The EHP sequence and the Goodwillie tower

Mark Behrens
MIT

problem: $\pi_*(S^n)$

<u>problem</u>: $\pi_*(S^n)$

EHP sequence

Goodwillie tower

problem: $\pi_*(S^n)$

EHP sequence

Goodwillie tower

interrelated in a complex but beautiful way

problem: $\pi_*(S^n)$

EHP sequence

Goodwillie tower

interrelated in a complex but beautiful way

Today: everything is localized at $p=2$

Goodwillie Tower

Goodwillie Tower

$F : Top_* \longrightarrow Top_*$

Preserves

- w.e.
- filtered hocolim
- $F(*) = *$

Goodwillie Tower

$F : Top_* \longrightarrow Top_*$

Preserves

- w.e.
- filtered hocolim
- $F(*) = *$

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & P_3(F) \arrow[d] \\
 & P_2(F) \arrow[d] \\
F \arrow[r] \arrow[ur] \arrow[uur] & P_1(F)
\end{tikzcd}$$

$P_i(F) = i$-excisive

Goodwillie Tower

$F : Top_* \longrightarrow Top_*$

Preserves
- w.e.
- filtered hocolim
- $F(*) = *$

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & P_3(F) \arrow[d] \\
 & P_2(F) \arrow[d] \\
F \arrow[r] \arrow[ur] \arrow[uur] & P_1(F)
\end{tikzcd}$$

$P_i(F)$ = i-excisive

$F(X) \xrightarrow{\simeq} \varprojlim P_i(F)(X)$

if X is sufficiently highly connected

Goodwillie Tower

$F : Top_* \longrightarrow Top_*$

$$\begin{tikzcd}
\vdots \arrow[d] & \\
P_3(F) \arrow[d] & D_3(F) \arrow[l] \\
P_2(F) \arrow[d] & D_2(F) \arrow[l] \\
P_1(F) \arrow[r, equal] & D_1(F)
\end{tikzcd}$$

$D_i(F)(X) = \Omega^\infty \mathbb{D}_i(F)(X)$

$\mathbb{D}_i F : Top_* \longrightarrow Sp$

homogeneous deg i

$P_i(F) = i$-excisive

Goodwillie tower of Id

$Id : Top_* \longrightarrow Top_*$ $\qquad P_i(Id)(X) =: P_i(X)$

$$\begin{tikzcd}
X \arrow[d, dotted] & \\
\vdots \arrow[d] & \\
P_3(X) \arrow[d] & D_3(X) \arrow[l, hook'] \\
P_2(X) \arrow[d] & D_2(X) \arrow[l] \\
P_1(X) \arrow[r, equal] & D_1(X)
\end{tikzcd}$$

(converges if X is connected, complete)

Goodwillie tower of Id

$\mathrm{Id} : \mathrm{Top}_* \longrightarrow \mathrm{Top}_*$ $\qquad$ $P_i(\mathrm{Id})(X) =: P_i(X)$

$$\begin{tikzcd}
X \arrow[d, dotted] & \\
{} \arrow[d] & \\
P_3(X) \arrow[d] & D_3(X) \arrow[l] \\
P_2(X) \arrow[d] & D_2(X) \arrow[l] \\
P_1(X) \arrow[r, equal] & D_1(X)
\end{tikzcd}$$

(converges if X is connected, complete)

Get a spectral sequence (GSS)

$$E_1 = \pi_t D_i(X) \Rightarrow \pi_t X$$

[compute unstable homotopy groups from stable htpy gps]

GSS for S^n

Arone-Mahowald, Arone-Dwyer:

Thm:

$$\mathbb{D}_i(S^n) \simeq \begin{cases} *, & i \neq 2^k \\ \Sigma^{n-k} L(k)_n, & i = 2^k \end{cases}$$

$$L(0)_n = \underline{S}, \quad L(1)_n = P_n^\infty = \Sigma^{\infty} \mathbb{R}P^\infty / \mathbb{R}P^{n-1}$$

GSS for S^n

Arone-Mahowald, Arone-Dwyer:

Thm:

$$D_i(S^n) \simeq \begin{cases} *, & i \neq 2^k \\ \Sigma^{n-k} L(k)_n, & i = 2^k \end{cases}$$

$L(0)_n = \underset{\sim}{S}$, $L(1)_n = P_n^\infty = \Sigma^\infty \mathbb{R}P^\infty / \mathbb{R}P^{n-1}$

$L(k)_n$ are "well-known spectra"

studied by Kuhn, Mitchell, Priddy....

$$L(k)_n = e_{st} \Sigma^\infty (B\mathbb{F}_2{}^k)^{n\bar{\rho}}$$

GSS for S^n

The GSS takes the form:

$$\pi_t L(k)_n \Longrightarrow \pi_{n+t-k}(S^n)$$

GSS for S^n

The GSS takes the form:

$$\pi_t L(k)_n \Longrightarrow \pi_{n+t-k}(S^n)$$

Problems:

(1) compute $\pi_* L(k)_n$ [Stable htpy thy]

(2) compute diff'ls [Mysterious]

EHP sequence

Consider the sequence of functors

$$\mathrm{Id} \xrightarrow[E]{} \Omega\Sigma \xrightarrow[H]{} \Omega\Sigma Sq$$

$$Sq(X) = X \wedge X$$

$$\begin{tikzcd} \Sigma\Omega\Sigma X \simeq \Sigma \bigvee_{n\geq 1} X^{\wedge n} \arrow[r] \arrow[r, bend right, "\tilde{H}"'] & \Sigma X^{\wedge 2} \end{tikzcd}$$

EHP sequence

Consider the sequence of functors

$$\mathrm{Id} \xrightarrow[E]{} \Omega\Sigma_i \xrightarrow[H]{} \Omega\Sigma_i Sq$$

$$\Omega^2 S^{2m+1} \xrightarrow[P]{} S^m \xrightarrow[E]{} \Omega S^{m+1} \xrightarrow[H]{} \Omega S^{2m+1}$$

Fiber sequence

EHP sequence

Consider the sequence of functors

$$\mathrm{Id} \xrightarrow[E]{} \Omega\Sigma_i \xrightarrow[H]{} \Omega\Sigma_i Sq$$

$$\Omega^{m+1} S^{2m+1} \xrightarrow[P]{} \Omega^m S^m \xrightarrow[E]{} \Omega^{m+1} S^{m+1} \xrightarrow[H]{} \Omega^{m+1} S^{2m+1}$$

EHP sequence

Consider the sequence of functors

$$\mathrm{Id} \xrightarrow{E} \Omega\Sigma_i \xrightarrow{H} \Omega\Sigma_i S_q$$

$$\Omega^{m+2} S^{2m+1} \xrightarrow{P} \Omega^m S^m \xrightarrow{E} \Omega^{m+1} S^{m+1} \xrightarrow{H} \Omega^{m+1} S^{2m+1}$$

Writing:

$$\Omega S^1 \longrightarrow \Omega^2 S^2 \longrightarrow \cdots\cdots \longrightarrow QS^0$$

EHPSS

$$\bigoplus_{0 \le m} \pi_{t+m+1} S^{2m+1} \Longrightarrow \pi_t \underline{S}$$

EHP sequence

Consider the sequence of functors

$$\mathrm{Id} \xrightarrow{E} \Omega\Sigma \xrightarrow{H} \Omega\Sigma\mathrm{Sq}$$

$$\Omega^{m+2} S^{2m+1} \xrightarrow{P} \Omega^m S^m \xrightarrow{E} \Omega^{m+1} S^{m+1} \xrightarrow{H} \Omega^{m+1} S^{2m+1}$$

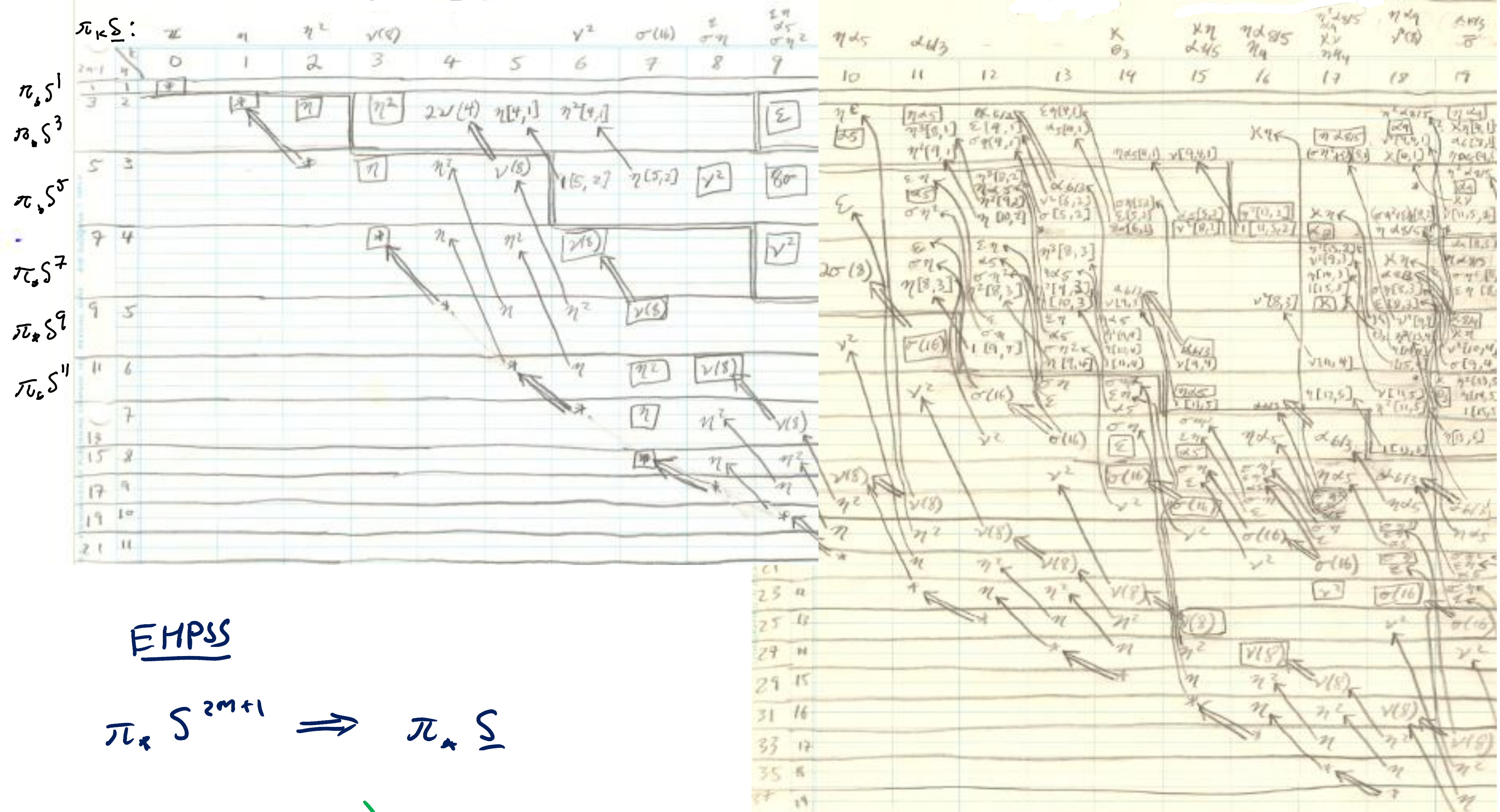
Writing:

$$\Omega S^1 \longrightarrow \Omega^2 S^2 \longrightarrow \cdots \Omega^n S^n \longrightarrow QS^0$$

