
770	771	772	773	774	775	776	777
778	779	780	781	782	783	784	785
786	787	788	789	790	791	792	793
794	795	796	797	798	799	800	801
802	803	804	805	806	807	808	809
810	811	812	813	814	815	816	817
818	819	820	821	822	823	824	825
826	827	828	829	830	831	832	833
834	835	836	837	838	839	840	841
842	843	844	845	846	847	848	849
850	851	852	853	854	855	856	857
858	859	860	861	862	863	864	865
866	867	868	869	870	871	872	873
874	875	876	877	878	879	880	881
882	883	884	885	886	887	888	889
890	891	892	893	894	895	896	897
898	899	900	901	902	903	904	905
906	907	908	909	910	911	912	913
914	915	916	917	918	919	920	921
922	923	924	925	926	927	928	929
930	931	932	933	934	935	936	937
938	939	940	941	942	943	944	945
946	947	948	949	950	951	952	953
954	955	956	957	958	959	960	961
962	963	964	965	966	967	968	969
970	971	972	973	974	975	976	977
978	979	980	981	982	983	984	985
986	987	988	989	990	991	992	993
994	995	996	997	998	999	1000	1001

EHPSS

$$\pi_{\star} S^{2m+1} \Rightarrow \pi_{\star} \underline{S}$$

$$(* =: \mathbb{Z})$$

Can lift diff'ls from APSS

$$\bigoplus_m \pi_* L^{(k-1)}_{2m+1} \implies \pi_* L^{(k)}_1$$

to the EMPSS

$$\pi_x L(1)_{2m+1} \Rightarrow \pi_b L(2)$$

$Q^{23} Q^1$	$1[2,2]$											
$Q^{25} Q^2$												
$Q^{27} Q^3$												
$Q^{29} Q^4$												
	4	5	6	7	8	9	10	11	12	13		

$Q^{23} Q^1$	$2[2,1]$	$3[2,0]$	$4[2,0]$	$5[2,0]$	$6[2,0]$	$7[2,0]$	$8[2,0]$	$9[2,0]$	$10[2,0]$	$11[2,0]$	$12[2,0]$	$13[2,0]$
$Q^{25} Q^2$	$1[5,2]$	$2[5,2]$	$3[5,2]$	$4[5,2]$	$5[5,2]$	$6[5,2]$	$7[5,2]$	$8[5,2]$	$9[5,2]$	$10[5,2]$	$11[5,2]$	$12[5,2]$
$Q^{27} Q^3$	$1[9,3]$	$2[9,3]$	$3[9,3]$	$4[9,3]$	$5[9,3]$	$6[9,3]$	$7[9,3]$	$8[9,3]$	$9[9,3]$	$10[9,3]$	$11[9,3]$	$12[9,3]$
$Q^{29} Q^4$	$1[13,4]$	$2[13,4]$	$3[13,4]$	$4[13,4]$	$5[13,4]$	$6[13,4]$	$7[13,4]$	$8[13,4]$	$9[13,4]$	$10[13,4]$	$11[13,4]$	$12[13,4]$
$Q^{31} Q^5$												
$Q^{33} Q^6$												
	14	15	16	17	18	19	20	21	22	23	24	25

$\pi_{\star} \underline{S}$:

π	π^1	π^2	$v(\pi)$	v^2	$\sigma(\pi)$	σ^2	σ^3														
0	1	2	3	4	5	6	7														
1	*																				
2		*																			
3			*																		
4				*																	
5					*																
6						*															
7							*														
8								*													
9									*												
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18																		*			
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20																				*	
21																					*

π	π^1	π^2	π^3	π^4	π^5	π^6	π^7	π^8	π^9	π^{10}	π^{11}	π^{12}	π^{13}	π^{14}	π^{15}	π^{16}	π^{17}	π^{18}	π^{19}	
10	11	12	13	14	15	16	17	18												
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EHPSS

$$\pi_{\star} S^{2m+1} \Rightarrow \pi_{\star} \underline{S}$$

($\star =: \mathbb{Z}$)

$$\pi_{\star} L(2)_{2n+1} \implies \pi_{\star} L(3)$$

	11	12	13	14	15	16	17	18	19	20		21
\mathbb{Z}^2	$[2,2]$					$[2,2]$	$[2,2]$	$[2,2]$	$[2,2]$	$[2,2]$		$[2,2]$
\mathbb{Z}^3												
\mathbb{Z}^4												

The table shows a sequence of elements in \mathbb{Z}^2 across columns 11 to 21. The elements are $[2,2]$ in columns 11, 16, 17, 18, 19, and 21. Purple arrows point from the $[2,2]$ in column 19 to the $[2,2]$ in column 17, and from the $[2,2]$ in column 21 to the $[2,2]$ in column 19.

