Exotic spheres and topological modular forms

Mark Behrens (MIT)

(joint with Mike Hill, Mike Hopkins, and Mark Mahowald)

Poincaré Conjecture

Q: Is every homotopy n-sphere homeomorphic
to an n-sphere?

A: Yes!

- $n = 2$: easy.
- $n \geq 5$: (Smale, 1961) h-cobordism theorem
- $n = 4$: (Freedman, 1982)
- $n = 3$: (Perelman, 2003)

Smooth Poincaré Conjecture

Q: Is every homotopy n-sphere diffeomorphic to an n-sphere?

A: Depends on n.

- $n = 2$: True - easy.
- $n = 7$: (Milnor, 1956) False – produced a smooth manifold which was homeomorphic but not diffeomorphic to S^7! [exotic sphere]
- $n \geq 5$: (Kervaire-Milnor, 1963) – `often’ false.
  (true for $n = 5{,}6$).
- $n = 3$: (Perelman, 2003) True.
- $n = 4$: Unknown.

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- $n \geq 5$: (Kervaire-Milnor, 1963) – `often' false. (true for $n = 5,6$).
- $n = 3$: (Perelman, 2003) True.
- $n = 4$: Unknown.

← Goal for this talk

Main Question

For which *n* do there exist exotic *n*-spheres?

Kervaire-Milnor

Θ_n $\coloneqq$ {oriented smooth homotopy *n*-spheres}/h-cobordism

(note: if $n \neq 4$, h-cobordant $\Leftrightarrow$ oriented diffeomorphic)

For $n \not\equiv 2(4)$:

$$0 \to \Theta_n^{bp} \to \Theta_n \to \frac{\pi_n^s}{Im\, J} \to 0$$

Kervaire-Milnor

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Θ_n^{bp} = subgroup of those which bound a parallelizable manifold

π_n^s = stable homotopy groups of spheres

$J\colon \pi_n(SO) \to \pi_n^s$ is the J-homomorphism.

Kervaire-Milnor

Θ_n $:=$ {oriented smooth homotopy *n*-spheres}/h-cobordism

$\pi_n^s = \Omega_n^{fr}$

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framed surgery

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Θ_n ≔ {oriented smooth homotopy *n*-spheres}/h-cobordism

For $n \not\equiv 2(4)$:

$$0 \to \Theta_n^{bp} \to \Theta_n \to \frac{\pi_n^s}{Im\, J} \to 0$$

For $n \equiv 2(4)$:

$$0 \to \Theta_n^{bp} \to \Theta_n \to \frac{\pi_n^s}{Im\, J} \to \mathbb{Z}/2 \to \Theta_{n-1}^{bp} \to 0$$

$$\rotatebox{90}{$\in$} \qquad \rotatebox{90}{$\in$}$$

$$[M] \longmapsto \Phi_K(M)$$

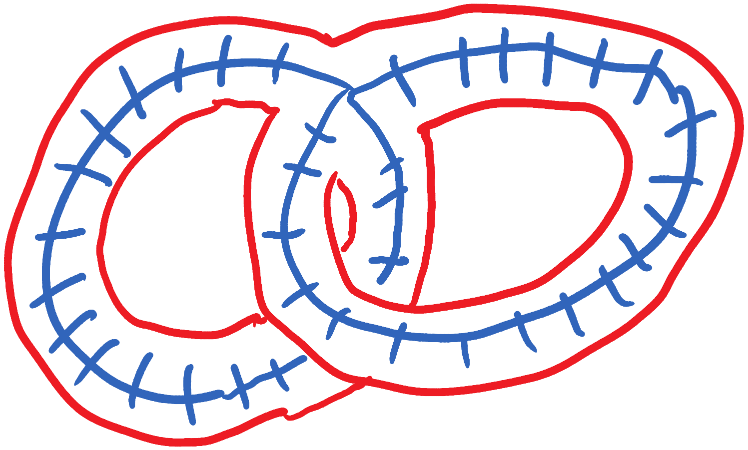
Kervaire Invariant

Θ_n^{bp}

- Trivial for *n* even
- Cyclic for *n* odd

Θ_n^{bp}

- Trivial for *n* even
- Cyclic for *n* odd
  - Generated by boundary of an explicit parallelizable manifold given by plumbing construction



Θ_n^{bp}

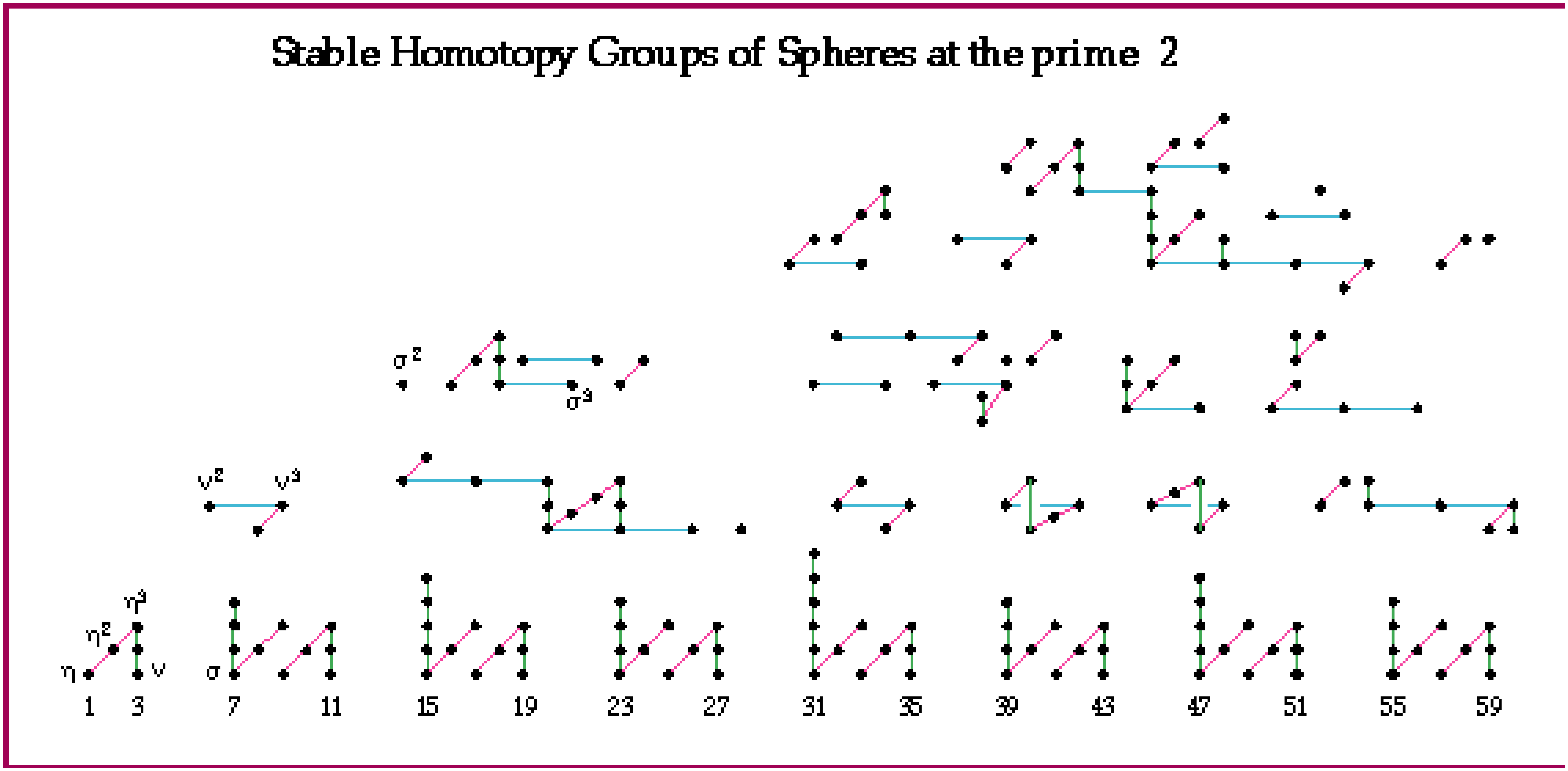
- Trivial for *n* even
- Cyclic for *n* odd:

$$|\Theta_n^{bp}| = \begin{cases} 2^{2k}(2^{2k+1}-1)num\left(\frac{4B_{k+1}}{k+1}\right), n = 4k+3 \\ \mathbb{Z}/_2, \qquad n \equiv 1(4), \exists M^{n+1}\ with\ \Phi_K = 1 \\ 0, \qquad n \equiv 1(4), \nexists M^{n+1}\ with\ \Phi_K = 1 \end{cases}$$

Upshot: *n* even ⇒ bp gives no exotic spheres

$n \equiv 3\ (4)$ ⇒ bp gives exotic spheres ($n \geq 7$)

$n \equiv 1\ (4)$ ⇒ bp gives exotic sphere only if there are no M^{n+1} with $\Phi_K = 1$

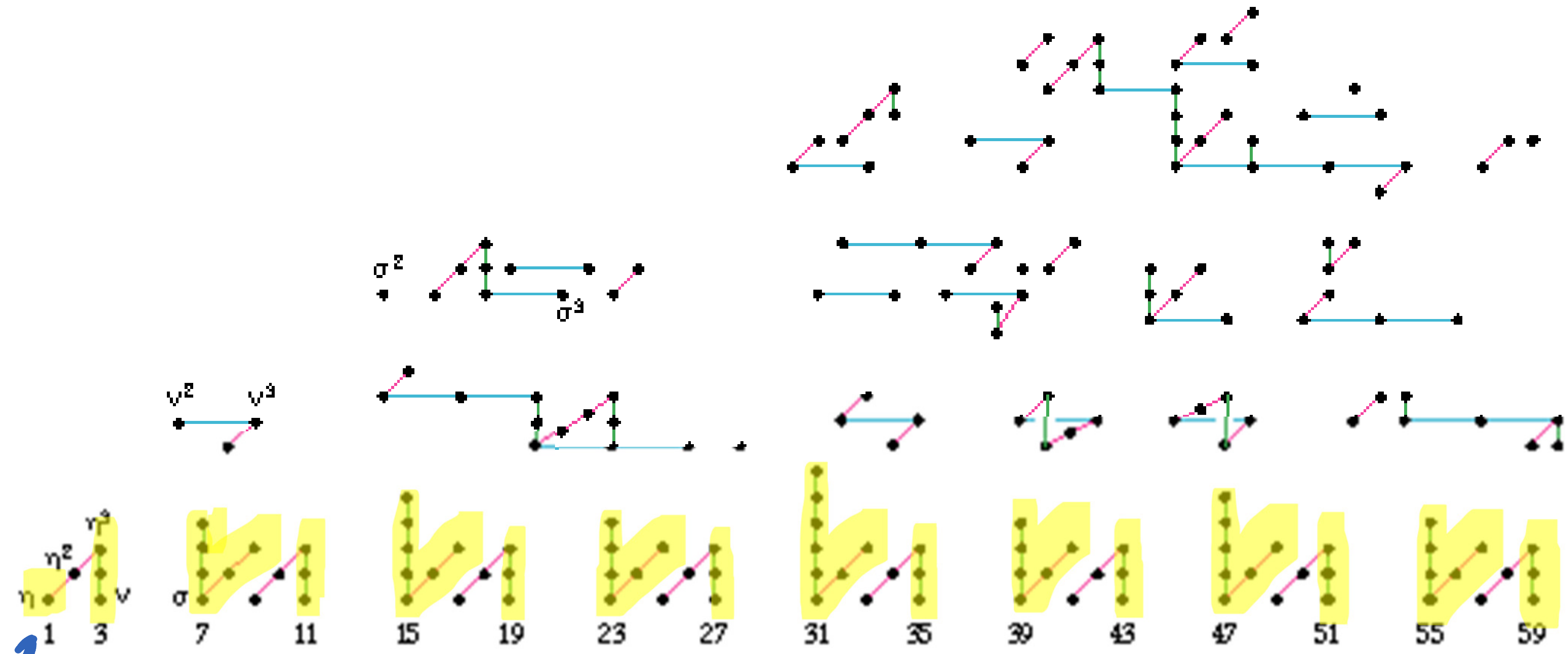


Computation: Mahowald-Tangora-Kochman
Picture: A. Hatcher

- Each dot represents a factor of 2, vertical lines indicate additive extensions

  e.g.: $(\pi_3^s)_{(2)} = \mathbb{Z}_8$, $(\pi_8^s)_{(2)} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$
- Vertical arrangement of dots is arbitrary, but meant to suggest patterns

Stable Homotopy Groups of Spheres at the prime 2



Im J

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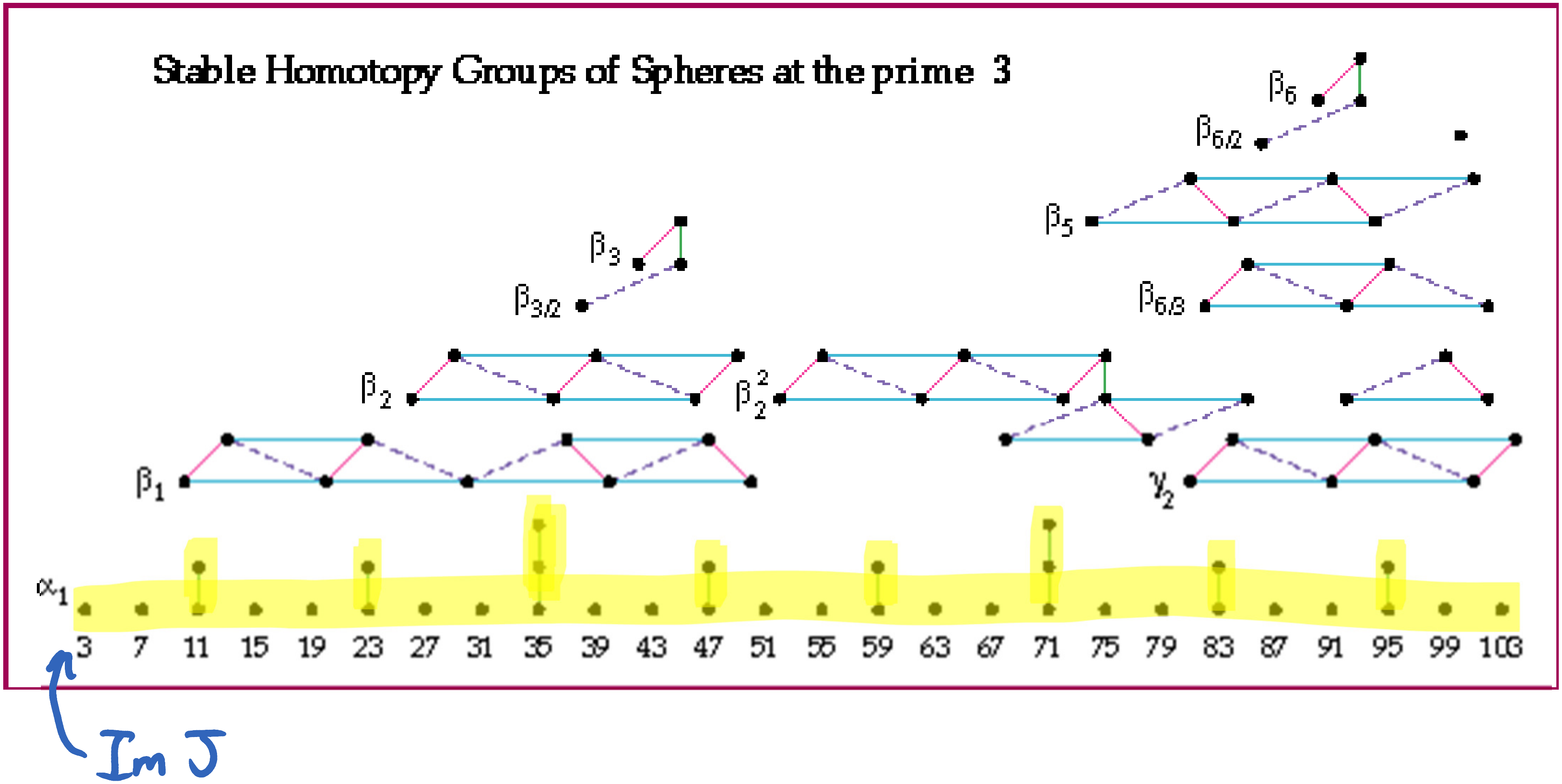
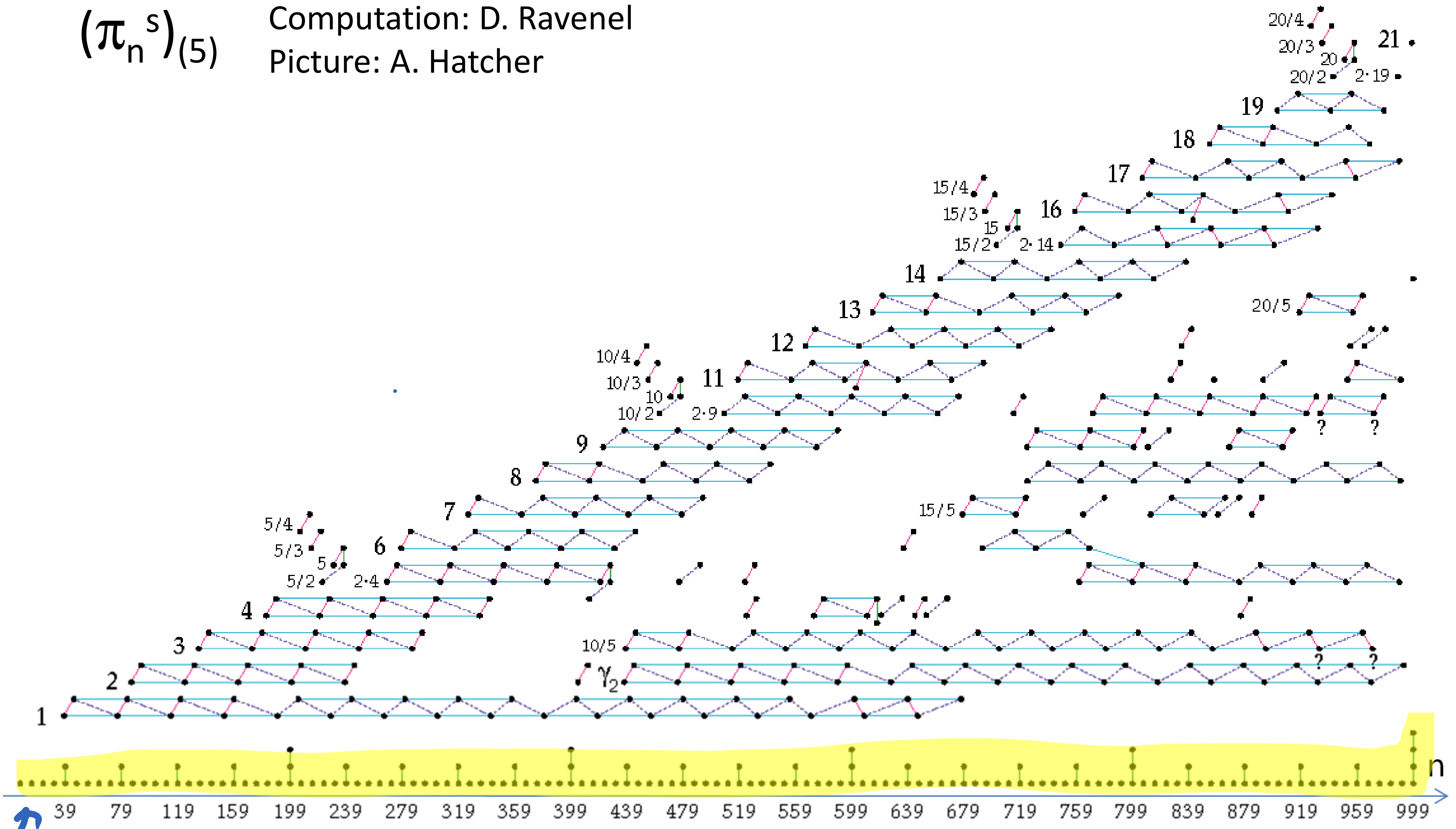
Computation: Nakamura -Tangora
Picture: A. Hatcher