EHPSS

$$\bigoplus_{0 \le m} \pi_{t+m+1} S^{2m+1} \Longrightarrow \pi_t \underline{S}$$

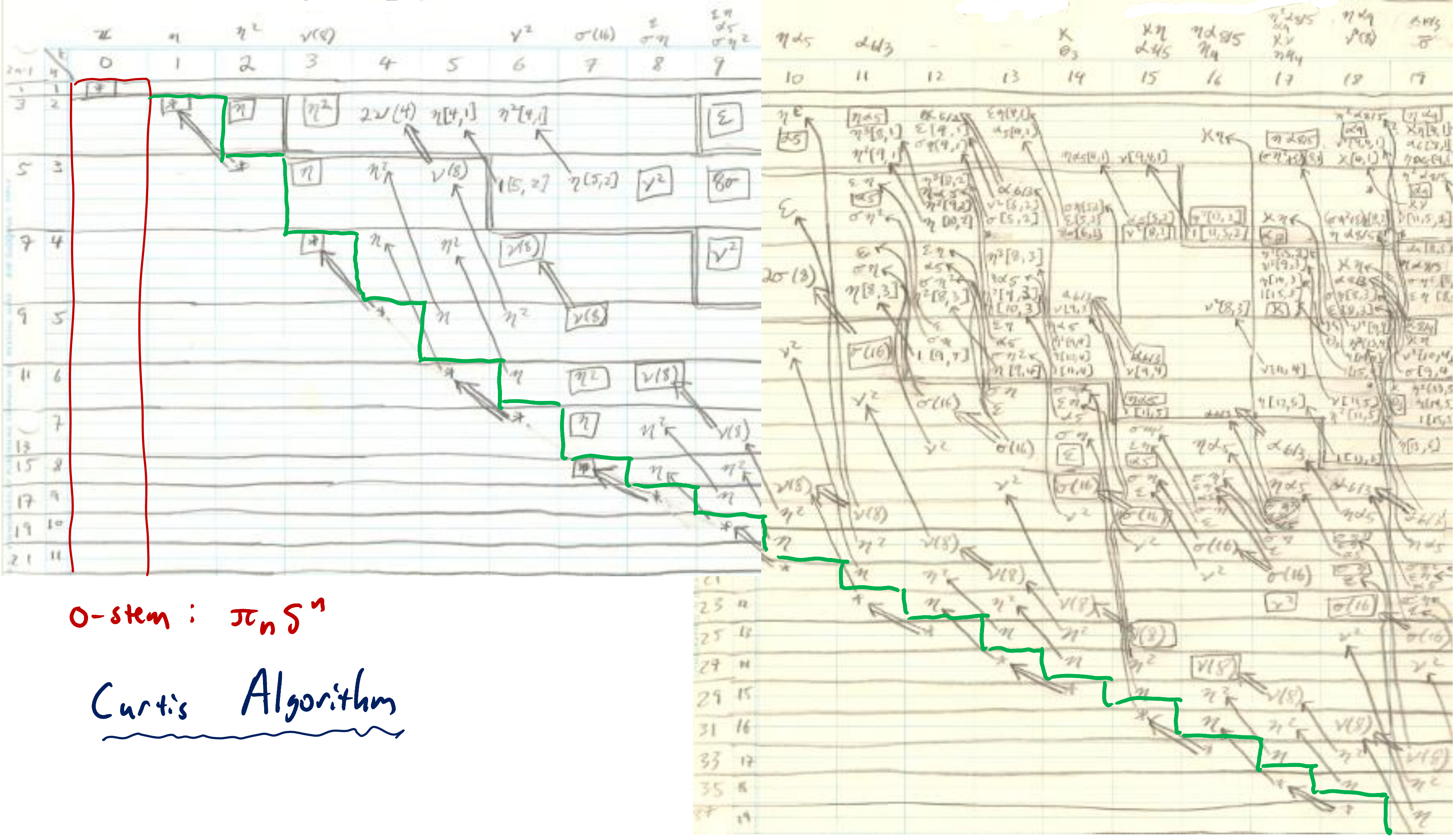
$$\bigoplus_{0 \le m < n} \pi_{t+m+1} S^{2m+1} \Longrightarrow \pi_{t+n} S^n \qquad 1 \le m < n$$



EHPSS

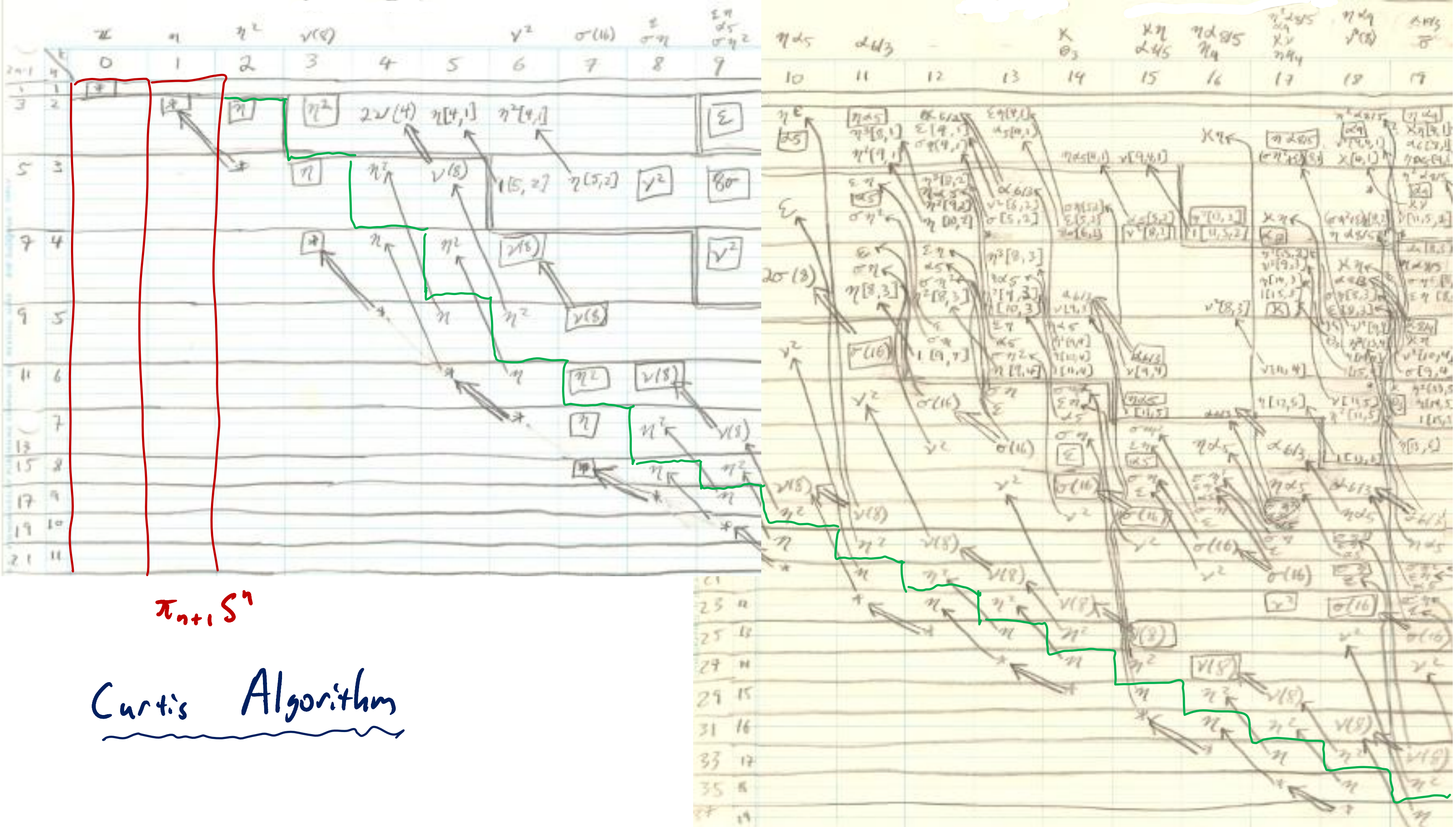
$$\pi_* S^{2m+1} \Longrightarrow \pi_* \underline{S}$$