$\pi_{\star} \underline{S}$:

π	η	η^2	$v(\eta)$	v^2	$\sigma(\eta)$	σ^2	σ^3
0	1	2	3	4	5	6	7
1	2	η	η^2	$2v(4)$	$\eta[4,1]$	$\eta^2[4,1]$	Σ
3	3	η	η^2	$v(8)$	$v(8)$	$v(8)$	Σ
5	4	η	η^2	$v(8)$	$v(8)$	$v(8)$	Σ
7	5	η	η^2	$v(8)$	$v(8)$	$v(8)$	Σ
9	6	η	η^2	$v(8)$	$v(8)$	$v(8)$	Σ
11	7	η	η^2	$v(8)$	$v(8)$	$v(8)$	Σ
13	8	η	η^2	$v(8)$	$v(8)$	$v(8)$	Σ
15	9	η	η^2	$v(8)$	$v(8)$	$v(8)$	Σ
17	10	η	η^2	$v(8)$	$v(8)$	$v(8)$	Σ
19	11	η	η^2	$v(8)$	$v(8)$	$v(8)$	Σ
21	12	η	η^2	$v(8)$	$v(8)$	$v(8)$	Σ

η_{d5}	d_{d3}	η_{d5}	η_{d5}	η_{d5}	η_{d5}	η_{d5}	η_{d5}
10	11	12	13	14	15	16	17
18	19	20	21	22	23	24	25
26	27	28	29	30	31	32	33
34	35	36	37	38	39	40	41
42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57
58	59	60	61	62	63	64	65
66	67	68	69	70	71	72	73
74	75	76	77	78	79	80	81
82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97
98	99	100	101	102	103	104	105
106	107	108	109	110	111	112	113
114	115	116	117	118	119	120	121
122	123	124	125	126	127	128	129
130	131	132	133	134	135	136	137
138	139	140	141	142	143	144	145
146	147	148	149	150	151	152	153
154	155	156	157	158	159	160	161
162	163	164	165	166	167	168	169
170	171	172	173	174	175	176	177
178	179	180	181	182	183	184	185
186	187	188	189	190	191	192	193
194	195	196	197	198	199	200	201
202	203	204	205	206	207	208	209
210	211	212	213	214	215	216	217
218	219	220	221	222	223	224	225
226	227	228	229	230	231	232	233
234	235	236	237	238	239	240	241
242	243	244	245	246	247	248	249
250	251	252	253	254	255	256	257
258	259	260	261	262	263	264	265
266	267	268	269	270	271	272	273
274	275	276	277	278	279	280	281
282	283	284	285	286	287	288	289
290	291	292	293	294	295	296	297
298	299	300	301	302	303	304	305
306	307	308	309	310	311	312	313
314	315	316	317	318	319	320	321
322	323	324	325	326	327	328	329
330	331	332	333	334	335	336	337
338	339	340	341	342	343	344	345
346	347	348	349	350	351	352	353
354	355	356	357	358	359	360	361
362	363	364	365	366	367	368	369
370	371	372	373	374	375	376	377
378	379	380	381	382	383	384	385
386	387	388	389	390	391	392	393
394	395	396	397	398	399	400	401
402	403	404	405	406	407	408	409
410	411	412	413	414	415	416	417
418	419	420	421	422	423	424	425
426	427	428	429	430	431	432	433
434	435	436	437	438	439	440	441
442	443	444	445	446	447	448	449
450	451	452	453	454	455	456	457
458	459	460	461	462	463	464	465
466	467	468	469	470	471	472	473
474	475	476	477	478	479	480	481
482	483	484	485	486	487	488	489
490	491	492	493	494	495	496	497
498	499	500	501	502	503	504	505
506	507	508	509	510	511	512	513
514	515	516	517	518	519	520	521
522	523	524	525	526	527	528	529
530	531	532	533	534	535	536	537
538	539	540	541	542	543	544	545
546	547	548	549	550	551	552	553
554	555	556	557	558	559	560	561
562	563	564	565	566	567	568	569
570	571	572	573	574	575	576	577
578	579	580	581	582	583	584	585
586	587	588	589	590	591	592	593
594	595	596	597	598	599	600	601
602	603	604	605	606	607	608	609
610	611	612	613	614	615	616	617
618	619	620	621	622	623	624	625
626	627	628	629	630	631	632	633
634	635	636	637	638	639	640	641
642	643	644	645	646	647	648	649
650	651	652	653	654	655	656	657
658	659	660	661	662	663	664	665
666	667	668	669	670	671	672	673
674	675	676	677	678	679	680	681
682	683	684	685	686	687	688	689
690	691	692	693	694	695	696	697
698	699	700	701	702	703	704	705
706	707	708	709	710	711	712	713
714	715	716	717	718	719	720	721
722	723	724	725	726	727	728	729
730	731	732	733	734	735	736	737
738	739	740	741	742	743	744	745
746	747	748	749	750	751	752	753
754	755	756	757	758	759	760	761
762	763	764	765	766	767	768	769
770	771	772	773	774	775	776	777
778	779	780	781	782	783	784	785
786	787	788	789	790	791	792	793
794	795	796	797	798	799	800	801
802	803	804	805	806	807	808	809
810	811	812	813	814	815	816	817
818	819	820	821	822	823	824	825
826	827	828	829	830	831	832	833
834	835	836	837	838	839	840	841
842	843	844	845	846	847	848	849
850	851	852	853	854	855	856	857
858	859	860	861	862	863	864	865
866	867	868	869	870	871	872	873
874	875	876	877	878	879	880	881
882	883	884	885	886	887	888	889
890	891	892	893	894	895	896	897
898	899	900	901	902	903	904	905
906	907	908	909	910	911	912	913
914	915	916	917	918	919	920	921
922	923	924	925	926	927	928	929
930	931	932	933	934	935	936	937
938	939	940	941	942	943	944	945
946	947	948	949	950	951	952	953
954	955	956	957	958	959	960	961
962	963	964	965	966	967	968	969
970	971	972	973	974	975	976	977
978	979	980	981	982	983	984	985
986	987	988	989	990	991	992	993
994	995	996	997	998	999	1000	1001

EHPSS

$$\pi_{\star} S^{2m+1} \Rightarrow \pi_{\star} \underline{S}$$

$$(*) =: \mathbb{Z}$$