$(\pi_n{}^s)_{(5)}$

Computation: D. Ravenel
Picture: A. Hatcher



Im J

Adams spectral sequence

$$Ext_A^{s,t}(\mathbb{Z}/p\,,\mathbb{Z}/p) \Rightarrow (\pi_{t-s}^{s})_p$$

[p=2]

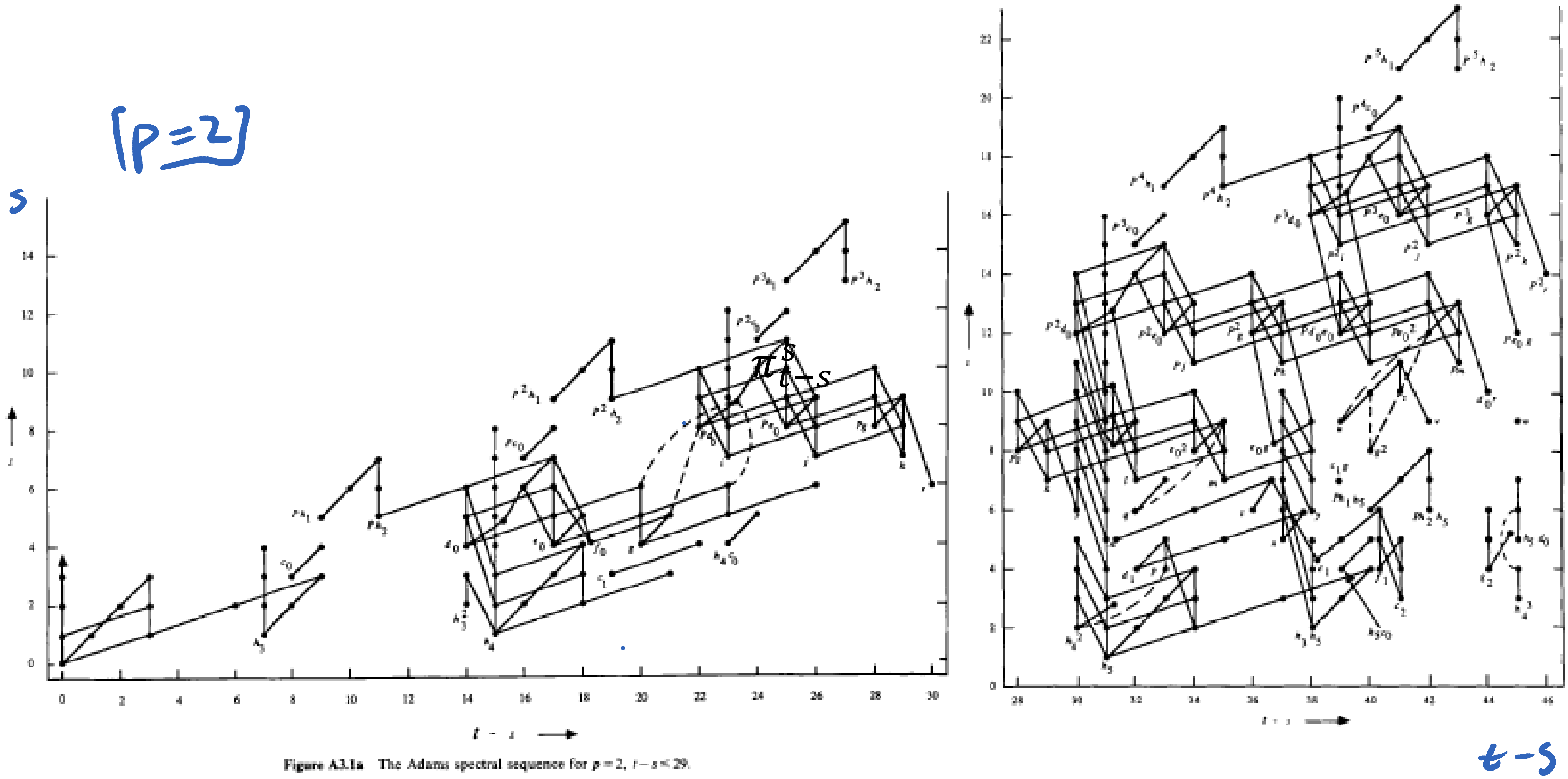


Figure A3.1a The Adams spectral sequence for $p = 2$, $t - s \leq 29$.

Adams spectral sequence

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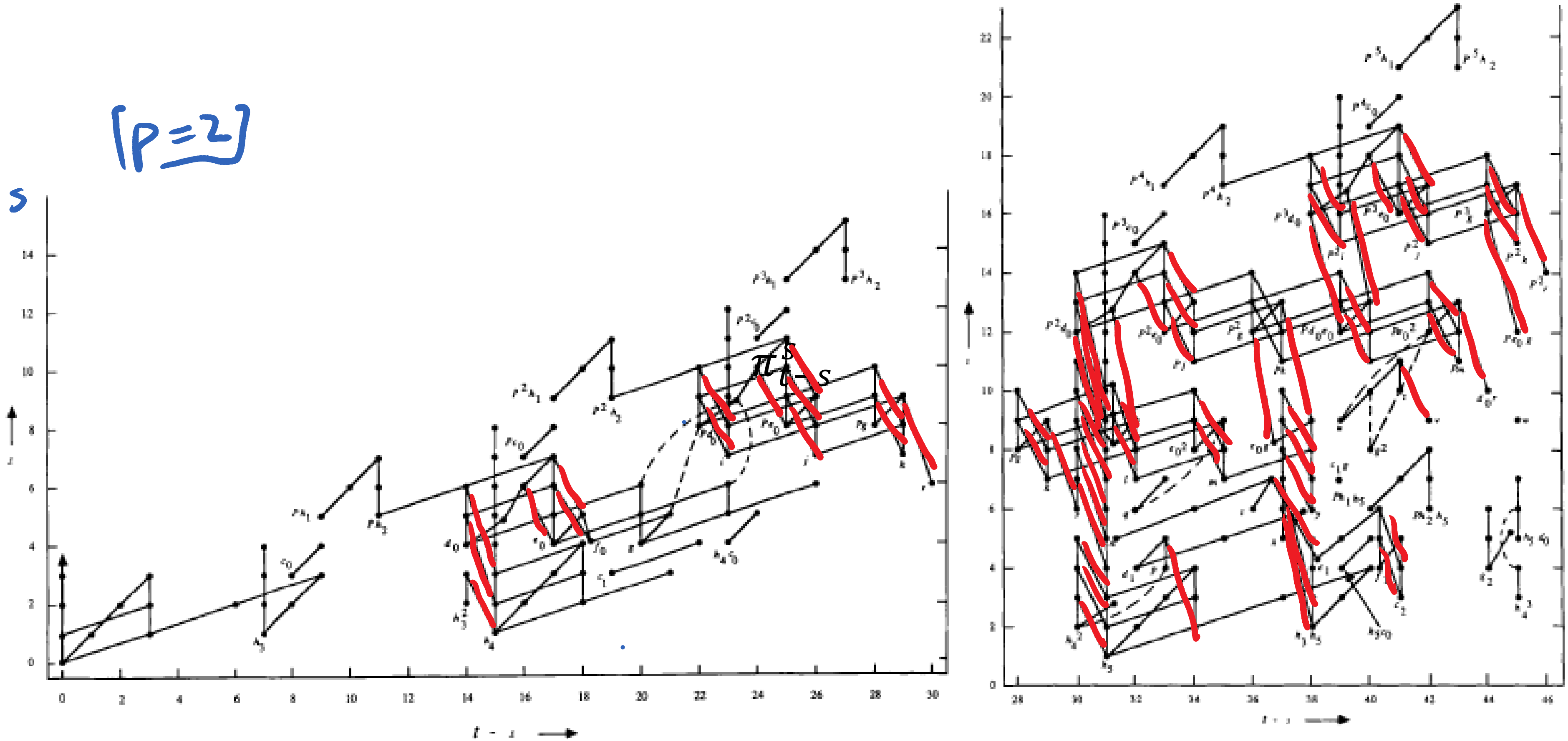

Figure A3.1a The Adams spectral sequence for $p = 2$, $t - s \leq 29$.

t-s

-Many differentials

-d_r differentials go back by 1 and up by r

Adams spectral sequence

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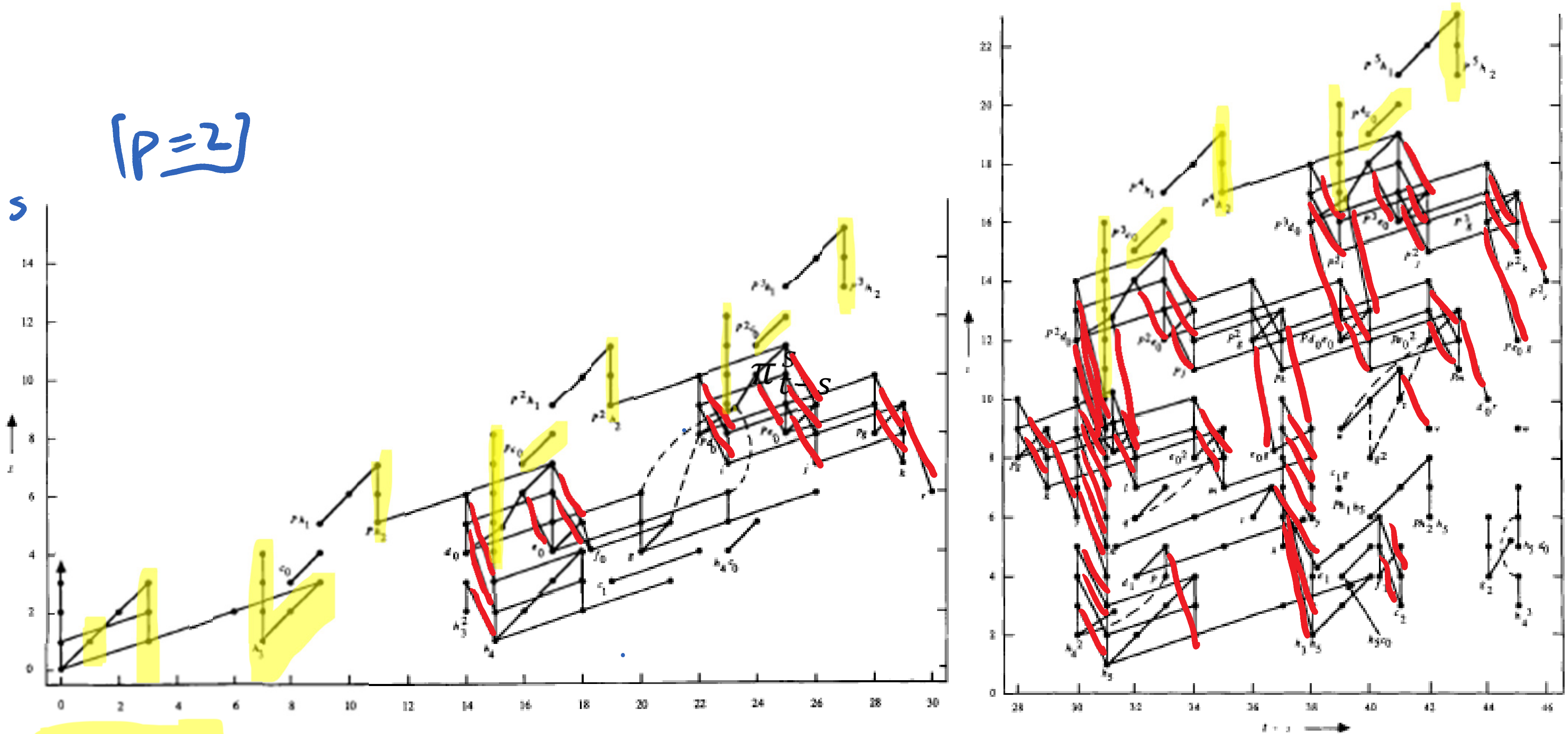