$$(* =: \mathbb{Z})$$

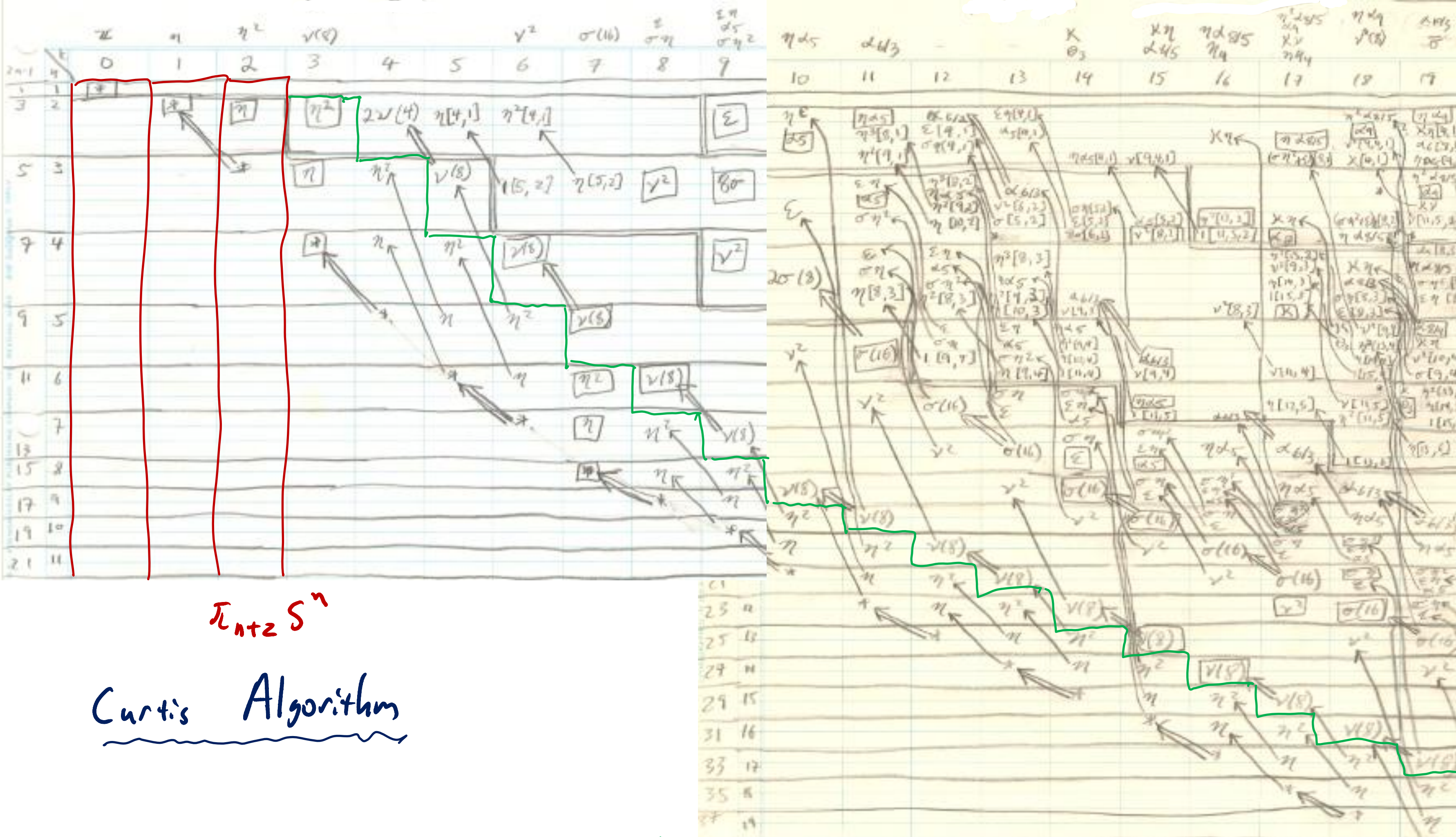


0-stem : $\pi_n S^n$

Curtis Algorithm

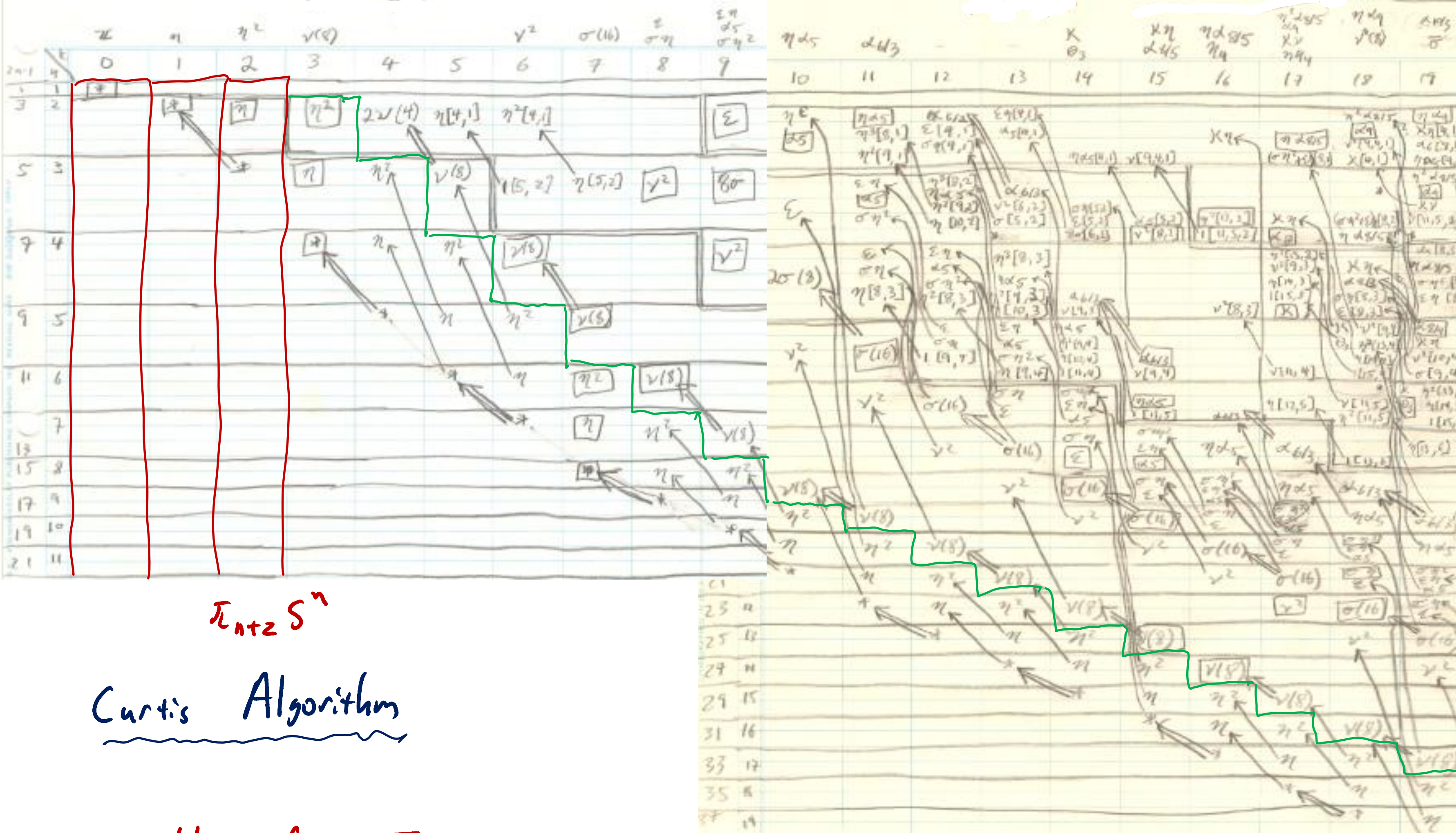


Curtis Algorithm



$\pi_{n+2} S^n$

Curtis Algorithm



$\pi_{n+2} S^n$

Curtis Algorithm

Main problem for EHPSS:

Compute diff'ls!

Back to $L(k)_n$

EHP sequence: $\Omega^n \Sigma^n \longrightarrow \Omega^{n+1} \Sigma^{n+1} \longrightarrow \Omega^{n+1} \Sigma^{n+1} S_q$

(exact on spheres)

Back to $L(k)_n$

EHP sequence: $\Omega^n\Sigma^n \longrightarrow \Omega^{n+1}\Sigma^{n+1} \longrightarrow \Omega^{n+1}\Sigma^{n+1}S_q$

(exact on spheres)

Apply Goodwillie tower! $\mathbb{D}_{2^k}(\Omega^n\Sigma^n) \longrightarrow \mathbb{D}_{2^k}(\Omega^{n+1}\Sigma^{n+1}) \longrightarrow \mathbb{D}_{2^k}(\Omega^{n+1}\Sigma^{n+1}S_q)$

(exact on spheres)

Back to $L(k)_n$

EHP sequence: $\Omega^n\Sigma^n \longrightarrow \Omega^{n+1}\Sigma^{n+1} \longrightarrow \Omega^{n+1}\Sigma^{n+1}S_q$

(exact on spheres)

Apply Goodwillie tower!

(exact on spheres)

$$\begin{array}{ccccc} \mathbb{D}_{2^k}(\Omega^n\Sigma^n) & \longrightarrow & \mathbb{D}_{2^k}(\Omega^{n+1}\Sigma^{n+1}) & \longrightarrow & \mathbb{D}_{2^k}(\Omega^{n+1}\Sigma^{n+1}S_q) \\ \wr\wr & & \wr\wr & & \wr\wr \\ \Sigma^{-n}\circ\mathbb{D}_{2^k}\circ\Sigma^n & & \Sigma^{-n-1}\circ\mathbb{D}_{2^k}\circ\Sigma^{n+1} & & \underbrace{\Sigma^{-n-1}\circ\mathbb{D}_{2^{k-1}}\circ\Sigma^{n+1}\circ S_q} \end{array}$$

$$f(x) = \sum a_k x^{2^k}$$

$$\Rightarrow \quad f(x^2) = \sum a_{k-1} x^{2^k}$$

Back to $L(k)_n$

EHP sequence:
(exact on spheres)

$$\Omega^n\Sigma^n \longrightarrow \Omega^{n+1}\Sigma^{n+1} \longrightarrow \Omega^{n+1}\Sigma^{n+1}S_q$$

Apply Goodwillie tower!
(exact on spheres)

$$\begin{array}{ccccc} \mathbb{D}_{2^k}(\Omega^n\Sigma^n) & \longrightarrow & \mathbb{D}_{2^k}(\Omega^{n+1}\Sigma^{n+1}) & \longrightarrow & \mathbb{D}_{2^k}(\Omega^{n+1}\Sigma^{n+1}S_q) \\ \wr\!\wr & & \wr\!\wr & & \wr\!\wr \\ \Sigma^{-n}\circ\mathbb{D}_{2^k}\circ\Sigma^n & & \Sigma^{-n-1}\circ\mathbb{D}_{2^k}\circ\Sigma^{n+1} & & \Sigma^{-n-1}\circ\mathbb{D}_{2^{k-1}}\circ\Sigma^{n+1}\circ S_q \end{array}$$

Cofiber sequence

$$\Sigma^n L(k-1)_{2n+1} \longrightarrow L(k)_n \longrightarrow L(k)_{n+1} \longrightarrow \Sigma^{n+1} L(k-1)_{2n+1}$$

[originally due to Kuhn and Takayasu]

Back to $L(k)_n$

EHP sequence: $\Omega^n\Sigma^n \longrightarrow \Omega^{n+1}\Sigma^{n+1} \longrightarrow \Omega^{n+1}\Sigma^{n+1}S_q$

(exact on spheres)

Apply Goodwillie tower!

(exact on spheres)

$$\begin{array}{ccccc} \mathbb{D}_{2^k}(\Omega^n\Sigma^n) & \longrightarrow & \mathbb{D}_{2^k}(\Omega^{n+1}\Sigma^{n+1}) & \longrightarrow & \mathbb{D}_{2^k}(\Omega^{n+1}\Sigma^{n+1}S_q) \\ \wr\wr & & \wr\wr & & \wr\wr \\ \Sigma^{-n}\circ\mathbb{D}_{2^k}\circ\Sigma^n & & \Sigma^{-n-1}\circ\mathbb{D}_{2^k}\circ\Sigma^{n+1} & & \Sigma^{-n-1}\circ\mathbb{D}_{2^{k-1}}\circ\Sigma^{n+1}\circ S_q \end{array}$$

Cofiber sequence

$$\Sigma^n L(k-1)_{2n+1} \longrightarrow L(k)_n \longrightarrow L(k)_{n+1} \longrightarrow \Sigma^n L(k-1)_{2n+1}$$

e.g. $k=1$

$$S^n \longrightarrow P_n^\infty \longrightarrow P_{n+1}^\infty \longrightarrow S^{n+1}$$

Homology of $L(k)_n$

$H_* = H_*(-;\mathbb{F}_2)$

$$H_* L(k)_n \cong \mathbb{F}_2 \left\{ Q^I \;\middle|\; \begin{array}{c} I = (i_1, \ldots, i_k) \text{ c.u.} \\ i_k \geq n \end{array} \right\}$$

$Q^I = Q^{i_1} \cdots Q^{i_k}$ sequence of D-L operations

c.u. = "completely unadmissible"

$$i_j > 2 i_{j+1}$$

Homology of $L(k)_n$

$H_* = H_*(-;\mathbb{F}_2)$

$$H_* L(k)_n \cong \mathbb{F}_2 \left\{ Q^I \;\middle|\; \begin{array}{c} I = (i_1, \dots, i_k) \text{ c.u.} \\ i_k \geq n \end{array} \right\}$$

$Q^I = Q^{i_1} \cdots Q^{i_k}$ sequence of D-L operations

c.u. = "completely unadmissible"

$$i_j > 2 i_{j+1}$$

Dual action of $\mathcal{A}$: Nishida Relations

Homology of $L(k)_n$

$$H_* L(k)_n \cong \mathbb{F}_2 \left\{ Q^I \;\middle|\; \begin{array}{l} I = (i_1, \dots, i_k) \text{ c.u.} \\ i_k \geq n \end{array} \right\}$$

$$0 \to H_* \Sigma^n L(k-1)_{2n+1} \longrightarrow H_* L(k)_n \longrightarrow H_* L(k)_{n+1} \to 0$$

$$\sigma^n Q^{i_1} \cdots Q^{i_{k-1}} \longmapsto Q^{i_1} \cdots Q^{i_{k-1}} Q^n$$

$$Q^{i_1} \cdots Q^{i_k} \longmapsto Q^{i_1} \cdots Q^{i_k}$$

$$i_k > n$$

Computing $\pi_* L(k)_n$: "AHSS's"

$$L(k)^m_n \longrightarrow L(k)_n \longrightarrow L(k)_{m+1}$$

$$\varinjlim_m L(k)^m_n = L(k)_n$$

Computing $\pi_* L(k)_n$: "AHSS's"

$$L(k)_n^m \to L(k)_n \longrightarrow L(k)_{m+1}$$

$$\varinjlim_m L(k)_n^m = L(k)_n$$

AHSS:

$$E_1 = \bigoplus_{m \geq n} \pi_t \Sigma^m L(k-1)_{2m+1} \Longrightarrow \pi_t L(k)_n$$

Computing $\pi_* L(k)_n$: "AHSS's"

Iterated AHSS:

$$\underbrace{\bigoplus \pi_t(S^{|I|}) \Rightarrow \cdots\cdots \Rightarrow \pi_t L(k)_n}_{k \text{ spectral sequences}}$$

$I = (i_1, \ldots, i_k)$
$i_k \geq n$

AHSS:

$$E_1 = \bigoplus_{m \geq n} \pi_t \Sigma^m L(k-1)_{2m+1} \Rightarrow \pi_t L(k)_n$$

Computing $\pi_* L(k)_n$: "AHSS's"

Iterated AHSS:

$$\bigoplus \pi_t(\underline{S}^{|I|}) \Rightarrow \cdots\cdots \Rightarrow \pi_t L(k)_n$$

$$\alpha \Rightarrow \cdots\cdots \Rightarrow \alpha[i_1,\dots,i_k]$$

$I=(i_1,\dots,i_k)$

$i_k \geq n$

k spectral sequences

AHSS:

$$E_1 = \bigoplus_{m\geq n} \pi_t \Sigma^m L(k-1)_{2m+1} \Longrightarrow \pi_t L(k)_n$$

=

Computing $\pi_* L(k)_n$: "AHSS's"

Iterated AHSS:

$$\alpha \Rightarrow \cdots\cdots \Longrightarrow \alpha[i_1, \ldots, i_k]$$

$$\bigoplus \pi_t(\underline{S}^{|I|}) \Rightarrow \underbrace{\cdots\cdots}_{k \text{ spectral seqences}} \Longrightarrow \pi_t L(k)_n$$

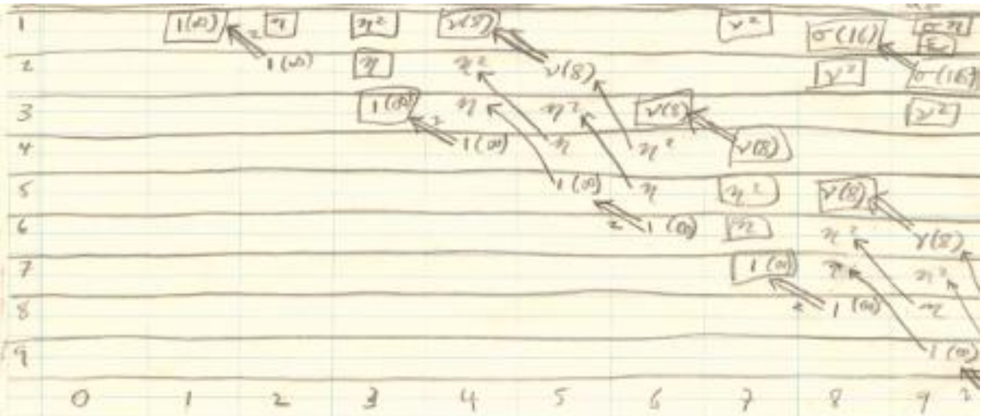
$I = (i_1, \ldots, i_k)$
$i_k \geq n$

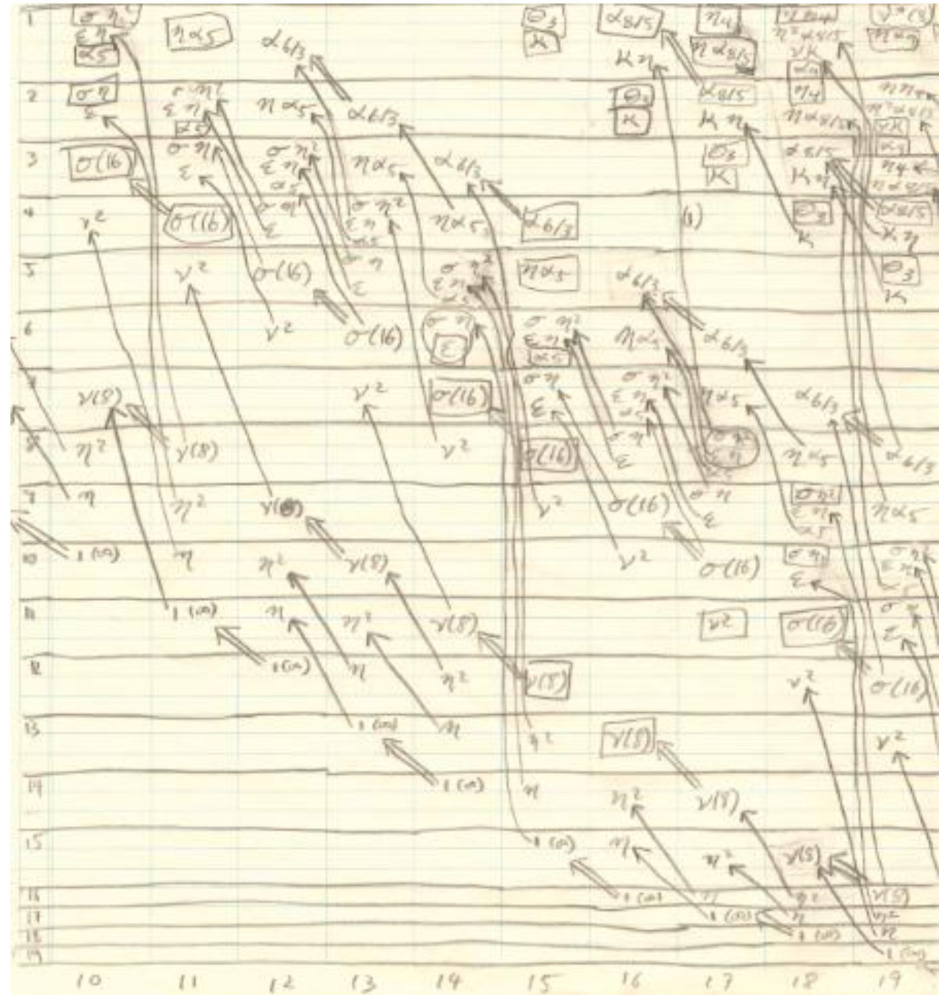
AHSS:

$$E_1 = \bigoplus_{m \geq n} \pi_t \Sigma^m L(k-1)_{2m+1} \Longrightarrow \pi_t L(k)_n$$

Diff'ls: attaching maps between "cells" $Q^{i_1} \cdots Q^{i_k}$ [largely determined by Steenrod action]

AHSS : $\pi_* \underline{S} \Longrightarrow \pi_* L(1)$,





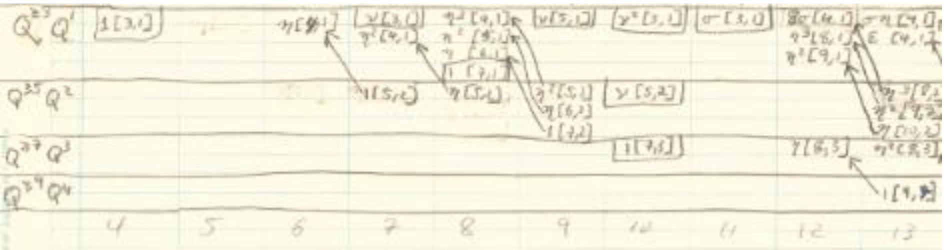
$1[1]$ $\eta[1]$ $\eta^2[1]$

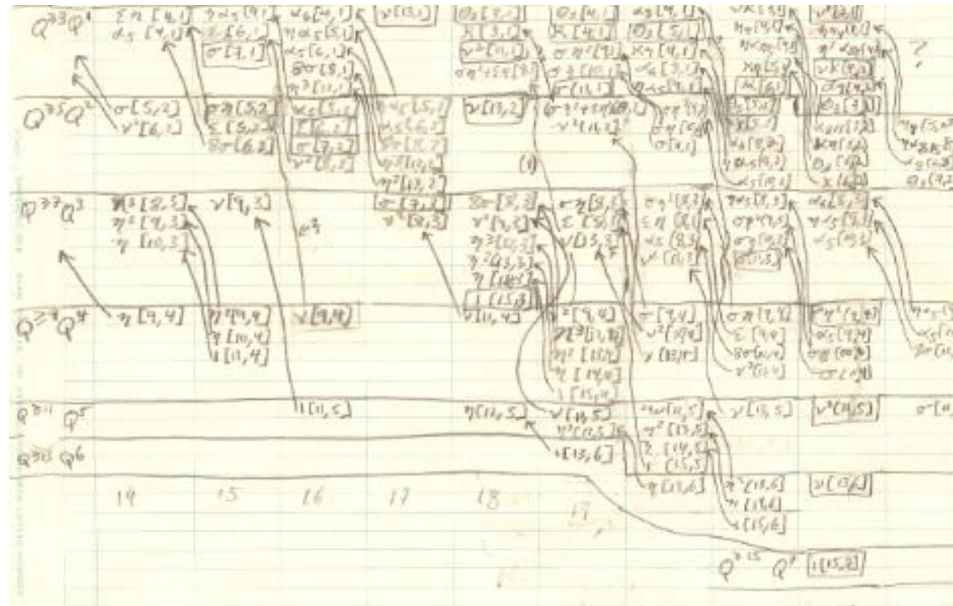
$\eta[2]$

$1[3]$

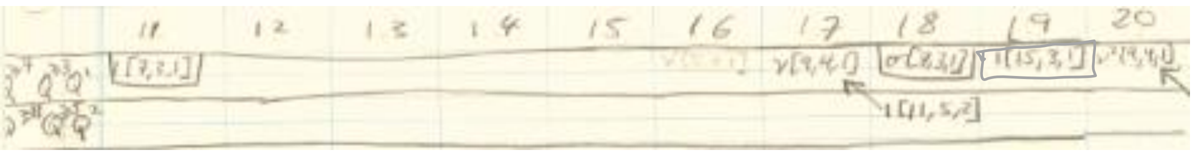
$\pi_1 P^\infty$ $\pi_2 P^\infty$ $\pi_3 P^\infty$

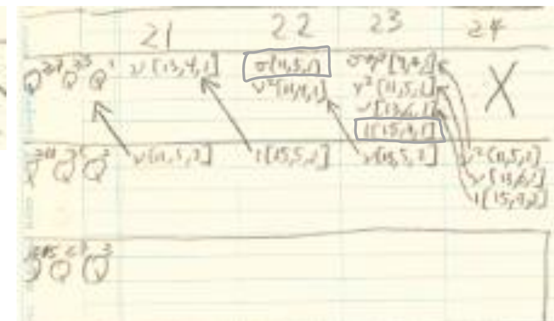
$\pi_* L(1)_{2m+1} \Longrightarrow \pi_* L(2)$





$$\pi_* L(2)_{2n+1} \Longrightarrow \pi_* L(3)$$





Summary so far:

We "understand" $E_1^{GSS} = \bigoplus_k \pi_* L(k)_n$

Need to understand diff'ls!

Summary so far:

We "understand" $E_1^{GSS} = \bigoplus_k \pi_* L(k)_n$

Need to understand diff'ls!