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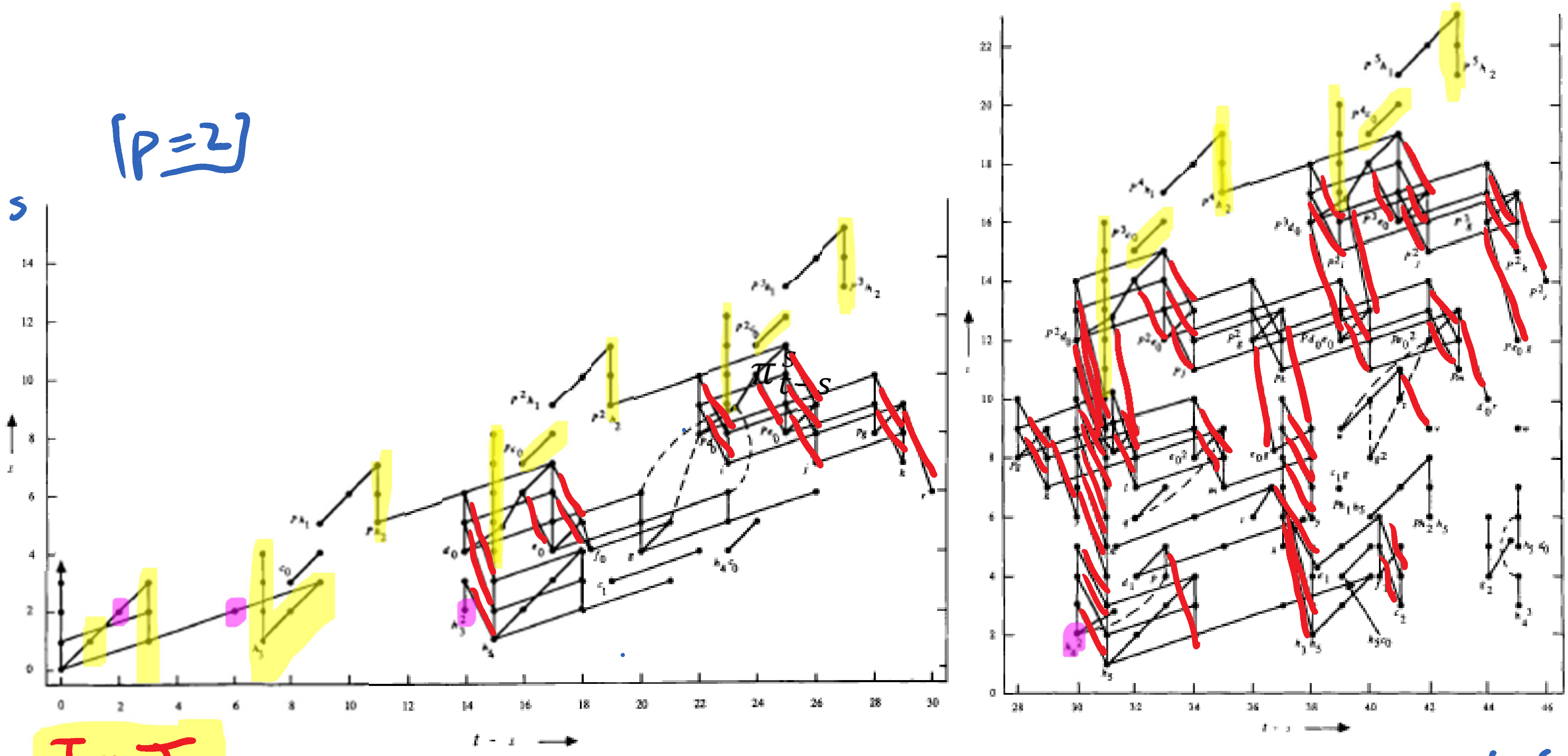

Figure A3.1a The Adams spectral sequence for $p = 2$, $t - s \leq 29$.

t−s

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🟪 = Kervaire Invariant 1 .

Kervaire Invariant

$$\Phi_K : \pi_n^s \to \mathbb{Z}/2$$

Browder:

$$(\Phi_K \neq 0) \Rightarrow \left(n = 2^k - 2\right)$$

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Computation in ASS: $\Phi_K \neq 0$ for

$$n \in \{2, 6, 14, 30, \underset{\substack{\uparrow \\ \text{Barratt-Jones-Mahowald '84}}}{62}\}$$

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Computation in ASS: $\Phi_K \neq 0$ for

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Hill-Hopkins-Ravenel:

$\Phi_K = 0$ for all $n \geq 254$

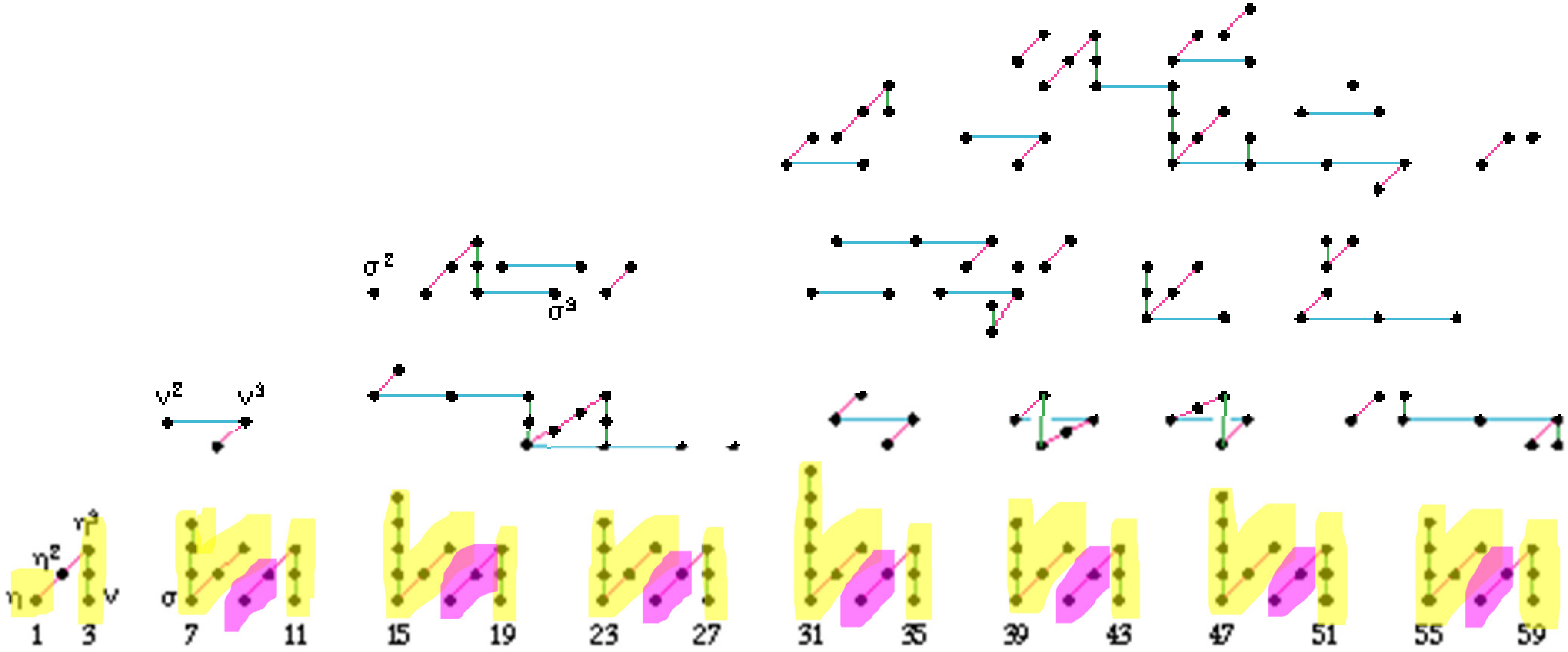
(Note: the case of $n = 126$ is still open)

Summary: Exotic spheres

$\Theta_n \neq 0$ if:

- $\Theta_n^{bp} \neq 0$:
  - $n \equiv 3\ (4)$ and $n \geq 7$
  - $n \equiv 1\ (4)$ and $n \notin \{1,5,13,29,61,125?\}$ [Kervaire]
- Remains to check: is $\frac{\pi_n^s}{Im\ J} \neq 0$ for
  - n even
  - $n \in \{1,5,13,29,61,125?\}$

Stable Homotopy Groups of Spheres at the prime 2



🟨 = Im J

🟪 = 8-fold periodic ⇒ $\frac{\pi_n^s}{\mathrm{Im} J} \neq 0$ for $n = 8k+2$

Summary: Exotic spheres

$\Theta_n \neq 0$ if:

- $\Theta_n^{bp} \neq 0$:
  - $n \equiv 3\ (4)$ and $n \geq 7$
  - $n \equiv 1\ (4)$ and $n \notin \{1,5,13,29,61,125?\}$
- $\frac{\pi_n^s}{Im\ J} \neq 0$ for $n \equiv 2\ (8)$
- Remains to check: is $\frac{\pi_n^s}{Im\ J} \neq 0$ for
  - $n \equiv 0\ (4)$ or $n \equiv -2\ (8)$
  - $n \in \{1,5,13,29,61,125?\}$

Low dimensional computations

- Limitation: only know $(\pi_n^s)_2$ for $n \leq 63$

- $\left(\frac{\pi_n^s}{Im\ J}\right)_p = 0$ in this range for $p \geq 7$.

Low dimensional computations

Non-trivial elements in $Coker\ J$:

$$n \equiv 0\ (4)$$

Stem	p = 2	p = 3	p = 5
4	0	0	0
8	ε	0	0
12	0	0	0
16	η4	0	0
20	κbar	β1^2	0
24	h4 ε η	0	0
28	ε κbar	0	0
32	q	0	0
36	t	β2 β1	0
40	κbar^2	β1^4	0
44	g2	0	0
48	e0 r	0	0
52	κbar q	β2^2	0
56	κbar t	0	0
60	kbar^3	0	0

Low dimensional computations

Non-trivial elements in $Coker\ J$:
$n \equiv -2\ (8)$

🟥 = kervaire inv 1

Stem	p = 2	p = 3	p = 5
6	ν^2	0	0
14	k	0	0
22	ε k	0	0
30	θ4	β1^3	0
38	y	β3/2	β1
46	w η	β2 β1^2	0
54	ν2^8 ν^2	0	0
62	h5 n	β2^2 β1	0

Low dimensional computations

Non-trivial elements in $Coker\ J$:

$n \in \{1,5,13,29,61\}$ [where $\Theta_n^{bp} = 0$ because of Kervaire classes]

Stem	**p = 2**	**p = 3**	**p = 5**
1	0	0	0
5	0	0	0
13	0	β1 α1	0
29	0	β2 α1	0
61	0	β4 α1	0

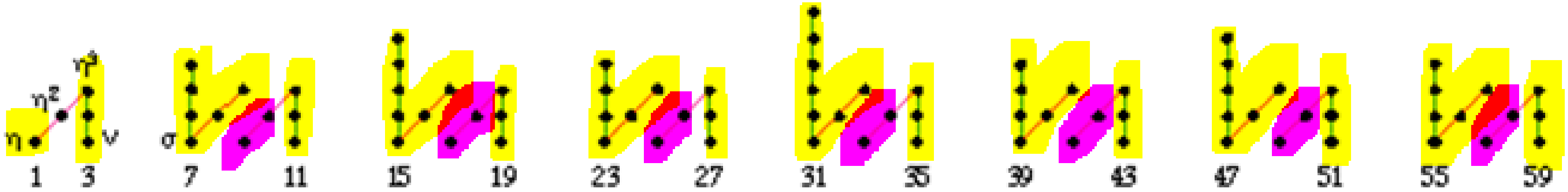
Low dimensional computations

Conclusion

For $n \leq 63$, the only n for which $\Theta_n = 0$ are:

$$1,2,3,4,5,6,12,61$$

Beyond low dimensions…



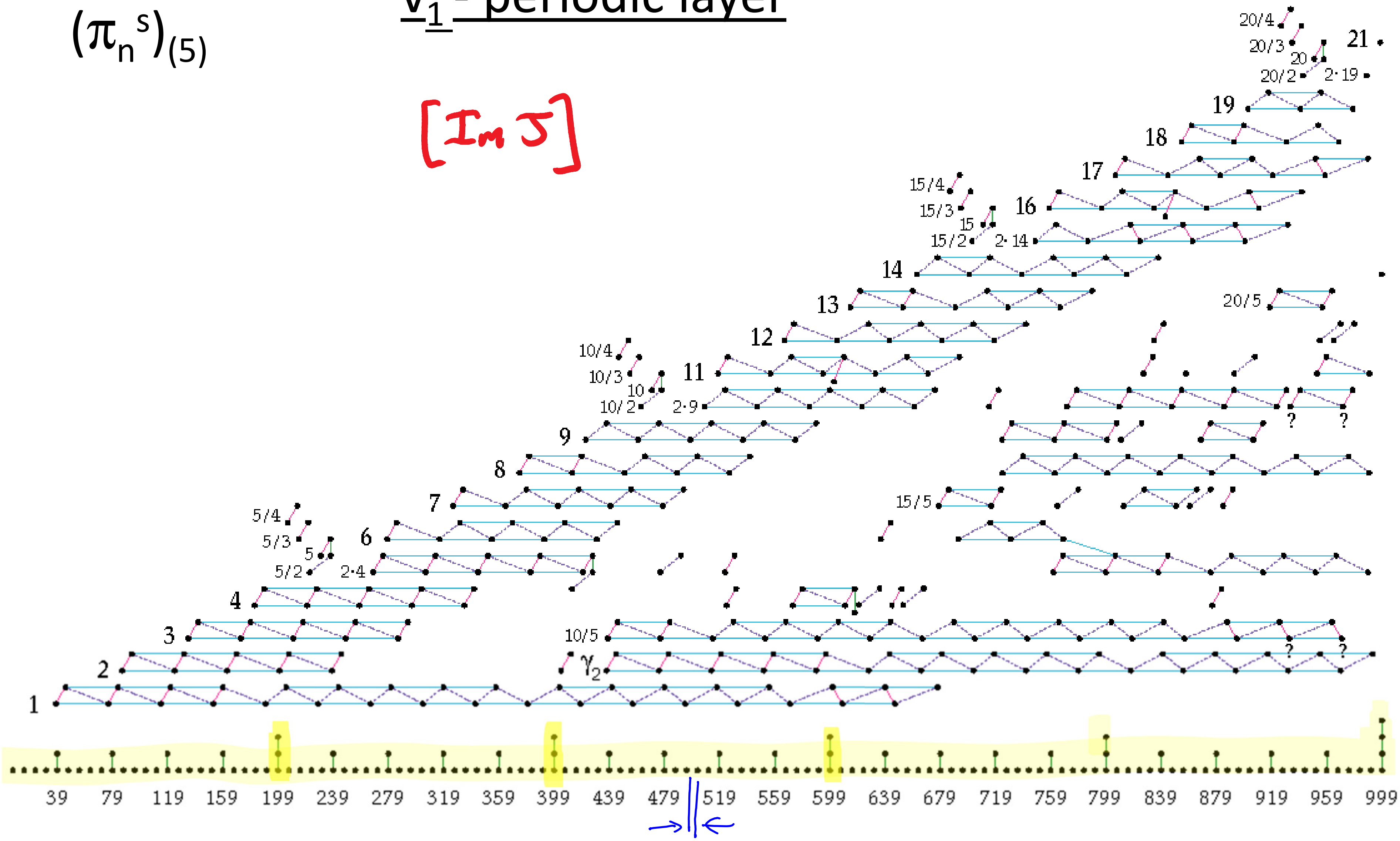
<u>Strategy</u>: try to demonstrate Coker J is non-zero in certain dimensions by producing infinite periodic families such as the one above.