First: Understand meaning of "Goodwillie filtration"

[Note: Biedermann-Dwyer have a different perspective]

Understanding Goodwillie Filtration

Thm: Consider sequence of spectral sequences

$$\bigoplus_{k} \bigoplus_{\substack{(i_1,\dots,i_k) \\ i_k \geq n}} \pi_*(\underline{S}^{|I|}) \xRightarrow{\text{Iterated AHSS}} \cdots \Rightarrow \bigoplus_{k} \pi_* L(k)_n \xRightarrow{\text{GSS}} \pi_* S^n$$

$$\underset{\alpha}{\psi} \Rightarrow \cdots \Rightarrow \alpha[i_1,\dots,i_k] \Rightarrow \underset{\beta}{\psi}$$

Then:

$$\beta = E^{-l_k} P_{i_k} E^{-l_{k-1}} P_{i_{k-1}} E^{-l_{k-2}} P_{i_{k-2}} \cdots E^{-l_1} P_{i_1} \tilde{\alpha}$$

where $l_j = i_j - (2i_{j+1}+1)$, $l_k = i_k - n$, $\tilde{\alpha}$ stabilizes to α

Understanding Goodwillie filtration

i.e.,

$$
\begin{tikzcd}
\alpha & & \tilde{\alpha} \arrow[ll, mapsto] \arrow[d, mapsto] & & \\
& & \bullet & \bullet \arrow[l, mapsto] \arrow[d, mapsto] & \\
& & & \bullet & \bullet \arrow[l, mapsto] \arrow[d, mapsto] \\
& & & & \bullet \arrow[r, phantom, "\cdots"] & \bullet & \beta \arrow[l, mapsto]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\pi_* \underline{S} & \pi_* S^{2j_1+1} \arrow[l, "E^\infty"'] \arrow[d, "P"] & & & & \\
& \pi_* S^{j_1} & \pi_* S^{2j_2+1} \arrow[l, "E^{l_1}"] \arrow[d, "P"] & & & \\
& & \pi_* S^{j_2} & \pi_* S^{2j_3+1} \arrow[l, "E^{l_2}"] \arrow[d, "P", "d_1"'] & & \\
& & & \pi_* S^{j_3} \arrow[r, phantom, "\cdots"] & \pi_* S^{j_k} & \pi_* S^n \arrow[l, "E^{l_k}"]
\end{tikzcd}
$$

Slogan:

$$\left(\beta \text{ has goodwillie filtration } \geq 2^k\right) \iff \left(\beta \in \text{“}\mathrm{Im}\, P^k\text{”}\right)$$

Sketch of proof

Define fibers:

$$R_{2^k}(S^n) \longrightarrow S^n \longrightarrow P_{2^{k-1}}(S^n)$$

γ in Goodwillie filtration 2^k $\Longrightarrow$

$$\begin{array}{ccc} \tilde{\gamma} & \in & \pi_* R_{2^k}(S^{2n+1}) \\ \downarrow & & \downarrow \\ \gamma & \in & \pi_* S^{2n+1} \end{array}$$

Sketch of proof

γ in Goodwillie filtration 2^k $\Longrightarrow$

$$
\begin{tikzcd}
 & & & \delta[i_1,\dots,i_k] \neq 0 \\
 & & & \pi_* L(k)_{2n+1} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \\
\tilde{\gamma} \arrow[uurrr, mapsto] \arrow[dd, mapsto] & \in & \pi_* R_{2^k}(S^{2n+1}) \arrow[ur] \arrow[dd] & \\
 & & & \\
\gamma & \in & \pi_* S^{2n+1} &
\end{tikzcd}
$$

Sketch of proof

$$
\begin{tikzcd}
 & \delta[i_1,\dots,i_k,n] \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & & \\
 & \pi_* L(k+1)_n & & \delta[i_1,\dots,i_k] \arrow[ull, mapsto] \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] \\
 & & & \pi_* L(k)_{2n+1} \arrow[ull] \\
\pi_* R_{2^{k+1}}(S^n) \arrow[uur, bend left] \arrow[d] & & \tilde{\gamma} \in \arrow[uur, mapsto] \arrow[d, mapsto] & \pi_* R_{2^k}(S^{2n+1}) \arrow[u] \arrow[lll, bend left, "P"] \arrow[d] \\
\pi_* S^n \ni P(\gamma) & & \gamma \in \arrow[ll, mapsto] & \pi_* S^{2n+1} \arrow[lll, bend left, "P"]
\end{tikzcd}
$$

$\square$

Goodwillie diff'ls

Hopf invariant EHPSS:

$$\bigoplus_{n} \pi_{*} \Omega^{n+1} S^{2n+1} \Longrightarrow \pi_{*} \underline{S}$$

$$\begin{array}{ccc} \Psi & & \Psi \\ \alpha & \Longrightarrow & \beta \end{array}$$

$\alpha = HI(\beta)$

Goodwillie diff'ls

Hopf invariant EHPSS:

$$\bigoplus_n \pi_* \Omega^{n+1} S^{2n+1} \Longrightarrow \pi_* S$$

$$\alpha \in \bigoplus_n \pi_* \Omega^{n+1} S^{2n+1}, \qquad \beta \in \pi_* S, \qquad \alpha \Longrightarrow \beta$$

$$\alpha = HI(\beta)$$

$$\begin{tikzcd}
\tilde{\beta} \arrow[rr, mapsto] \arrow[d, bend right] & & \beta \\
\Omega^{n+1} S^{n+1} \arrow[rr] \arrow[d, "H"] & & QS^0 \\
\alpha \quad \Omega^{n+1} S^{2n+1} & &
\end{tikzcd}$$

Goodwillie diff'ls

$$QS^0 \xrightarrow[J-H]{} Q\mathbb{R}P^\infty$$

adjoint to

$$\Sigma^\infty QS^0 \longrightarrow \Sigma^\infty S^{\wedge 2}_{h\Sigma_2} \simeq \Sigma^\infty \mathbb{R}P^\infty$$

Goodwillie diff'ls

$$QS^0 \xrightarrow[\text{J-H}]{} Q\mathbb{R}P^\infty$$

adjoint to

$$\Sigma^\infty QS^0 \longrightarrow \Sigma^\infty \underline{S}^{\wedge 2}_{h\Sigma_2} \simeq \Sigma^\infty \mathbb{R}P^\infty$$

Stable Hopf invariant:

$$\begin{tikzcd}
 & & \pi_* \underline{S} \arrow[d, "\text{J-H}"] & \ni \beta \arrow[d, mapsto] \\
\bigoplus_m \pi_* \underline{S}^m \arrow[rr, Rightarrow, "\text{AHSS}"'] & & \pi_* P^\infty & \\
\ni \alpha \arrow[rrr, Rightarrow] & & & \text{J-H}(\beta)
\end{tikzcd}$$

$$\alpha = \text{SHI}(\beta)$$

Goodwillie diff'ls

Fundamental diagram

$$\begin{tikzcd}
\Omega^m S^m \arrow[r, "E"] \arrow[d, "JH"] & \Omega^{m+1} S^{m+1} \arrow[r, "H"] \arrow[d, "JH"] & \Omega^{m+1} S^{2m+1} \arrow[d, "E^\infty"] \\
Q\mathbb{R}P^{m-1} \arrow[r] & Q\mathbb{R}P^m \arrow[r] & QS^m
\end{tikzcd}$$

Deduce:

If $\quad E^\infty HI(\beta) \neq 0$

$$\Longrightarrow E^\infty HI(\beta) = SHI(\beta)$$

Goodwillie Diff'ls

Thm: For the Goodwillie spectral sequence

$$d_1(\alpha[i_1, \dots, i_k]) = SHI(\alpha)[m, i_1, \dots, i_k] + \text{lower terms}$$

where $SHI(\alpha)$ is carried by m-cell of $\mathbb{R}P^\infty$

Idea of proof:

First show the following diagram commutes:

$$\begin{tikzcd} QS \arrow[r, "J\text{-}H"'] \arrow[d, equal] & Q\mathbb{R}P^\infty \arrow[r] & Q\mathbb{R}P^\infty_n \arrow[d, equal] \\ \Omega^\infty L(0)_n \arrow[rr, "d_1"'] & & \Omega^\infty L(1)_n \end{tikzcd}$$

(This establishes the claim for $k=0$)

Idea of proof

$$
\begin{tikzcd}
\Omega D_{2^k}(S^{n+1}) \arrow[r, "d_1"] \arrow[d, phantom, "\cong" rotate=90] & D_{2^{k+1}}(S^{n+1}) \arrow[d, phantom, "\cong" rotate=90] \\
D_{2^k}(\Omega\Sigma_1)(S^n) \arrow[r, "d_1"] \arrow[d, "H"] & B D_{2^{k+1}}(\Omega\Sigma_1)(S^n) \arrow[d, "H"] \\
D_{2^k}(\Omega\Sigma_1 S_q)(S^n) \arrow[r, "d_1"'] \arrow[d, phantom, "\cong" rotate=90] & B D_{2^{k+1}}(\Omega\Sigma_1 S_q)(S^n) \arrow[d, phantom, "\cong" rotate=90] \\
\Omega D_{2^{k-1}}(S^{2n+1}) \arrow[r, "d_1"'] & D_{2^k}(S^{2n+1})
\end{tikzcd}
$$

(relates k to $k-1$)

$\square$

TABLE 5. The GSS for $\pi_{n+1}(S^1)$

n	$\pi_n(L(0))$	$\pi_{n-1}(L(1))$	$\pi_{n-2}(L(2))$	$\pi_{n-3}(L(3))$
0	$1(\infty)$	$1[1]$		
1	η	$\eta[1]$		
2	η^2	$\eta^2[1]$		
		$\eta[2]$		
		$1[3]$	$1[3,1]$	
3	4ν	$\nu[1]$		
	2ν			
	ν			
4				
5		$\nu[3]$	$\nu[3,1]$	
6	ν^2	$\nu^2[1]$		
		$\eta^3[4]$		
		$\eta^2[5]$		
		$\eta[6]$		
		$1[7]$	$1[7,1]$	
7	8σ	$\sigma[1]$		
	4σ			
	2σ	$\nu^2[2]$		
	σ	$\nu[5]$	$\nu[5,1]$	
8	ϵ	$\sigma\eta[1]$		
	$\sigma\eta$	$\epsilon[1]$	$\nu^2[3,1]$	
		$\nu^2[3]$	$\nu[5,2]$	
		$8\sigma[2]$	$1[7,3]$	$1[7,3,1]$
9	$\epsilon\eta$	$\sigma\eta^2[1]$	$\sigma[3,1]$	
	$\sigma\eta^2$	$\sigma\eta[2]$		
	α_5	$\sigma[3]$		
		$\alpha_5[1]$		
10	$\eta\alpha_5$	$\eta\alpha_5[1]$		
		$\alpha_5[2]$		
		$8\sigma[4]$		
11	α_6			
	$\alpha_{6/2}$			
	$\alpha_{6/3}$			
12				
13		$\epsilon[6]$	$\sigma[7,1]$	
		$\sigma[7]$		
	κ	$\theta_3[1]$		
	θ_3			

n	$\pi_n(L(0))$	$\pi_{n-1}(L(1))$	$\pi_{n-2}(L(2))$	$\pi_{n-3}(L(3))$
		$\epsilon[6]$		
		$\sigma[7]$	$\sigma[7,1]$	
14	κ	$\theta_3[1]$		
	θ_3	$\kappa[1]$		
		$\alpha_8[4]$		
		$\eta\alpha_5[5]$		
		$\alpha_5[6]$		
		$8\sigma[8]$	$\sigma[7,2]$	
		$\eta^3[12]$	$\epsilon[6,2]$	
15	$\eta\kappa$	$\theta_3[2]$		
	α_8	$\kappa[2]$		
	$\alpha_{8/2}$			
	$\alpha_{8/3}$			
	$\alpha_{8/4}$	$\nu[13]$	$\sigma[7,3]$	
	$\alpha_{8/5}$	$\alpha_{8/5}[1]$	$\nu[13,1]$	$\sigma[7,3,1]$
16	η_4	$\theta_3[3]$	$\theta_3[3,1]$	
	$\eta\alpha_{8/5}$	$\eta_4[1]$		
		$\nu^2[11]$		
		$\eta\alpha_{8/5}[1]$	$\nu^2[11,1]$	
		$\kappa[3]$	$\nu[13,2]$	
		$\alpha_8[2]$	$1[15,3]$	$1[15,3,1]$
17	$\eta\eta_4$	$\eta\eta_4[1]$	$\theta_3[4,1]$	
	$\eta^2\alpha_{8/5}$	$\eta_4[2]$		
	$\nu\kappa$	$\theta_3[4]$		
	α_9	$\sigma\eta^2[9]$		
		$\sigma\eta[10]$		
		$\sigma[11]$	$\kappa[4,1]$	
		$\alpha_9[1]$	$\sigma[11,1]$	
18	$4\nu^*$	$\nu\kappa[2]$		
	$2\nu^*$	$\nu^*[1]$		
	ν^*	$\theta_3[5]$		
	$\eta\alpha_9$	$\eta\alpha_9[1]$		
		$\alpha_9[2]$		
		$\alpha_8[4]$	$\theta_3[5,1]$	
19	$\overline{\sigma}$	$\overline{\sigma}[1]$		
	α_{10}	$\kappa\nu[3]$	$\kappa[6,1]$	
	$\alpha_{10/2}$	$\kappa\eta[5]$	$\theta_3[5,2]$	
	$\alpha_{10/3}$	$\kappa[6]$	$\sigma[11,3]$	$\sigma[11,3,1]$
	$4\overline{\kappa}$	$\overline{\kappa}[1]$	$\nu^*[3,1]$	
	$2\overline{\kappa}$	$\overline{\sigma}[2]$		
	$\overline{\kappa}$	$\nu^*[3]$		

only red diff'ls don't follow

Goodwillie - Whitehead conj

The GSS

$$E_1 = \bigoplus_k \pi_* L(k)_1 \Longrightarrow \pi_* S^1$$

collapses at E_2

(Arone
Dwyer
Kuhn
Lesh
Mahowald)

Table 6. The GSS for $\pi_{n+2}(S^2)$

n	$\pi_n(L(0)_2)$	$\pi_{n-1}(L(1)_2)$	$\pi_{n-2}(L(2)_2)$	$\pi_{n-3}(L(3)_2)$
0	$1(\infty)$			
1	η	$1(\infty)[2]$		
2		$\eta[2]$		
	η^2	$1[3]$		
3	2ν			
	ν			
	4ν			
4		$2\nu[2]$		
		$\nu[2]$		
5		$\nu[3]$	$1[5,2]$	
6	ν^2	$\eta^3[4]$		
		$\eta^2[5]$		
		$\eta[6]$		
		$1[7]$	$\eta[5,2]$	
7	8σ			
	4σ		$\eta^2[5,2]$	
	2σ	$\nu^2[2]$	$\eta[6,2]$	
	σ	$\nu[5]$	$1[7,2]$	
8	ϵ	$4\sigma[2]$		
	$\sigma\eta$	$2\sigma[2]$		
		$\sigma[2]$		
		$8\sigma[2]$	$\nu[5,2]$	
		$\nu^2[3]$	$1[7,3]$	
9	α_5	$\sigma\eta[2]$		
	$\sigma\eta^2$	$\sigma[3]$		
	$\epsilon\eta$			
10		$4\nu[8]$		
		$\alpha_5[2]$		
	$\eta\alpha_5$	$8\sigma[4]$		
11	$\alpha_{6/2}$		$\eta^2[9,2]$	
	$\alpha_{6/3}$		$\eta[10,2]$	
	α_6		$\eta^3[8,2]$	
12		$\epsilon\eta[4]$		
		$\alpha_{6/2}[2]$	$\sigma[5,2]$	
		$\alpha_{6/3}[2]$	$\nu^2[6,2]$	
13		$\epsilon[6]$	$\epsilon[5,2]$	
		$\sigma[7]$	$8\sigma[6,2]$	
14	κ	$\alpha_5[4]$	$\alpha_5[5,2]$	
	θ_3	$\eta\alpha_5[5]$		
		$\alpha_5[6]$	$\epsilon[6,2]$	
		$8\sigma[8]$	$\sigma[7,2]$	
		$4\nu[12]$	$\nu^2[8,2]$	
15	α_8	$\kappa[2]$	$\eta\alpha_5[5,2]$	
	$\alpha_{8/2}$	$\theta_3[2]$	$\alpha_5[6,2]$	
	$\alpha_{8/3}$		$8\sigma[8,2]$	
	$\alpha_{8/4}$		$\eta^3[12,2]$	
	$\alpha_{8/5}$		$\sigma[7,3]$	
	$\kappa\eta$	$\nu[13]$	$\eta^2[13,2]$	$1[11,5,2]$
16	η_4	$\alpha_{8/2}[2]$		
	$\eta\alpha_{8/5}$	$\alpha_{8/3}[2]$		
		$\alpha_{8/4}[2]$		
		$\alpha_{8/5}[2]$		
		$\theta_3[3]$		
		$(\sigma\eta^2+\epsilon\eta)[8]$		
		$\alpha_8[2]$		
		$\kappa[3]$	$\nu[13,2]$	
		$\nu^2[11]$	$1[15,3]$	
17	α_9	$\eta_4[2]$	$\nu^2[9,2]$	
	$\nu\kappa$	$\theta_3[4]$		
	$\eta\eta_4$	$\sigma\eta^2[9]$		
		$\sigma\eta[10]$		
	$\eta^2\alpha_{8/5}$	$\sigma[11]$	$(\sigma\eta^2+\epsilon\eta)[8,2]$	
18	$4\nu^*$	$\kappa\eta[4]$		
	$2\nu^*$	$\nu\kappa[2]$		
	ν^*	$\alpha_6[8]$		
		$\alpha_9[2]$		
		$\alpha_8[4]$	$\sigma\eta[10,2]$	
	$\eta\alpha_9$	$\theta_3[5]$	$\sigma[11,2]$	$\nu[11,5,2]$
19	$\alpha_{10/2}$	$2\nu^*[2]$	$\kappa[5,2]$	
	$\alpha_{10/3}$	$\nu^*[2]$	$\eta\alpha_5[9,2]$	
	$\overline{\sigma}$	$\nu\kappa[3]$	$\alpha_5[10,2]$	
		$\kappa\eta[5]$	$\alpha_6[8,2]$	
	α_{10}	$\kappa[6]$	$\sigma[11,3]$	$1[15,5,2]$
	$4\overline{\kappa}$	$\eta^2\alpha_{8/5}$	$\theta_3[6,2]$	
	$2\overline{\kappa}$	$\nu^*[3]$		
	$\overline{\kappa}$			

Longer diff'ls

Thm: Suppose

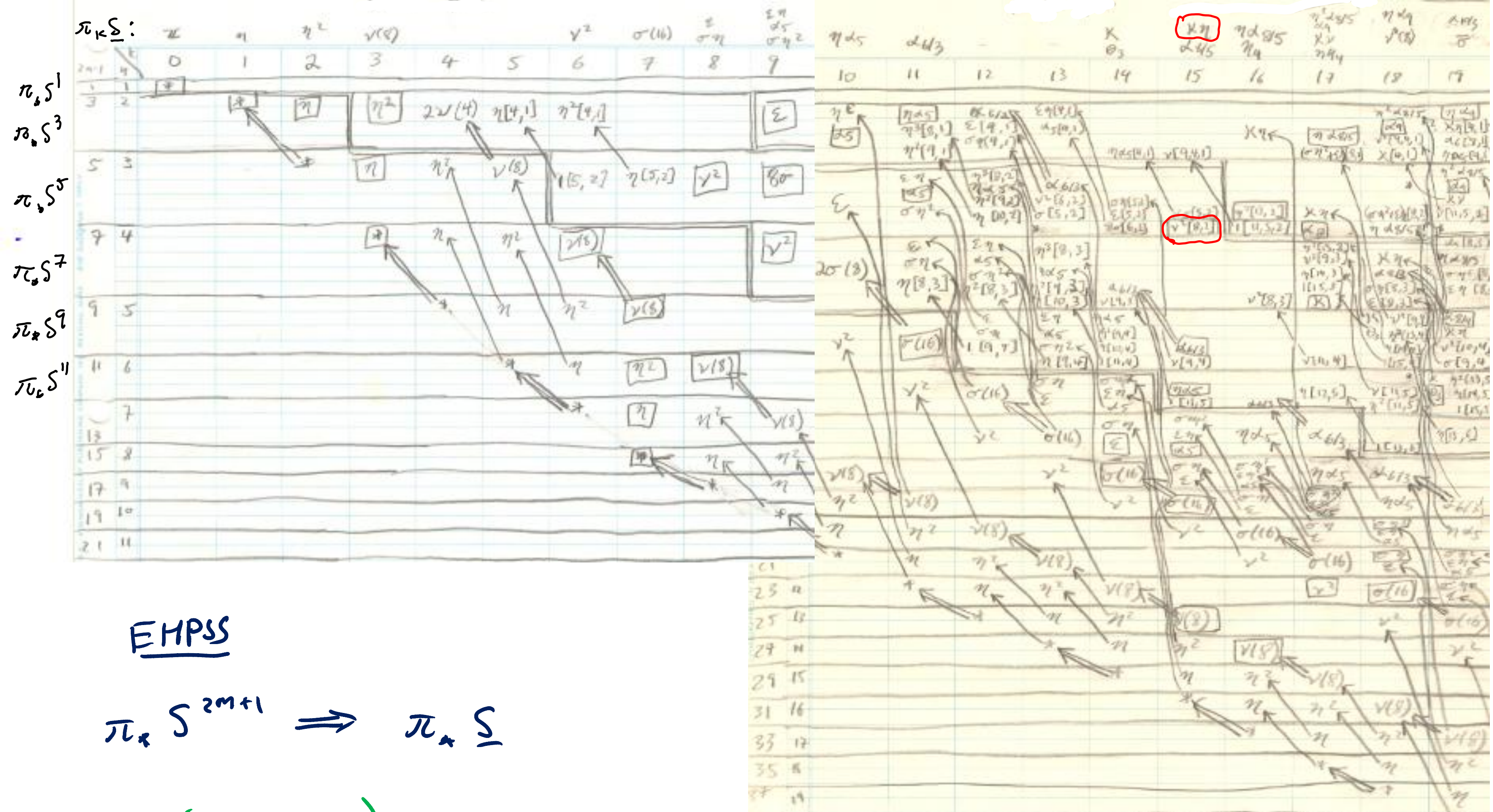
(1) $\alpha[i_1, \ldots, i_k]$ persists to E_r^{GSS}

(2) $HI(\alpha) \in \pi_* S^{2m+1}$ has Goodwillie filtration 2^{r-1} detected by $\gamma[j_1, \ldots, j_{r-1}]$

Then: $d_r(\alpha[i_1, \ldots, i_k]) = \gamma[j_1, \ldots, j_{r-1}, m, i_1, \ldots, i_k]$ + lower terms