Need to study <u>periodicity</u> in π_*^s

$(\pi_n^s)_{(5)}$

v_1 - periodic layer

[Im J]

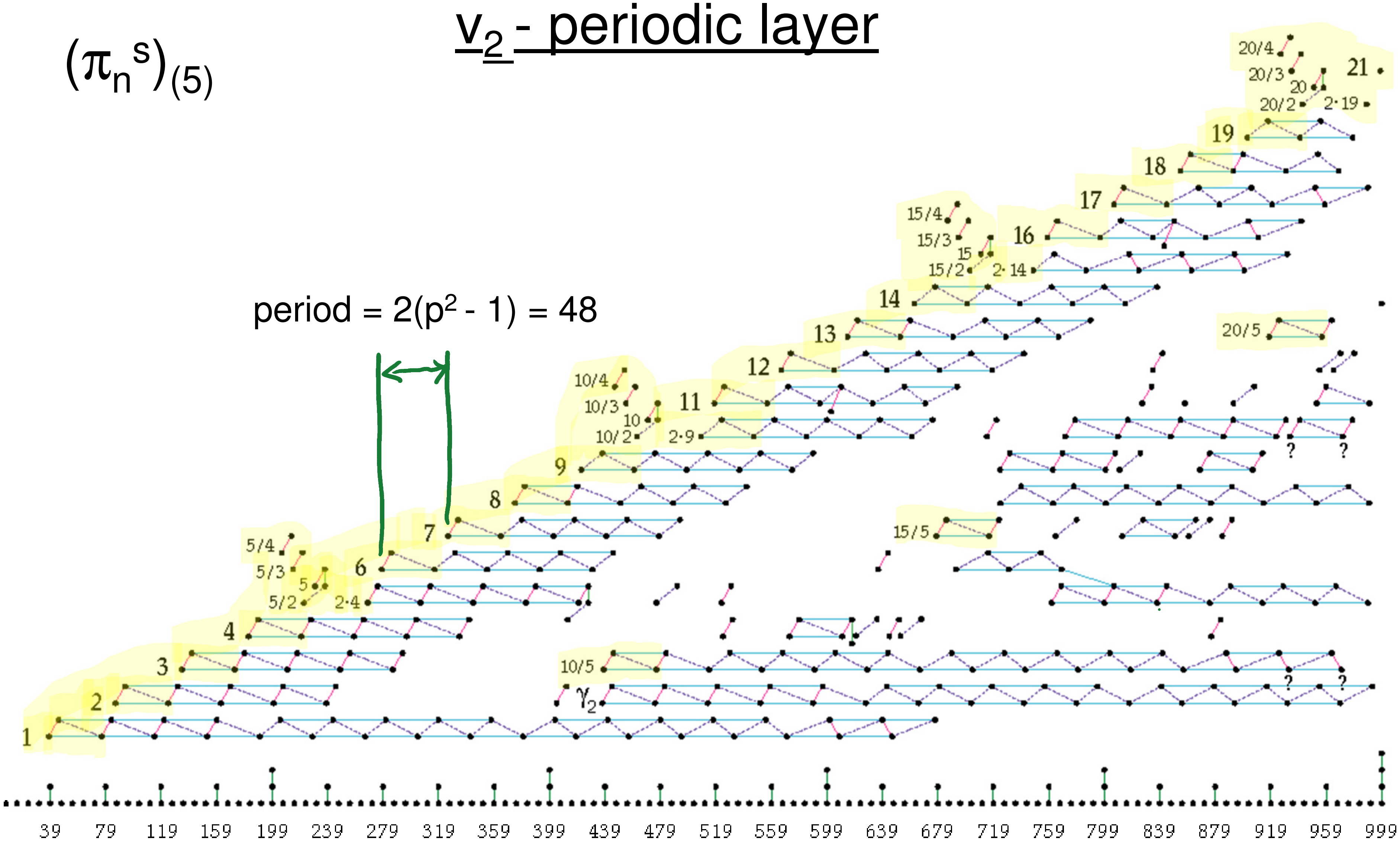


period = 2(p-1) = 8

$(\pi_n^s)_{(5)}$

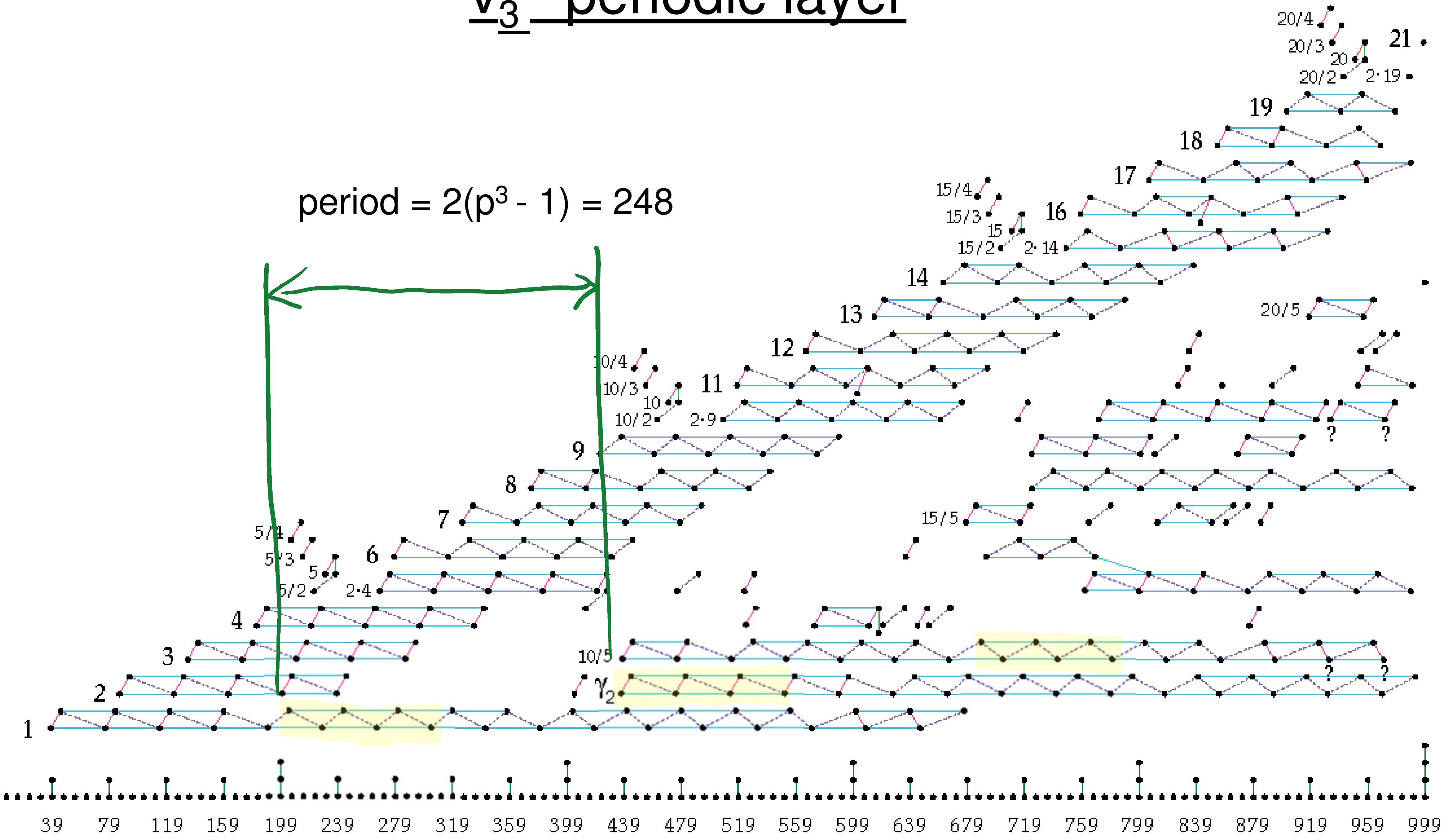
v_2 - periodic layer

period = $2(p^2 - 1) = 48$



v_3 - periodic layer

period = $2(p^3 - 1) = 248$



v_n-periodic families

Generalized Moore spectra:

$$M_{(i_0,i_1,\ldots,i_k)} = S/(p^{i_0}, v_1^{i_1}, \ldots, v_k^{i_k})$$

Desuspension (top cell in dim 0):

$$M^0_{(i_0,\ldots,i_k)} = \Sigma^{-d} M_{(i_0,i_1,\ldots,i_k)}$$

v_n-periodic families

$x \in \pi_t(S)$, $(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})$-torsion

$$\Updownarrow$$

$$\pi_t \underset{\substack{\rotatebox{90}{$\in$} \\ \tilde{x}}}{M}{}^0_{(i_0,\ldots,i_{n-1})} \to \underset{\substack{\rotatebox{90}{$\in$} \\ x}}{\pi_t S}$$

v_n-periodic families

$x \in \pi_t(S)$, $\quad (p^{i_0}, \dots, v_{n-1}^{i_{n-1}})$-torsion

$$\Updownarrow$$

$$\underset{\tilde{x}}{\underset{\rotatebox{90}{$\in$}}{\pi_t M^0_{(i_0,\dots,i_{n-1})}}} \to \underset{x}{\underset{\rotatebox{90}{$\in$}}{\pi_t S}}$$

Find a v_n-self map

$$\Sigma^d M^0_{(i_0,\dots,i_{n-1})} \xrightarrow{v} M^0_{(i_0,\dots,i_{n-1})}$$

v_n-periodic families

$x \in \pi_t(S)$, $\quad (p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})$-torsion

$$\Updownarrow$$

$$\pi_t M^0(\underset{\substack{\rotatebox{90}{$\in$} \\ \tilde{x}}}{i_0, \ldots, i_{n-1}}) \to \underset{\substack{\rotatebox{90}{$\in$} \\ x}}{\pi_t S}$$

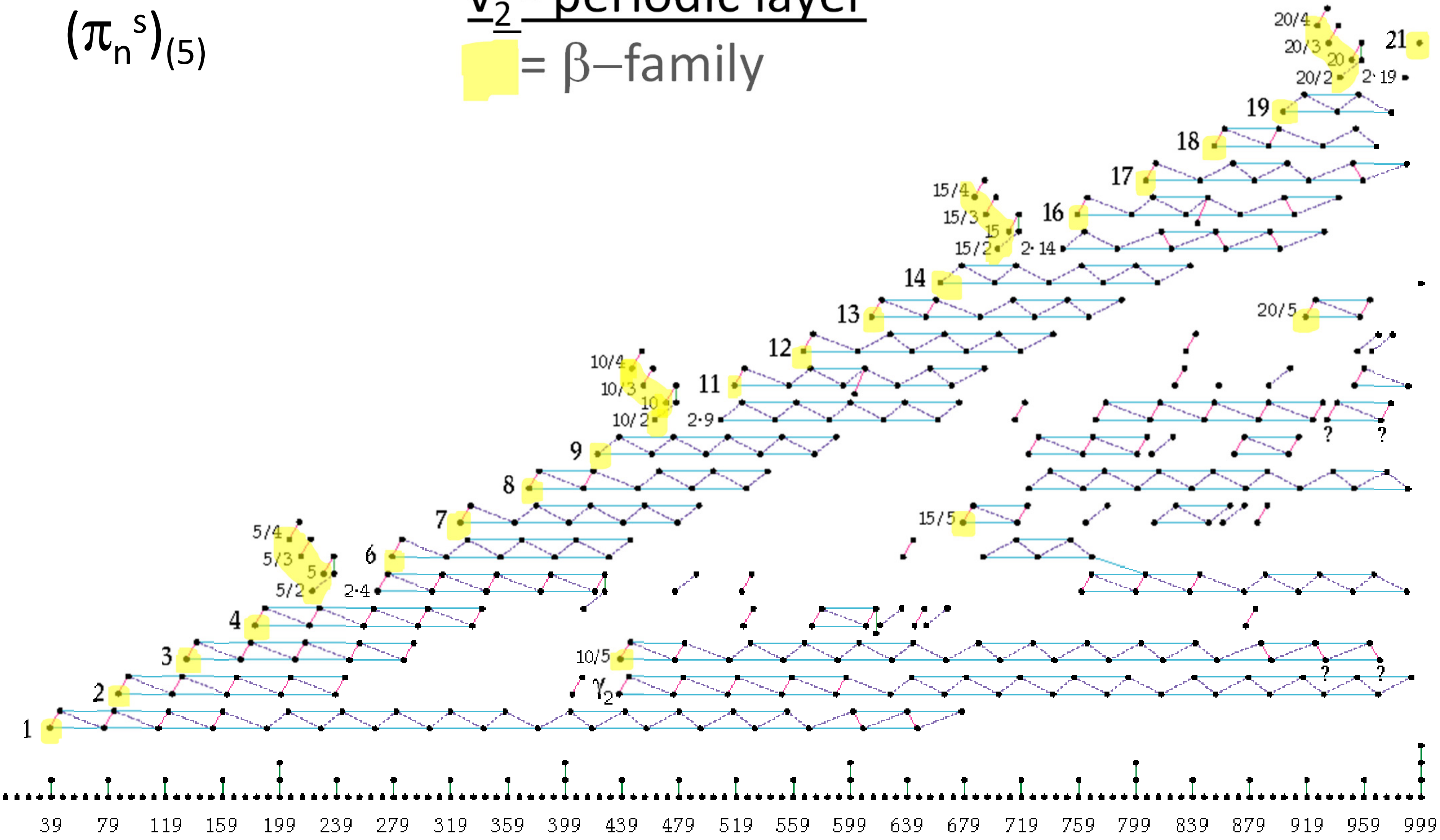
Get a periodic family:

$$\underset{\substack{\rotatebox{90}{$\in$} \\ \tilde{x}}}{\pi_t} M_{(i_0,\ldots,i_{n-1})} \xrightarrow{v^k} \pi_{t+kd} M^0_{(i_0,\ldots,i_{n-1})} \to \pi_{t+kd} \underset{\substack{\rotatebox{90}{$\in$} \\ x_k}}{S}$$

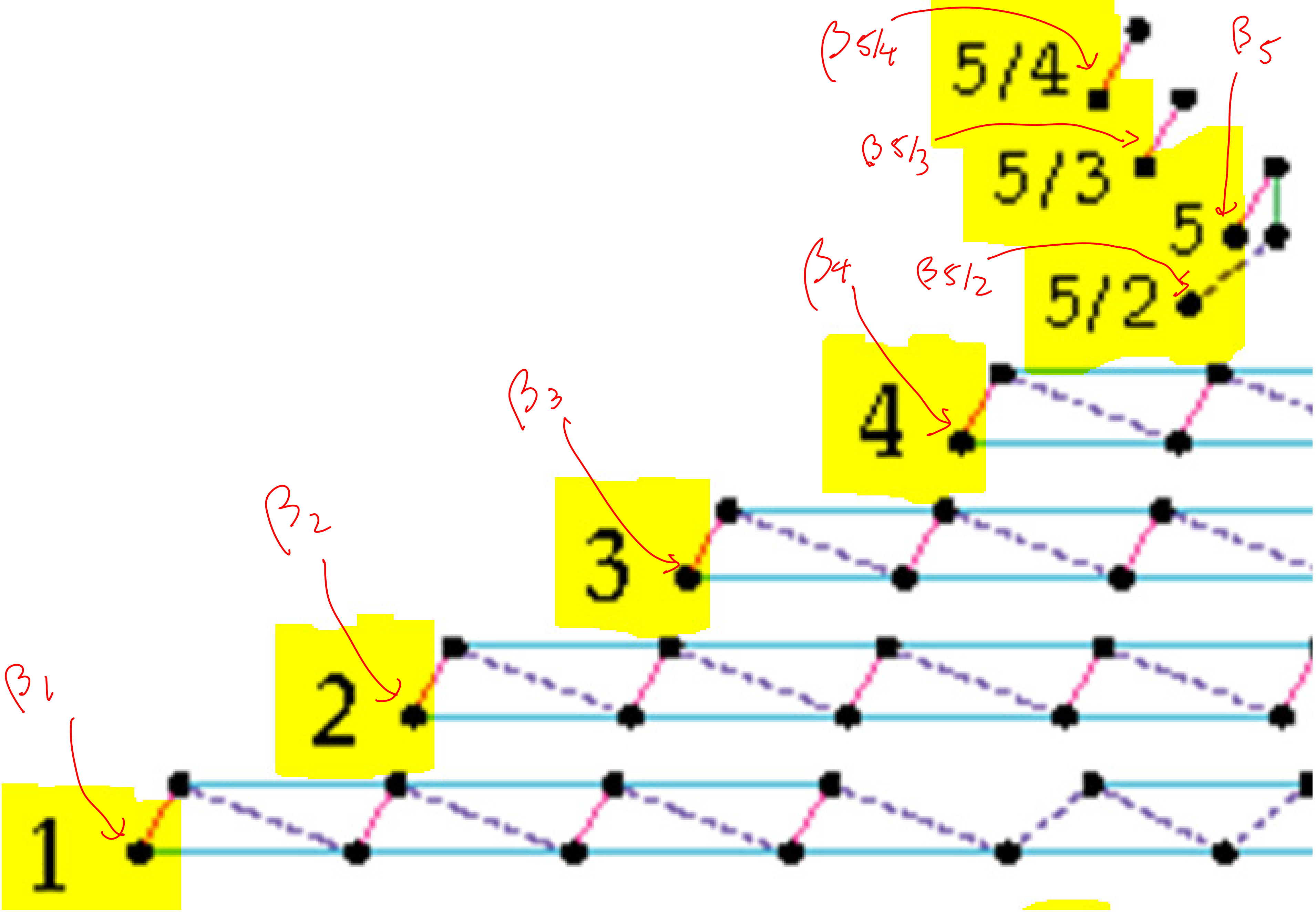
$(\pi_n{}^{s})_{(5)}$

v_2 - periodic layer

🟨 = β–family



Greek Letter Names (Miller-Ravenel-Wilson)

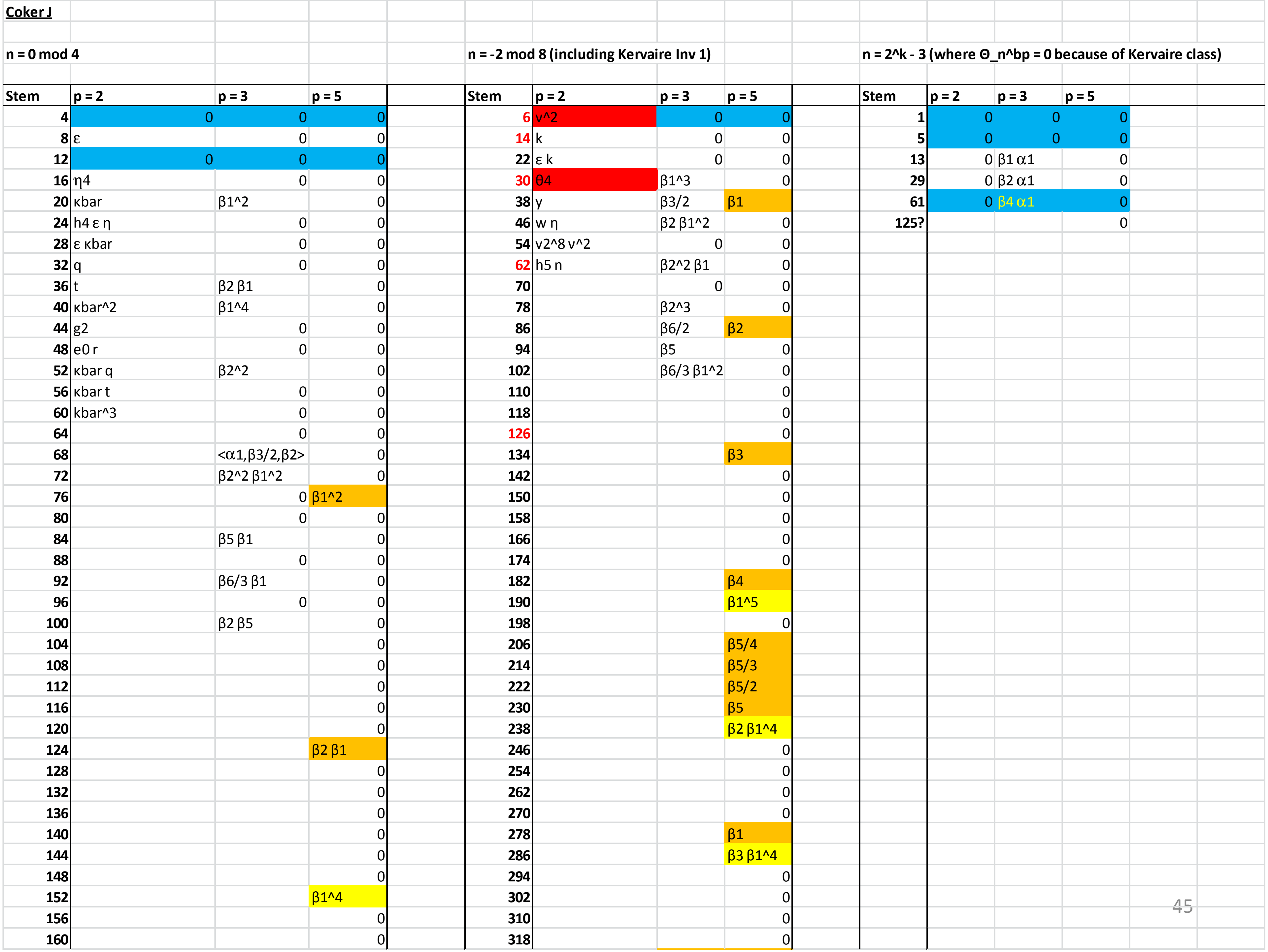


Exotic spheres from β-family

- β_k exists for $p \geq 5$ and $k \geq 1$ [Smith-Toda]

  $\Theta_n \neq 0$ for $n \equiv -2(p-1) - 2 \; mod \; 2(p^2-1)$

$$\Sigma^{2(p^2-1)} M^0_{1,1} \xrightarrow[v_2]{} M^0_{1,1}$$



Coker J

n = 0 mod 4

Stem	p = 2	p = 3	p = 5
4	0	0	0
8	ε	0	0
12	0	0	0
16	η4	0	0
20	κbar	β1^2	0
24	h4 ε η	0	0
28	ε κbar	0	0
32	q	0	0
36	t	β2 β1	0
40	κbar^2	β1^4	0
44	g2	0	0
48	e0 r	0	0
52	κbar q	β2^2	0
56	κbar t	0	0
60	kbar^3	0	0
64		0	0
68		<α1,β3/2,β2>	0
72		β2^2 β1^2	0
76		0	β1^2
80		0	0
84		β5 β1	0
88		0	0
92		β6/3 β1	0
96		0	0
100		β2 β5	0
104			0
108			0
112			0
116			0
120			0
124			β2 β1
128			0
132			0
136			0
140			0
144			0
148			0
152			β1^4
156			0
160			0

n = -2 mod 8 (including Kervaire Inv 1)

Stem	p = 2	p = 3	p = 5
6	ν^2	0	0
14	k	0	0
22	ε k	0	0
30	θ4	β1^3	0
38	y	β3/2	β1
46	w η	β2 β1^2	0
54	ν2^8 ν^2	0	0
62	h5 n	β2^2 β1	0
70		0	0
78		β2^3	0
86		β6/2	β2
94		β5	0
102		β6/3 β1^2	0
110			0
118			0
126			0
134			β3
142			0
150			0
158			0
166			0
174			0
182			β4
190			β1^5
198			0
206			β5/4
214			β5/3
222			β5/2
230			β5
238			β2 β1^4
246			0
254			0
262			0
270			0
278			β1
286			β3 β1^4
294			0
302			0
310			0
318			0

n = 2^k - 3 (where Θ_n^bp = 0 because of Kervaire class)

Stem	p = 2	p = 3	p = 5
1	0	0	0
5	0	0	0
13	0	β1 α1	0
29	0	β2 α1	0
61	0	β4 α1	0
125?			0

Exotic spheres from β-family

- β_k exists for $p \geq 5$ and $k \geq 1$
- [Smith-Toda]

  $\Theta_n \neq 0$ for $n \equiv -2(p-1) - 2\ mod\ 2(p^2-1)$

- β_k exists for $p = 3$ and $k \equiv 0,1,2,3,5,6\ (9)$

  [Shimomura]

  [B-Pemmaraju]

  $\Theta_n \neq 0$ for $n \equiv -6, 10, 26, 42; 74, 90 \text{ mod } 144$

$$\Sigma^{144} M^0_{1,1} \xrightarrow[v_2^9]{} M^0_{1,1}$$ [uses TMF]

Exotic spheres from β-family

- $\beta_k = \beta_{k/1,1}$ exists for $p \geq 5$ and $k \geq 1$
  [Smith-Toda]

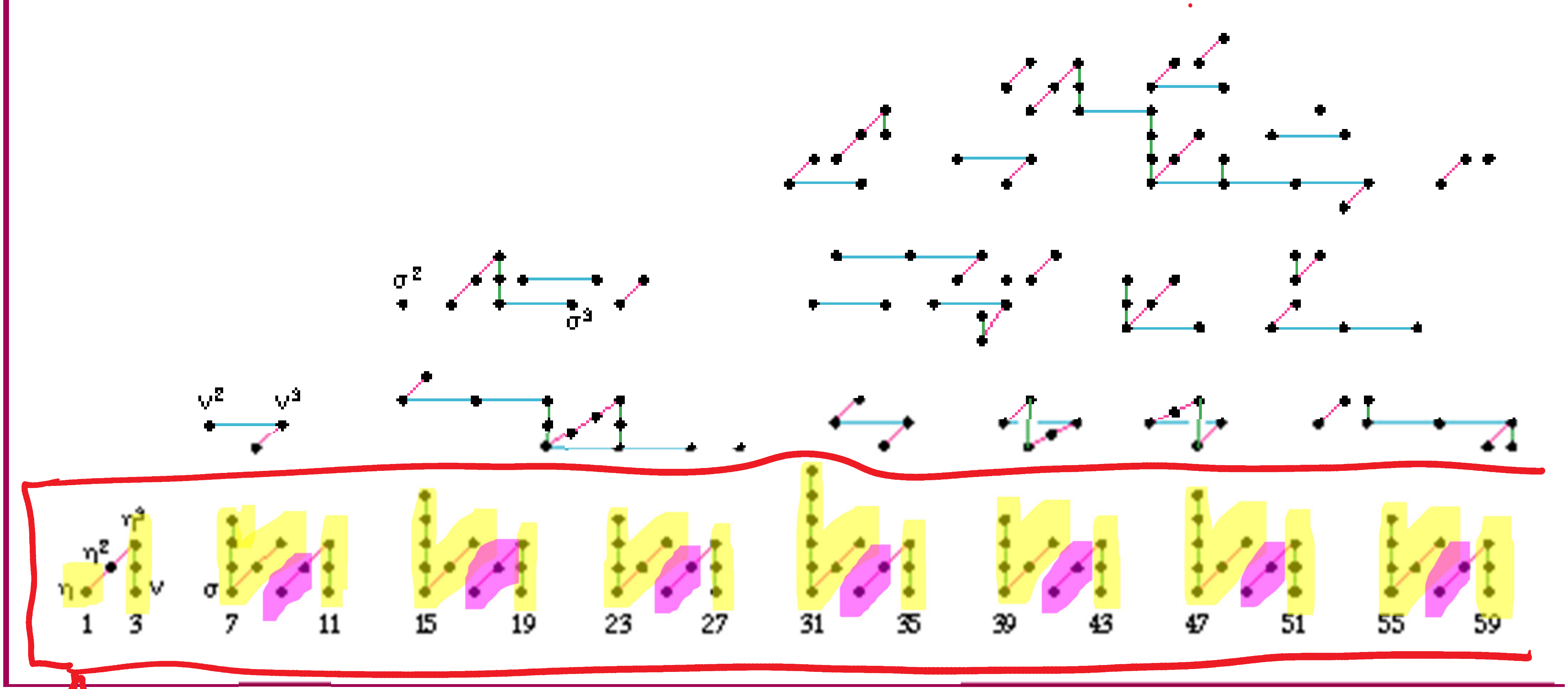
  $\Theta_n \neq 0$ for $n \equiv -2(p-1) - 2 \; mod \; 2(p^2 - 1)$

- β_k exists for $p = 3$ and $k \equiv 0,1,2,\underset{\substack{\uparrow \\ [\text{Shimomura}]}}{3},5,6 \; (9)$
  [B-Pemmaraju]

  $\Theta_n \neq 0$ for $n \equiv -\underbrace{6, 10, 26, 42; 74, 90}_{\text{all} \equiv 2 \; (8)} \text{ mod } 144$ ☹

Back to Θ_n:

Stable Homotopy Groups of Spheres at the prime 2



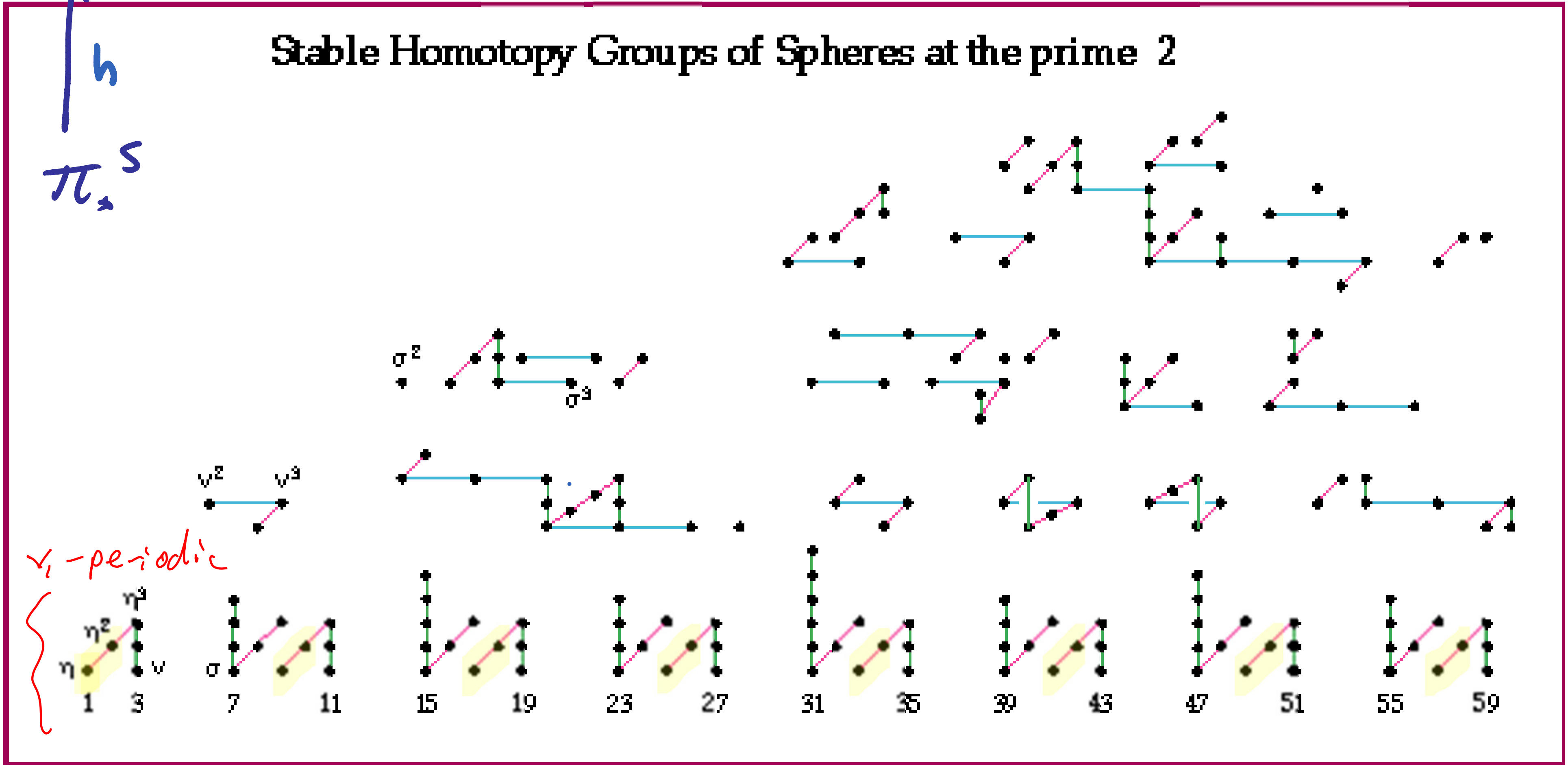
v_1-periodic

🟨 = Im J

🟪 $\Longrightarrow$ $\Theta_n \neq 0$ for $n \equiv 2\ (8)$

KO Hurewicz homomorphism

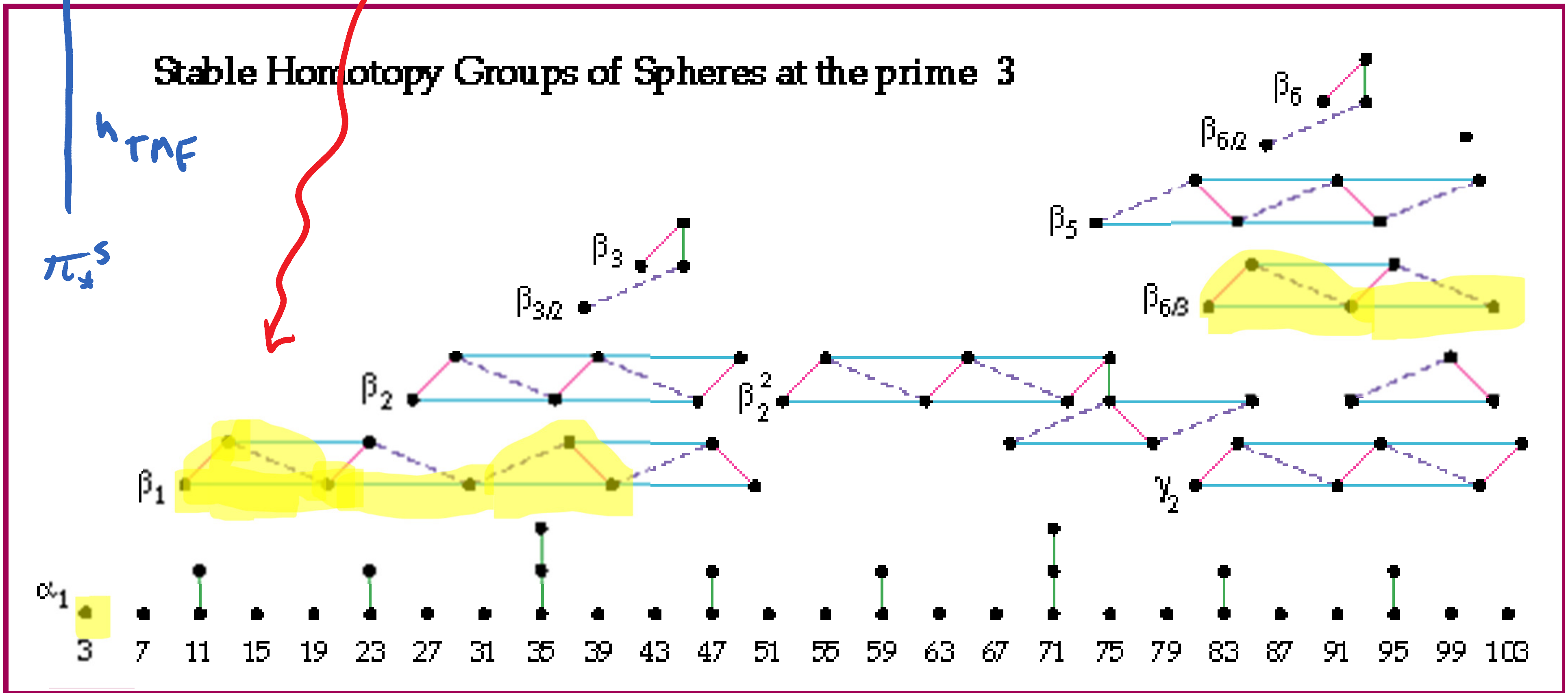
$$\begin{tikzcd} \pi_* KO = \mathbb{Z}\ \ \mathbb{Z}/2\ \ \mathbb{Z}/2\ \ 0\ \ \mathbb{Z}\ \ 0\ \ 0\ \ 0\ \ \mathbb{Z}\ \ \mathbb{Z}/2\ \ \mathbb{Z}/2\ \ 0\ \ \mathbb{Z}\ldots \\ \pi_*^s \arrow[u, "h"] \end{tikzcd}$$


Stable Homotopy Groups of Spheres at the prime 2

Hurewicz image of TMF (p = 3)

$$\begin{tikzcd}
\pi_* TMF_{(3)} \\
\pi_*^s \arrow[u, "h_{TMF}"']
\end{tikzcd}$$

144 periodic



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16	η4	0	0
20	κbar	β1^2	0
24	h4 ε η	0	0
28	ε κbar	0	0
32	q	0	0
36	t	β2 β1	0
40	κbar^2	β1^4	0
44	g2	0	0
48	e0 r	0	0
52	κbar q	β2^2	0
56	κbar t	0	0
60	κbar^3	0	0
64		0	0
68		<α1,β3/2,β2>	0
72		β2^2 β1^2	0
76		0	β1^2
80		0	0
84		β5 β1	0
88		0	0
92		β6/3 β1	0
96		0	0
100		β2 β5	0
104			0
108			0
112		β6/3 β1^3	0
116			0
120			0
124			β2 β1
128			0
132			0
136			0
140			0
144			0
148			0
152			β1^4
156			0
160			0

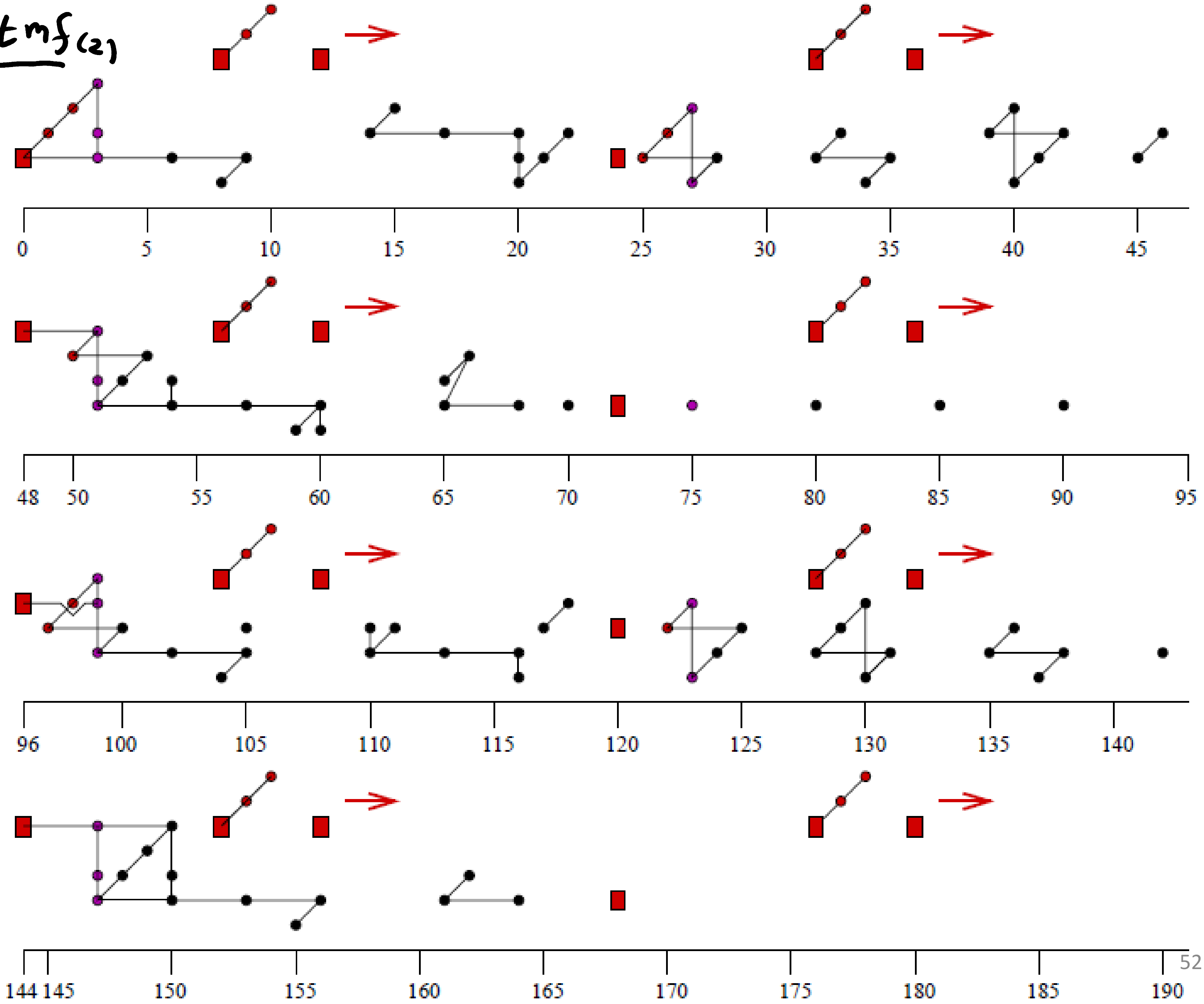
n = -2 mod 8 (including Kervaire Inv 1)

Stem	p = 2	p = 3	p = 5
6	ν^2	0	0
14	k	0	0
22	ε k	0	0
30	θ4	β1^3	0
38	y	β3/2	β1
46	w η	β2 β1^2	0
54	v2^8 ν^2	0	0
62	h5 n	β2^2 β1	0
70		0	0
78		β2^3	0
86		β6/2	β2
94		β5	0
102		β6/3 β1^2	0
110			0
118			0
126			0
134			β3
142			0
150			0
158			0
166			0
174		β1^3	0
182		β3/2	β4
190		β2 β1^2	β1^5
198			0
206		β2^2 β1	β5/4
214			β5/3
222		β2^3	β5/2
230		β6/2	β5
238		β5	β2 β1^4
246		β6/3 β1^2	0
254			0
262			0
270			0
278			β1
286			β3 β1^4
294			0
302			0
310			0
318		β1^3	0

n = 2^k - 3 (where Θ_n^bp = 0 because of Kervaire class)

Stem	p = 2	p = 3	p = 5
1	0	0	0
5	0	0	0
13	0	β1 α1	0
29	0	β2 α1	0
61	0	β4 α1	0
125?			0

$\pi_* tmf_{(2)}$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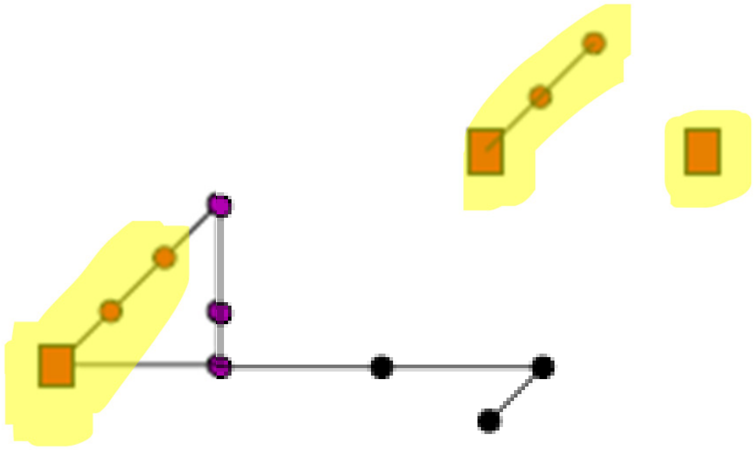
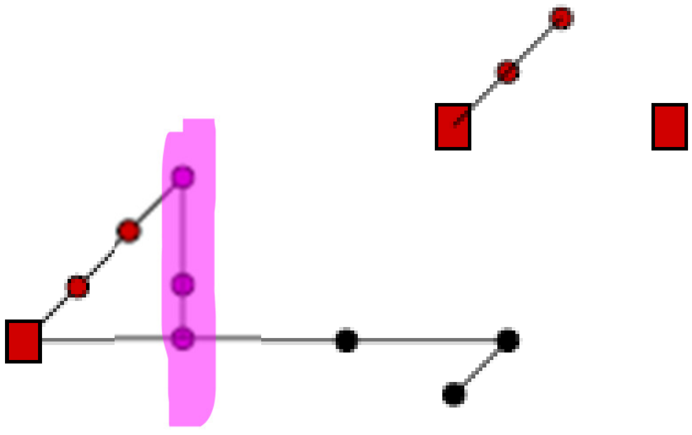
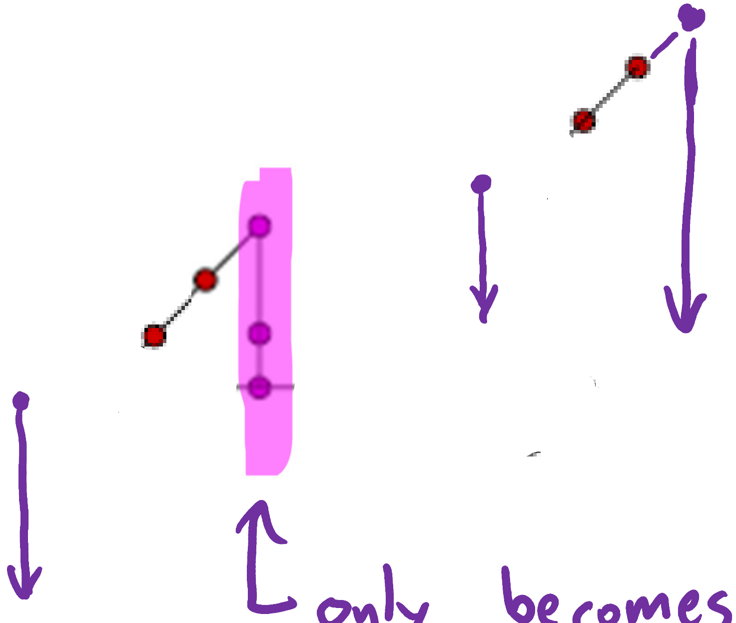