TABLE 6. The GSS for $\pi_{n+2}(S^2)$

n	$\pi_n(L(0)_2)$	$\pi_{n-1}(L(1)_2)$	$\pi_{n-2}(L(2)_2)$	$\pi_{n-3}(L(3)_2)$
0	$1(\infty)$			
1	η	$1(\infty)[2]$		
2		$\eta[2]$		
	η^2	$1[3]$		
3	2ν			
	ν			
	4ν			
4		$2\nu[2]$		
		$\nu[2]$		
5		$\nu[3]$	$1[5,2]$	
6	ν^2	$\eta^3[4]$		
		$\eta^2[5]$		
		$\eta[6]$		
		$1[7]$	$\eta[5,2]$	
7	8σ			
	4σ		$\eta^2[5,2]$	
	2σ	$\nu^2[2]$	$\eta[6,2]$	
	σ	$\nu[5]$	$1[7,2]$	
8	ϵ	$4\sigma[2]$		
	$\sigma\eta$	$2\sigma[2]$		
		$\sigma[2]$		
		$8\sigma[2]$	$\nu[5,2]$	
		$\nu^2[3]$	$1[7,3]$	
9	α_5	$\sigma\eta[2]$		
	$\sigma\eta^2$	$\sigma[3]$		
	$\epsilon\eta$			
10		$4\nu[8]$		
		$\alpha_5[2]$		
	$\eta\alpha_5$	$8\sigma[4]$		
11	$\alpha_{6/2}$		$\eta^2[9,2]$	
	$\alpha_{6/3}$		$\eta[10,2]$	
	α_6		$\eta^3[8,2]$	
12		$\epsilon\eta[4]$		
		$\alpha_{6/2}[2]$	$\sigma[5,2]$	
		$\alpha_{6/3}[2]$	$\nu^2[6,2]$	
13		$\epsilon[6]$	$\epsilon[5,2]$	
		$\sigma[7]$	$8\sigma[6,2]$	
14	κ	$\alpha_6[4]$	$\alpha_5[5,2]$	
	θ_3	$\eta\alpha_5[5]$		
		$\alpha_5[6]$	$\epsilon[6,2]$	
		$8\sigma[8]$	$\sigma[7,2]$	
		$4\nu[12]$	$\nu^2[8,2]$	
15	α_8	$\kappa[2]$	$\eta\alpha_5[5,2]$	
	$\alpha_{8/2}$	$\theta_3[2]$	$\alpha_5[6,2]$	
	$\alpha_{8/3}$		$8\sigma[8,2]$	
	$\alpha_{8/4}$		$\eta^3[12,2]$	
	$\alpha_{8/5}$		$\sigma[7,3]$	
	$\kappa\eta$	$\nu[13]$	$\eta^2[13,2]$	$1[11,5,2]$
16	η_4	$\alpha_{8/2}[2]$		
	$\eta\alpha_{8/5}$	$\alpha_{8/3}[2]$		
		$\alpha_{8/4}[2]$		
		$\alpha_{8/5}[2]$		
		$\theta_3[3]$		
		$(\sigma\eta^2+\epsilon\eta)[8]$		
		$\alpha_8[2]$		
		$\kappa[3]$	$\nu[13,2]$	
		$\nu^2[11]$	$1[15,3]$	
17	α_9	$\eta_4[2]$	$\nu^2[9,2]$	
	$\nu\kappa$	$\theta_3[4]$		
	$\eta\eta_4$	$\sigma\eta^2[9]$		
		$\sigma\eta[10]$		
	$\eta^2\alpha_{8/5}$	$\sigma[11]$	$(\sigma\eta^2+\epsilon\eta)[8,2]$	
18	$4\nu^*$	$\kappa\eta[4]$		
	$2\nu^*$	$\nu\kappa[2]$		
	ν^*	$\alpha_6[8]$		
		$\alpha_9[2]$		
		$\alpha_8[4]$	$\sigma\eta[10,2]$	
	$\eta\alpha_9$	$\theta_3[5]$	$\sigma[11,2]$	$\nu[11,5,2]$
19	$\alpha_{10/2}$	$2\nu^*[2]$	$\kappa[5,2]$	
	$\alpha_{10/3}$	$\nu^*[2]$	$\eta\alpha_5[9,2]$	
	$\overline{\sigma}$	$\nu\kappa[3]$	$\alpha_5[10,2]$	
		$\kappa\eta[5]$	$\alpha_6[8,2]$	
	α_{10}	$\kappa[6]$	$\sigma[11,3]$	$1[15,5,2]$
	$4\overline{\kappa}$	$\eta^2\alpha_{8/5}$	$\theta_3[6,2]$	
	$2\overline{\kappa}$	$\nu^*[3]$		
	$\overline{\kappa}$			



EHPSS

$$\pi_* S^{2m+1} \Longrightarrow \pi_* \underline{S}$$

$$(* =: \mathbb{Z})$$

TABLE 6. The GSS for $\pi_{n+2}(S^2)$

n	$\pi_n(L(0)_2)$	$\pi_{n-1}(L(1)_2)$	$\pi_{n-2}(L(2)_2)$	$\pi_{n-3}(L(3)_2)$
0	$1(\infty)$			
1	η	$1(\infty)[2]$		
2		$\eta[2]$		
	η^2	$1[3]$		
3	2ν			
	ν			
	4ν			
4		$2\nu[2]$		
		$\nu[2]$		
5		$\nu[3]$	$1[5,2]$	
6	ν^2	$\eta^3[4]$		
		$\eta^2[5]$		
		$\eta[6]$		
		$1[7]$	$\eta[5,2]$	
7	8σ			
	4σ		$\eta^2[5,2]$	
	2σ	$\nu^2[2]$	$\eta[6,2]$	
	σ	$\nu[5]$	$1[7,2]$	
8	ϵ	$4\sigma[2]$		
	$\sigma\eta$	$2\sigma[2]$		
		$\sigma[2]$		
		$8\sigma[2]$	$\nu[5,2]$	
		$\nu^2[3]$	$1[7,3]$	
9	α_5	$\sigma\eta[2]$		
	$\sigma\eta^2$	$\sigma[3]$		
	$\epsilon\eta$			
10		$4\nu[8]$		
		$\alpha_5[2]$		
	$\eta\alpha_5$	$8\sigma[4]$		
11	$\alpha_{6/2}$		$\eta^2[9,2]$	
	$\alpha_{6/3}$		$\eta[10,2]$	
	α_6		$\eta^3[8,2]$	
12		$\epsilon\eta[4]$		
		$\alpha_{6/2}[2]$	$\sigma[5,2]$	
		$\alpha_{6/3}[2]$	$\nu^2[6,2]$	
13		$\epsilon[6]$	$\epsilon[5,2]$	
		$\sigma[7]$	$8\sigma[6,2]$	
14	κ	$\alpha_6[4]$	$\alpha_5[5,2]$	
	θ_3	$\eta\alpha_5[5]$		
		$\alpha_5[6]$	$\epsilon[6,2]$	
		$8\sigma[8]$	$\sigma[7,2]$	
		$4\nu[12]$	$\nu^2[8,2]$	
15	α_8	$\kappa[2]$	$\eta\alpha_5[5,2]$	
	$\alpha_{8/2}$	$\theta_3[2]$	$\alpha_5[6,2]$	
	$\alpha_{8/3}$		$8\sigma[8,2]$	
	$\alpha_{8/4}$		$\eta^3[12,2]$	
	$\alpha_{8/5}$		$\sigma[7,3]$	
	$\kappa\eta$	$\nu[13]$	$\eta^2[13,2]$	$1[11,5,2]$
16	η_4	$\alpha_{8/2}[2]$		
	$\eta\alpha_{8/5}$	$\alpha_{8/3}[2]$		
		$\alpha_{8/4}[2]$		
		$\alpha_{8/5}[2]$		
		$\theta_3[3]$		
		$(\sigma\eta^2+\epsilon\eta)[8]$		
		$\alpha_8[2]$		
		$\kappa[3]$	$\nu[13,2]$	
		$\nu^2[11]$	$1[15,3]$	
17	α_9	$\eta_4[2]$	$\nu^2[9,2]$	
	$\nu\kappa$	$\theta_3[4]$		
	$\eta\eta_4$			
		$\sigma\eta^2[9]$		
		$\sigma\eta[10]$		
	$\eta^2\alpha_{8/5}$	$\sigma[11]$	$(\sigma\eta^2+\epsilon\eta)[8,2]$	
18	$4\nu^*$	$\kappa\eta[4]$		
	$2\nu^*$	$\nu\kappa[2]$		
	ν^*	$\alpha_6[8]$		
		$\alpha_9[2]$		
		$\alpha_8[4]$	$\sigma\eta[10,2]$	
	$\eta\alpha_9$	$\theta_3[5]$	$\sigma[11,2]$	$\nu[11,5,2]$
19	$\alpha_{10/2}$	$2\nu^*[2]$	$\kappa[5,2]$	
	$\alpha_{10/3}$	$\nu^*[2]$	$\eta\alpha_5[9,2]$	
	$\overline{\sigma}$		$\alpha_5[10,2]$	
		$\nu\kappa[3]$		
		$\kappa\eta[5]$	$\alpha_6[8,2]$	
	α_{10}	$\kappa[6]$	$\sigma[11,3]$	$1[15,5,2]$
	$4\overline{\kappa}$	$\eta^2\alpha_{8/5}$	$\theta_3[6,2]$	
	$2\overline{\kappa}$	$\nu^*[3]$		
	$\overline{\kappa}$			

Summary:

EHP sequence: E_1 term = potential Hopf invariants

diff'ls = "Whitehead products"

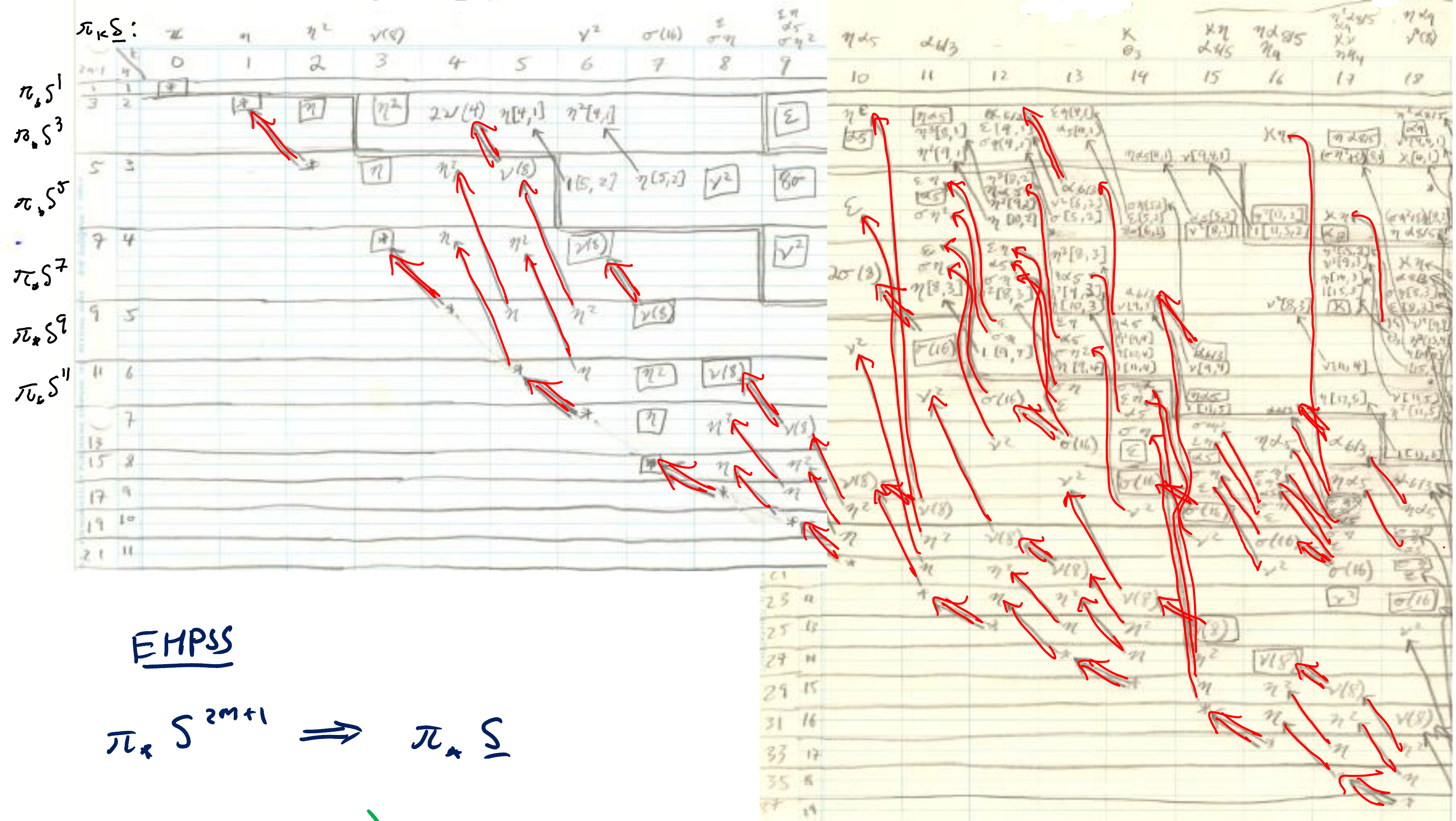
GSS: E_1-term = potential "Whitehead products"

diff'ls = Hopf invariants

Computing diff'ls in EHPSS

$$\begin{tikzcd}
\Omega^m S^m \arrow[r] \arrow[d, "JH"] & \Omega^{m+1} S^{m+1} \arrow[r] \arrow[d, "JH"] & \Omega^{m+1} S^{2m+1} \arrow[d, "E^\infty"] \\
Q\mathbb{R}P^{m-1} \arrow[r] & Q\mathbb{R}P^{m} \arrow[r] & QS^m
\end{tikzcd}$$

Consequence: Can "lift" diff'ls from the AHSS for $\mathbb{R}P^\infty$ to the EHPSS