$= v_1$-periodic

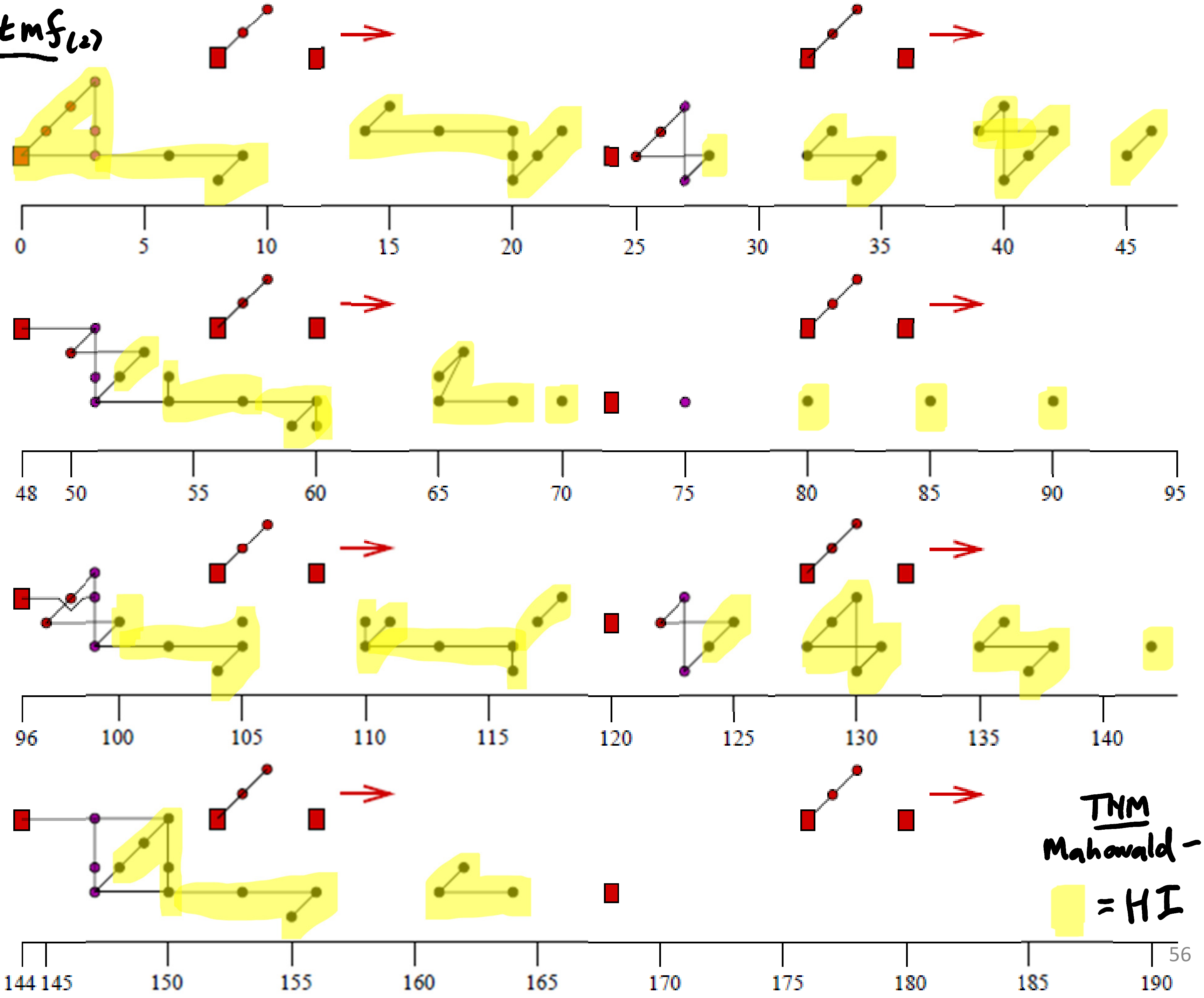


only becomes v_1-periodic in $M_1 tmf$



only becomes v_1-periodic in $M_1 tmf$

$\pi_* tmf_{(2)}$

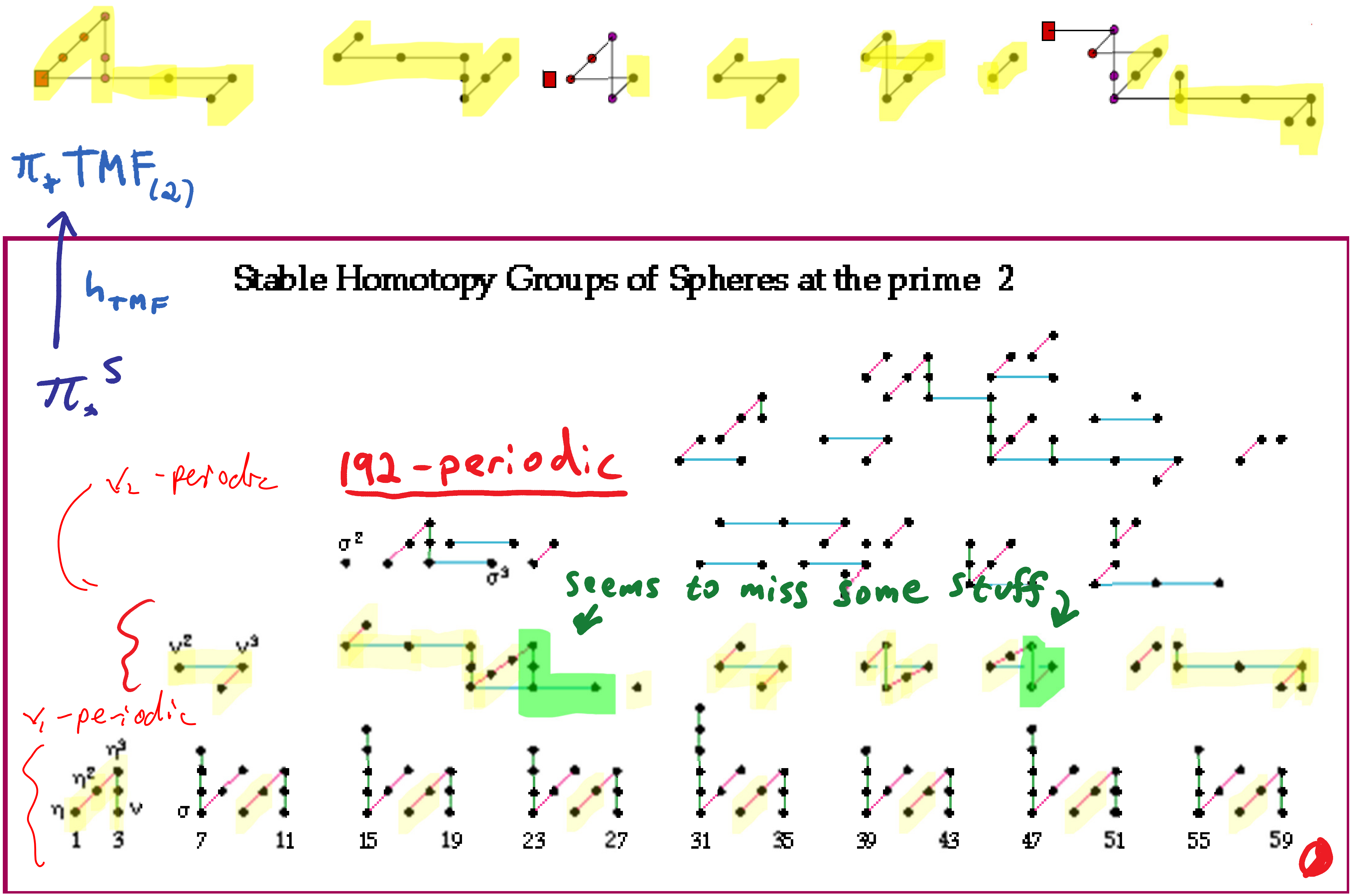


TMM

Mahowald-B

🟨 = HI

Hurewicz image of TMF (p = 2)



$\pi_* TMF_{(2)}$

h_{TMF}

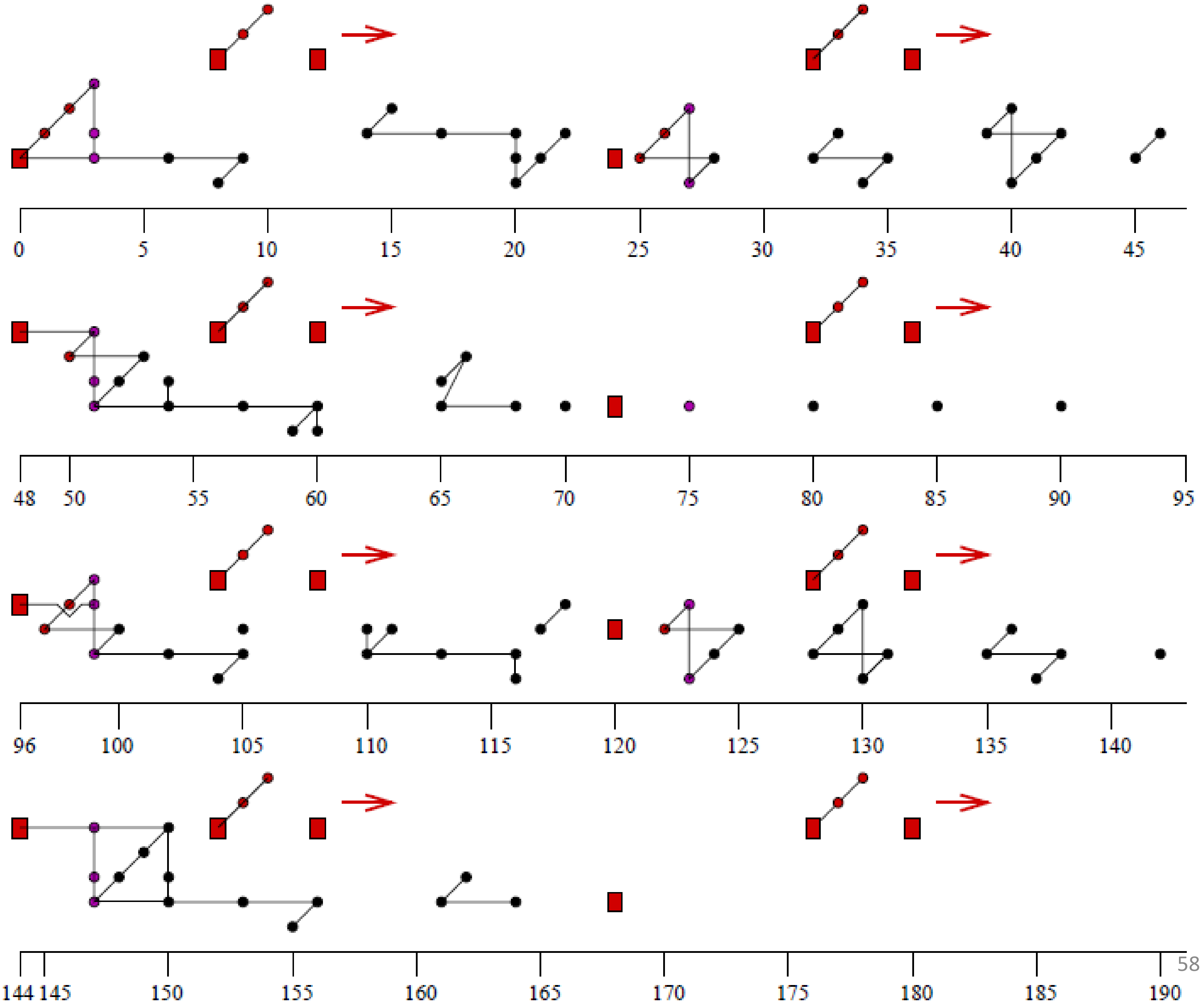
π_*^S

192-periodic

v_2-periodic

v_1-periodic

Seems to miss some stuff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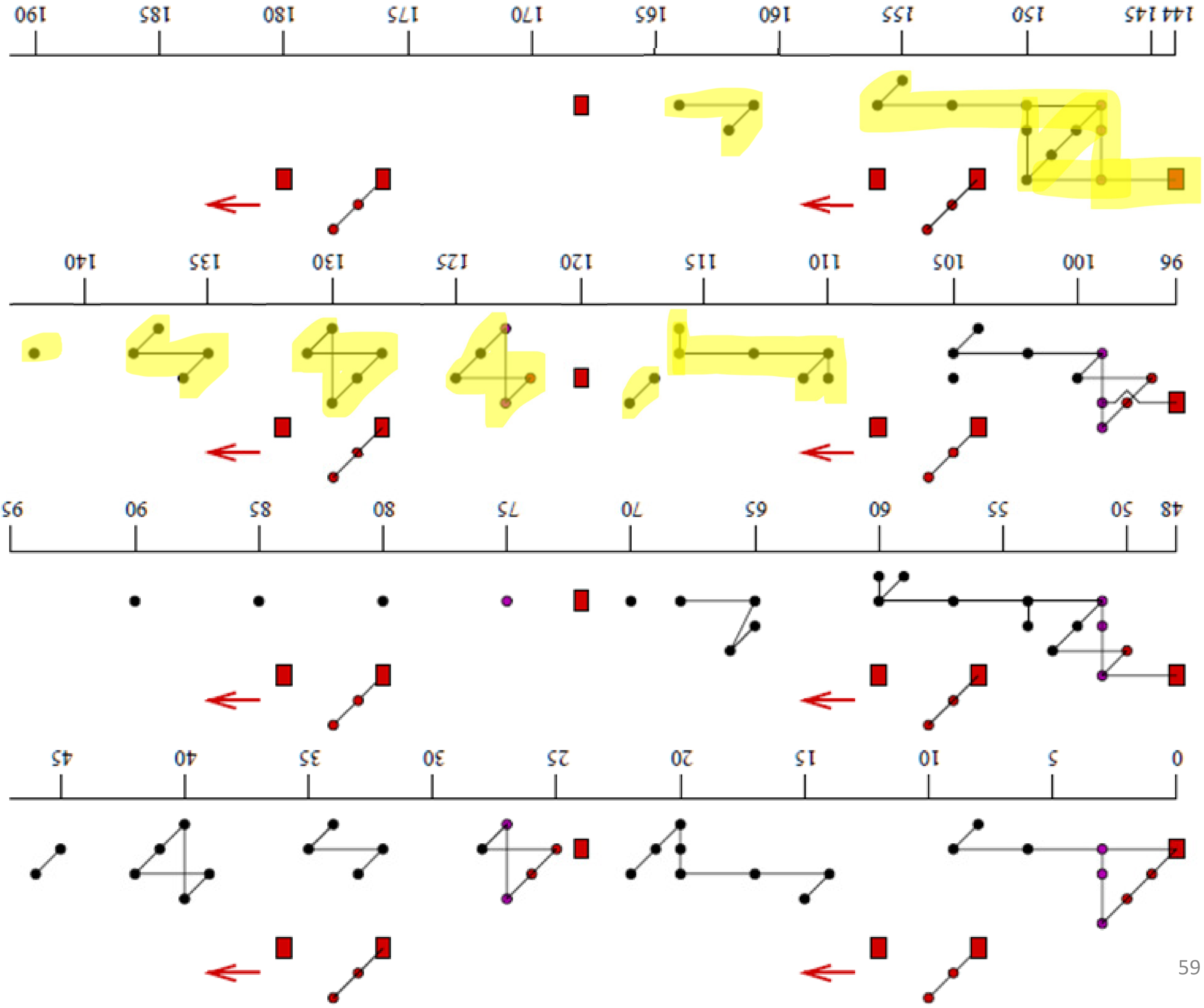
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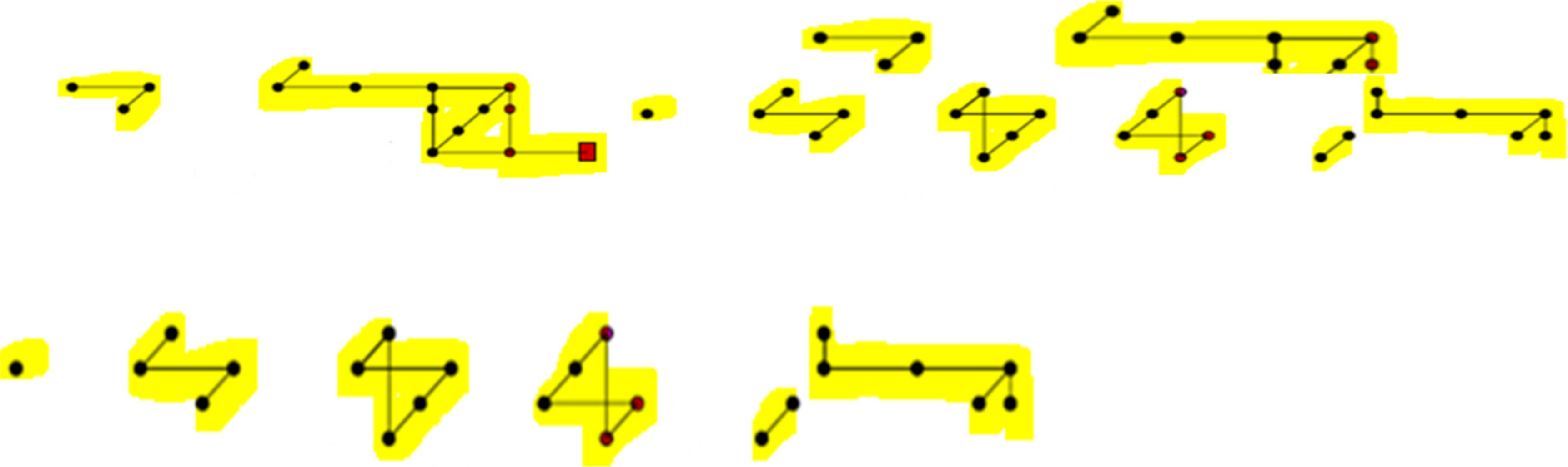
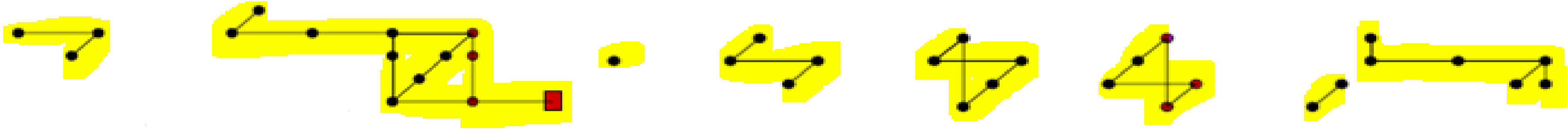
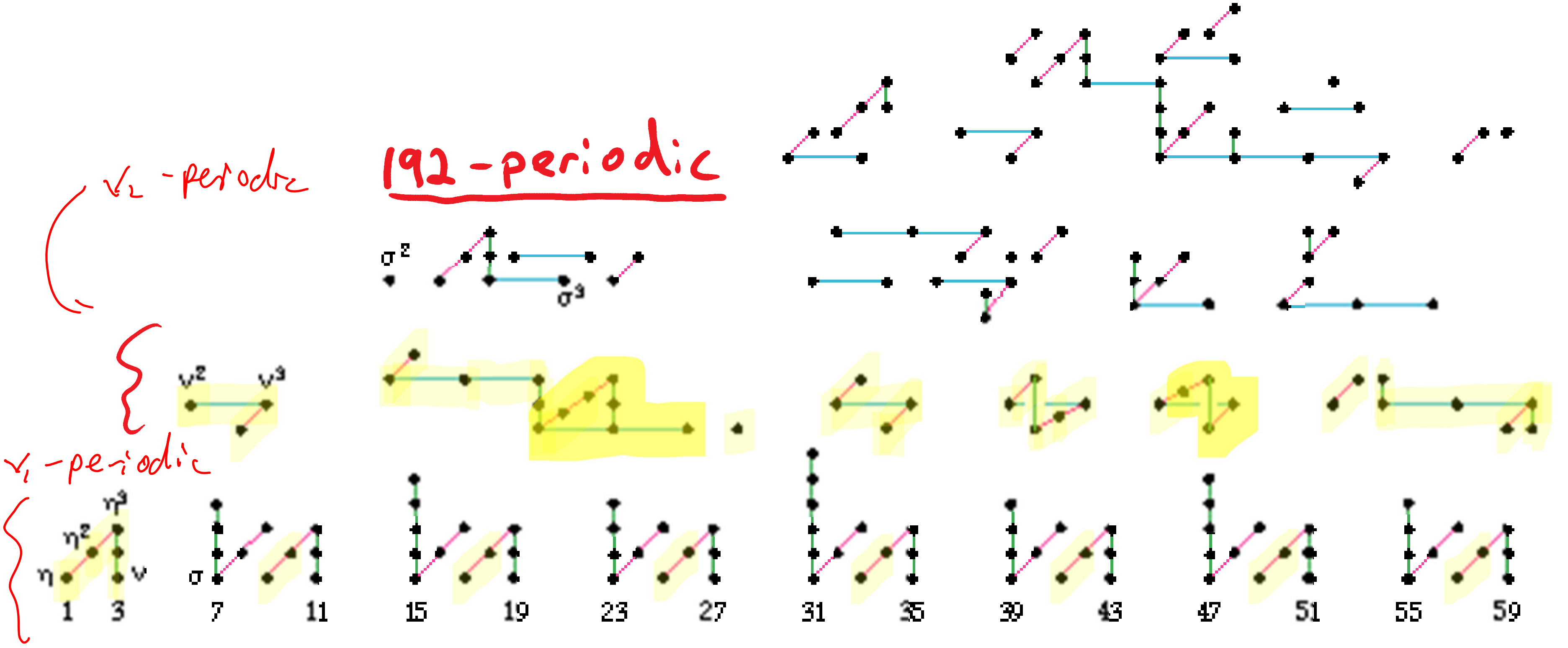