EHPSS

$$\pi_* S^{2m+1} \Longrightarrow \pi_* \underline{S}$$

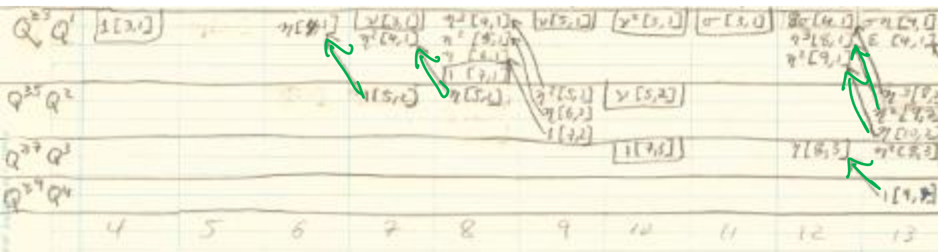
$(* =: \mathbb{Z})$

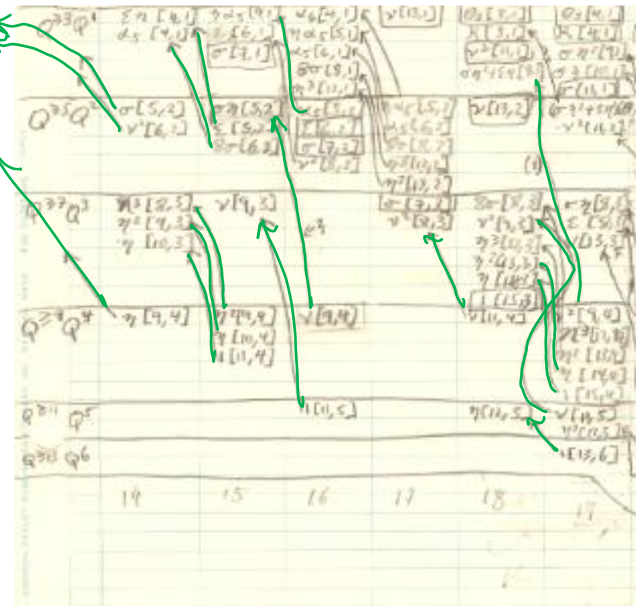
Can lift diff'ls from AHSS

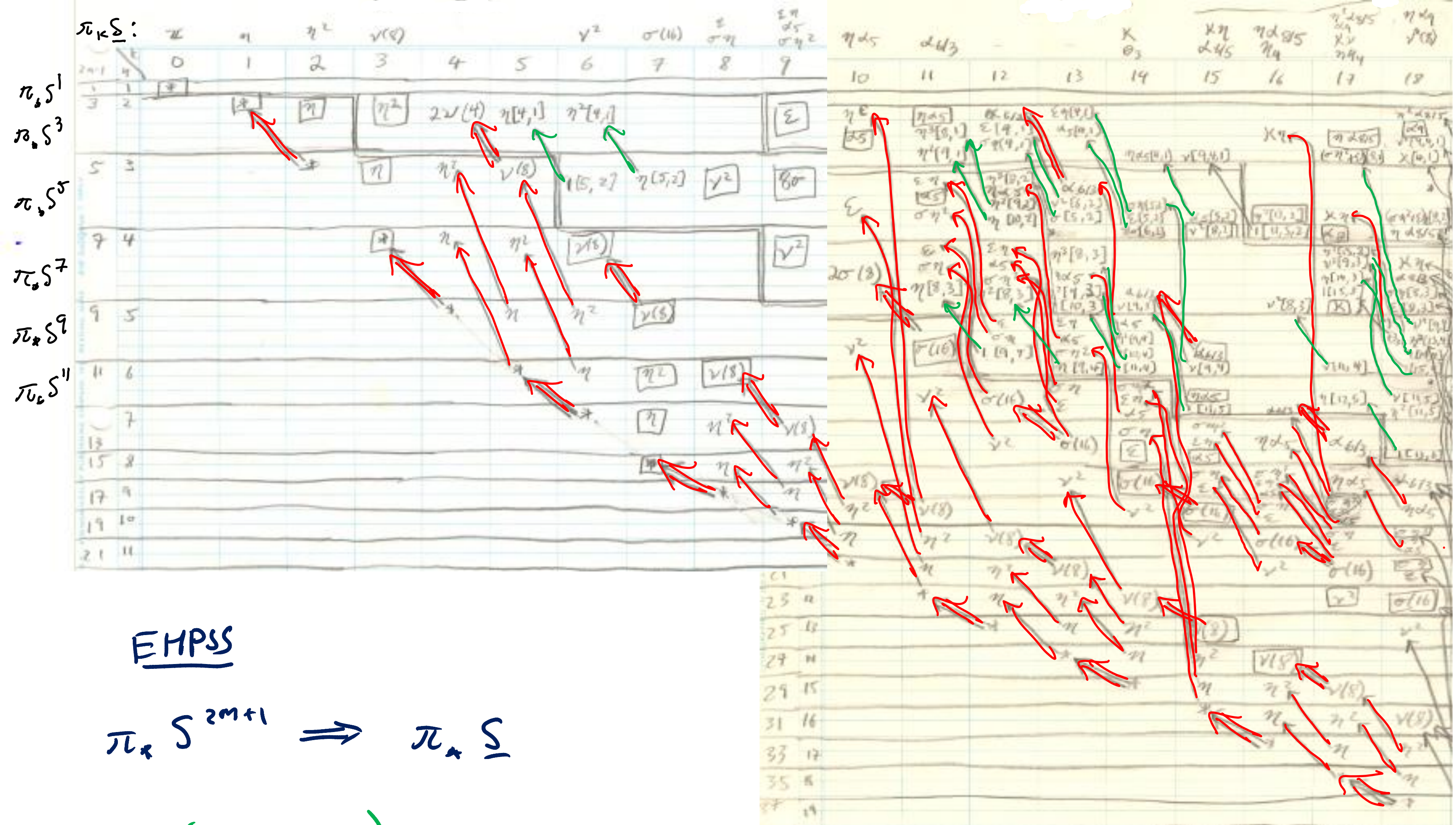
$$\bigoplus_{m} \pi_* L(k-1)_{2m+1} \Longrightarrow \pi_* L(k)_1$$

to the EHPSS

$$\pi_* L(1)_{2m+1} \Longrightarrow \pi_* L(2)$$





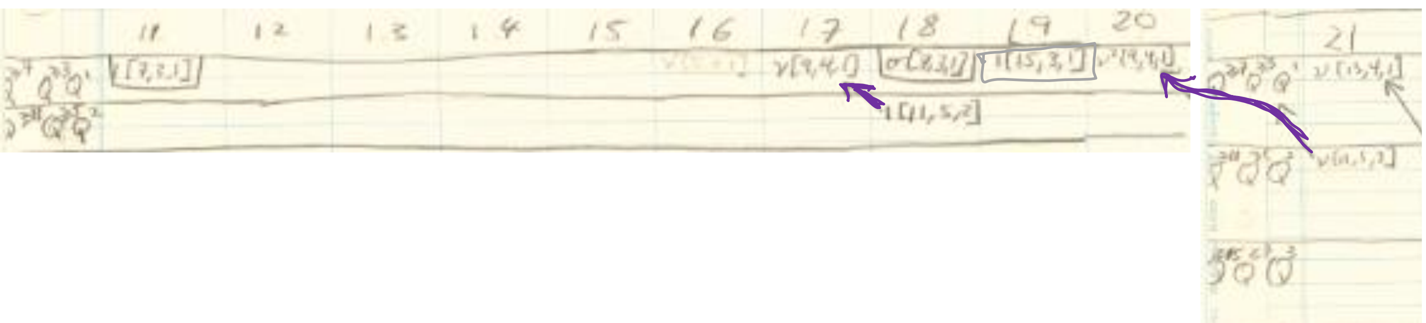


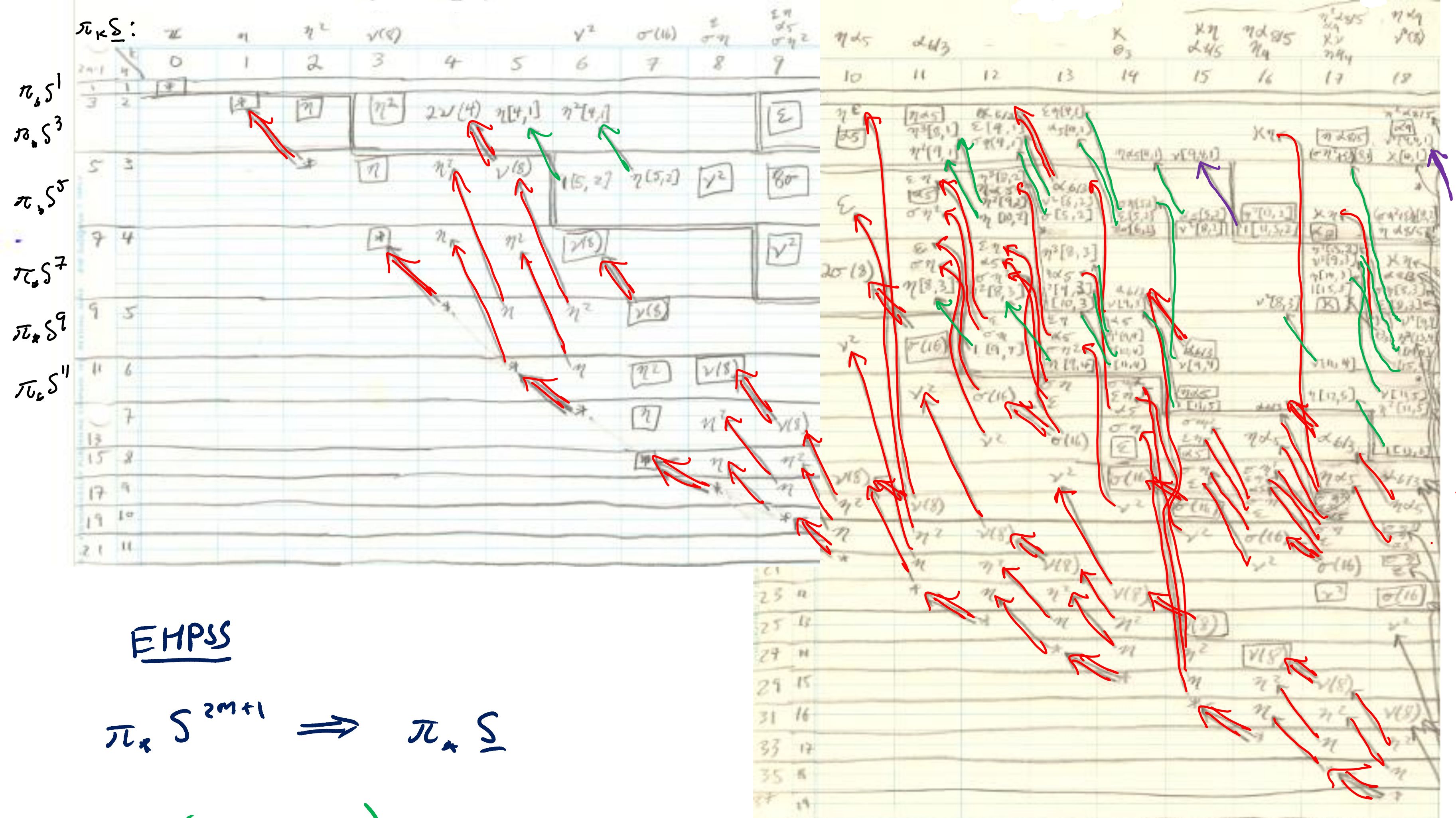
EHPSS

$$\pi_* S^{2m+1} \Rightarrow \pi_* \underline{S}$$

$(* =: \mathbb{Z})$

$$\pi_* L(2)_{2n+1} \Longrightarrow \pi_* L(3)$$





EHPSS

$$\pi_* S^{2m+1} \Rightarrow \pi_* \underline{S}$$

$(* =: \mathbb{Z})$