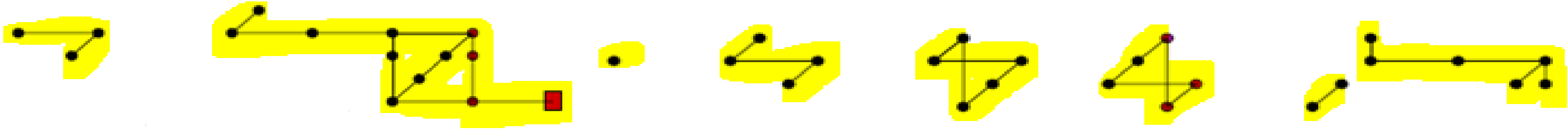
Stable Homotopy Groups of Spheres at the prime 2



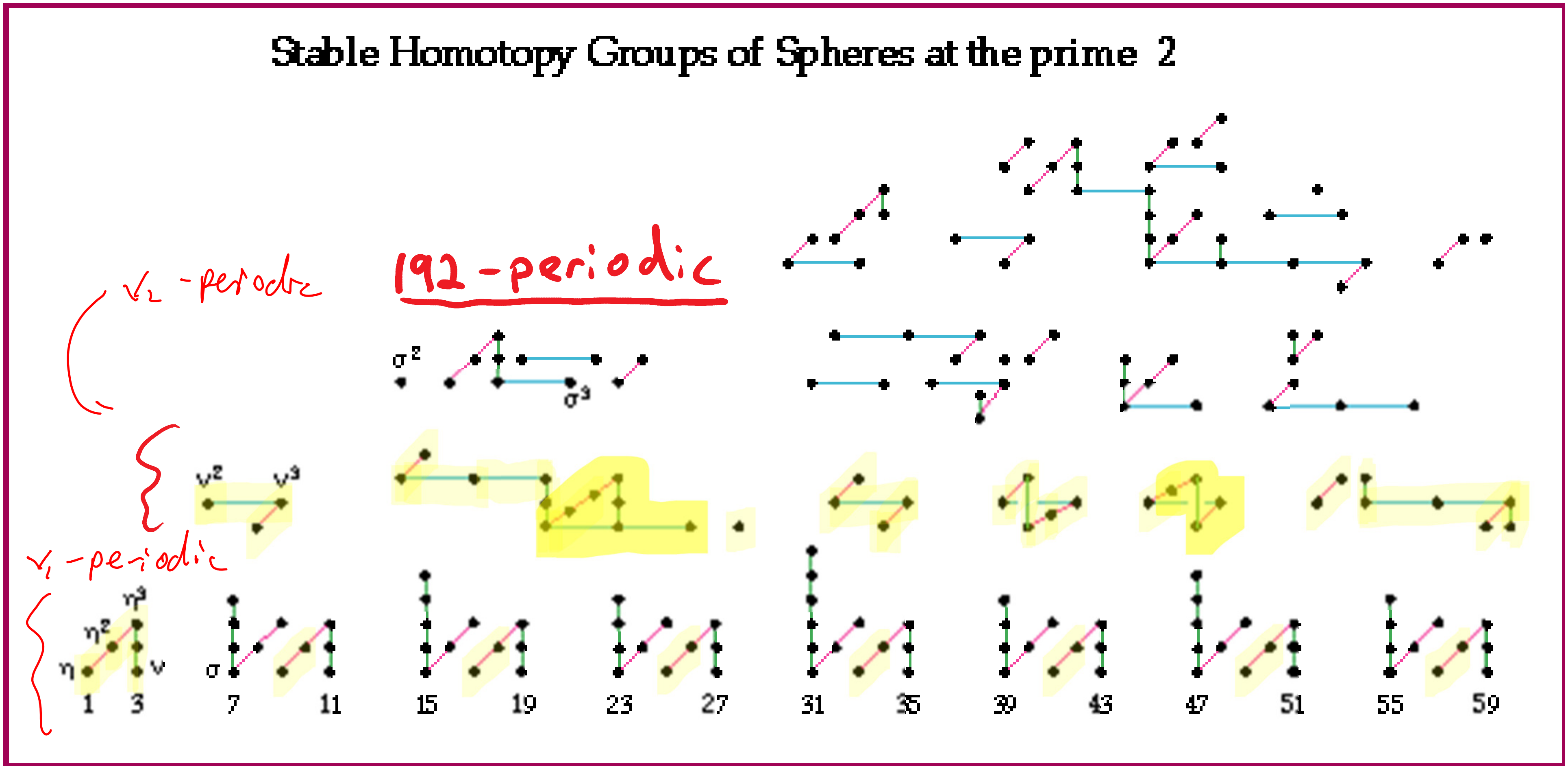
192-periodic

v_2-periodic

v_1-periodic



What the *e$# ??



Stable Homotopy Groups of Spheres at the prime 2

192-periodic

v_2-periodic

v_1-periodic

Plan to determine Hurewicz Image of tmf

Plan to determine Hurewicz Image of tmf

$y \in \pi_* tmf, \; * < 192$

(1) Try to construct element x of π_*^s

$$\begin{tikzcd}
\pi_*^s \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \pi_* tmf \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
x & y \arrow[l, mapsto]
\end{tikzcd}$$

Plan to determine Hurewicz Image of tmf

$y \in \pi_* tmf$, $* < 192$, y v_2-periodic

(1) Try to construct element x of π_*^s

$$\begin{tikzcd}
\pi_*^s \arrow[r] & \pi_* tmf \\
x \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & y \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

(2) Determine i, j s.t. $\begin{cases} 2^i x = 0 \\ v_1^j x = 0 \end{cases}$

$$\begin{tikzcd}
\pi_*^s M^0_{i,j} \arrow[r] & \pi_*^s \\
\tilde{x} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & x \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

Plan to determine Hurewicz Image of tmf

$y \in \pi_* tmf$, $* < 192$, y v_2-periodic

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x \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, mapsto] & y \arrow[u, phantom, "\rotatebox{90}{$\in$}"]
\end{tikzcd}$$

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$$\begin{cases} 2^i x = 0 \\ v_1^j x = 0 \end{cases}$$

$$\begin{tikzcd}
\pi_*^s M^0_{i,j} \arrow[r] & \pi_*^s \\
\tilde{x} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, mapsto] & x \arrow[u, phantom, "\rotatebox{90}{$\in$}"]
\end{tikzcd}$$

(3) Produce

$$v_2^{32} : \Sigma^{192} M_{i,j} \to M_{i,j}$$

Plan to determine Hurewicz Image of tmf

$y \in \pi_* tmf$, $* < 192$, y v_2-periodic

(1) Try to construct element x of π_*^s

$$\begin{tikzcd}
\pi_*^s \arrow[r] & \pi_* tmf \\
\rotatebox[origin=c]{90}{$\in$} & \rotatebox[origin=c]{90}{$\in$} \\
x \arrow[r, mapsto] & y
\end{tikzcd}$$

(2) Determine i, j s.t. $\begin{cases} 2^i x = 0 \\ v_1^j x = 0 \end{cases}$

$$\begin{tikzcd}
\pi_*^s M^0_{ij} \arrow[r] & \pi_*^s \\
\rotatebox[origin=c]{90}{$\in$} & \rotatebox[origin=c]{90}{$\in$} \\
\tilde{x} \arrow[r, mapsto] & x
\end{tikzcd}$$

(3) Produce

$$v_2^{32} : \Sigma^{192} M_{ij} \longrightarrow M_{ij}$$

(4) Get a 192-periodic family

$$\begin{tikzcd}
\pi_* M^0_{ij} \arrow[r] & \pi_*^s \arrow[r] & \pi_* tmf \\
\rotatebox[origin=c]{90}{$\in$} & \rotatebox[origin=c]{90}{$\in$} & \rotatebox[origin=c]{90}{$\in$} \\
v_2^{32k} \tilde{x} \arrow[r, mapsto] & v_2^{32k} x \arrow[r, mapsto] & v_2^{32k} y
\end{tikzcd}$$

v_2-periodicity at the prime 2

Thm [B-Hill-Hopkins-Mahowald]

$$\exists \quad v_2^{32} : \Sigma^{192} M^0_{1,4} \longrightarrow M^0_{1,4}$$

Uses TMF

v_2-periodicity at the prime 2

Thm [B-Hill-Hopkins-Mahowald]

$$\exists \quad v_2^{32} : \Sigma^{192} M^0_{1,4} \longrightarrow M^0_{1,4}$$

Uses TMF

Problem: Minimum i, j s.t. $\begin{cases} 2^i y = 0 \\ v_1^j y = 0 \end{cases}$

for $y \in \pi_* tmf$ (v_2-periodic)

is $(i,j) = (3,8)$

v_2-periodicity at the prime 2

Thm [B-Hill-Hopkins-Mahowald]

$$\exists \quad v_2^{32} : \Sigma^{192} M^0_{1,4} \longrightarrow M^0_{1,4}$$

Uses TMF

Thm [B-Mahowald]

$$\exists \quad v_2^{32} : \Sigma^{192} M^0_{3,8} \longrightarrow M^0_{3,8}$$

Allows for complete determination of Hurewicz image $p=2$

v_2-periodicity at the prime 2

Thm [B-Hill-Hopkins-Mahowald]

$$\exists \quad v_2^{32} : \Sigma^{192} M^0_{1,4} \longrightarrow M^0_{1,4}$$

proof of this thm:

"tmf-resolutions"

(AKA "eo_2-resolutions")

Thm [B-Mahowald]

$$\exists \quad v_2^{32} : \Sigma^{192} M^0_{3,8} \longrightarrow M^0_{3,8}$$

Allows for complete determination of Hurewicz image $p=2$

v_2^{32} on $M_{1,4}^0$

$\underline{bo}_n$ = $n^{\underline{th}}$ bo - Brown - Gitler spectrum

v_2^{32} on $M_{1,4}^0$

$\underline{bo}_n$ = nth bo-Brown-Gitler spectrum

$bo_n = H^*(\underline{bo}_n;\mathbb{F}_2)$ Module over $\mathcal{A}$

v_2^{32} on $M_{1,4}^0$

$\underline{bo}_n$ = $n^{\underline{th}}$ bo-Brown-Gitler spectrum

$bo_n = H^*(\underline{bo}_n; \mathbb{F}_2)$ Module over $\mathcal{A}$

$$\mathcal{A}/\!/_{A(2)} = H^*(tmf)$$

$$\underset{A(2)}{\cong} \bigoplus_{n \geqslant 0} \Sigma^{8n} bo_n$$

v_2^{32} on $M_{1,4}^0$

$\underline{bo}_n$ = nth bo - Brown - Gitler spectrum

bo_n = $H^*(\underline{bo}_n ; \mathbb{F}_2)$ Module over $\mathcal{A}$

$$\mathcal{A}/\!\!/_{A(2)} = H^*(tmf)$$

$$\underset{A(2)}{\cong} \bigoplus_{n \geq 0} \Sigma^{8n} bo_n$$

SS: <u>algebraic tmf - resolution</u>

$$\mathrm{Ext}_{A(2)}(bo_{n_1} \otimes \cdots \otimes bo_{n_s} \otimes M) \Rightarrow \mathrm{Ext}_{\mathcal{A}}(M)$$

v_2^{32} on $M_{1,4}^0$

$$\mathrm{Ext}_{A(2)}(bo_{n_1} \otimes \cdots \otimes bo_{n_s} \otimes M_{1,4}) \Rightarrow \mathrm{Ext}_A(M_{1,4})$$

v_2^{32} on $M_{1,4}^0$

$$\mathrm{Ext}_{A(2)}(bo_{n_1} \otimes \cdots \otimes bo_{n_s} \otimes M_{1,4}) \Rightarrow \mathrm{Ext}_A(M_{1,4})$$

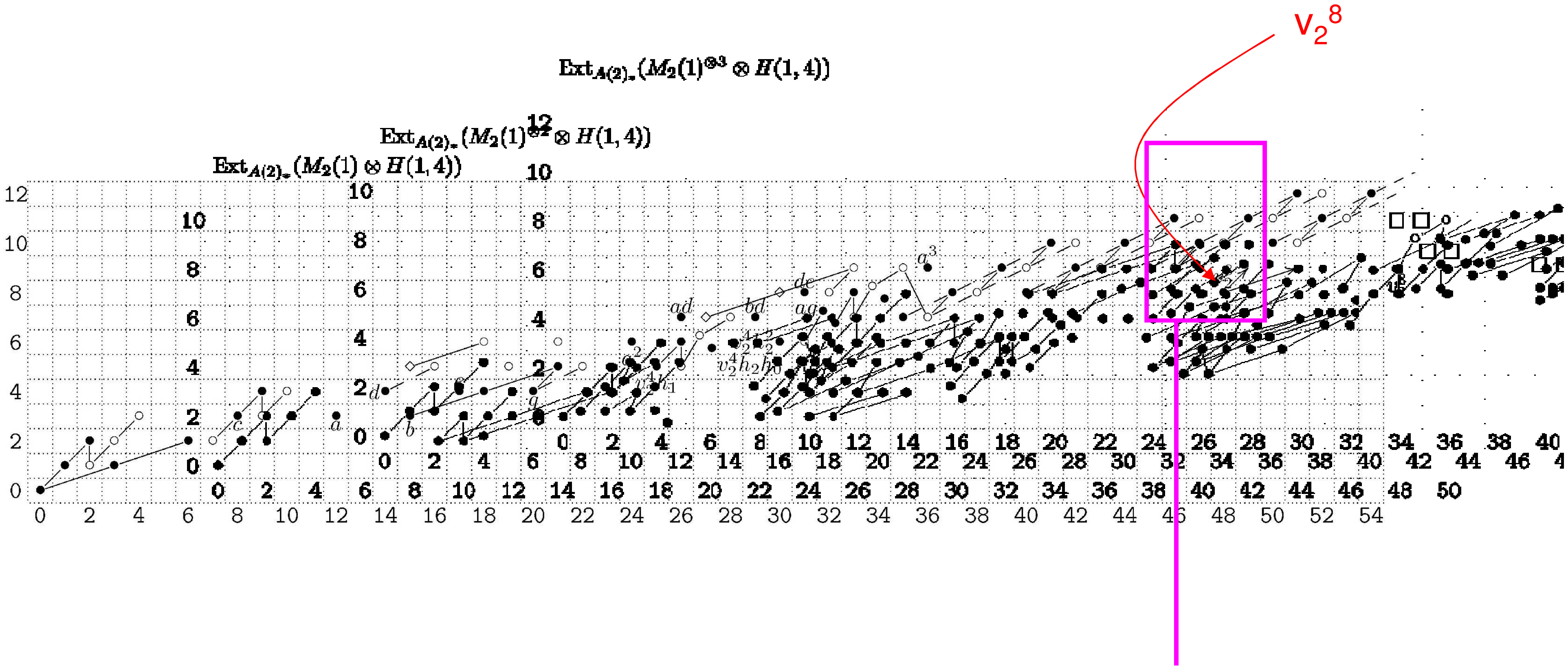
$$v_2^{32} \in \mathrm{Ext}_{A(2)}(M_{1,4})$$

Vanishing lines

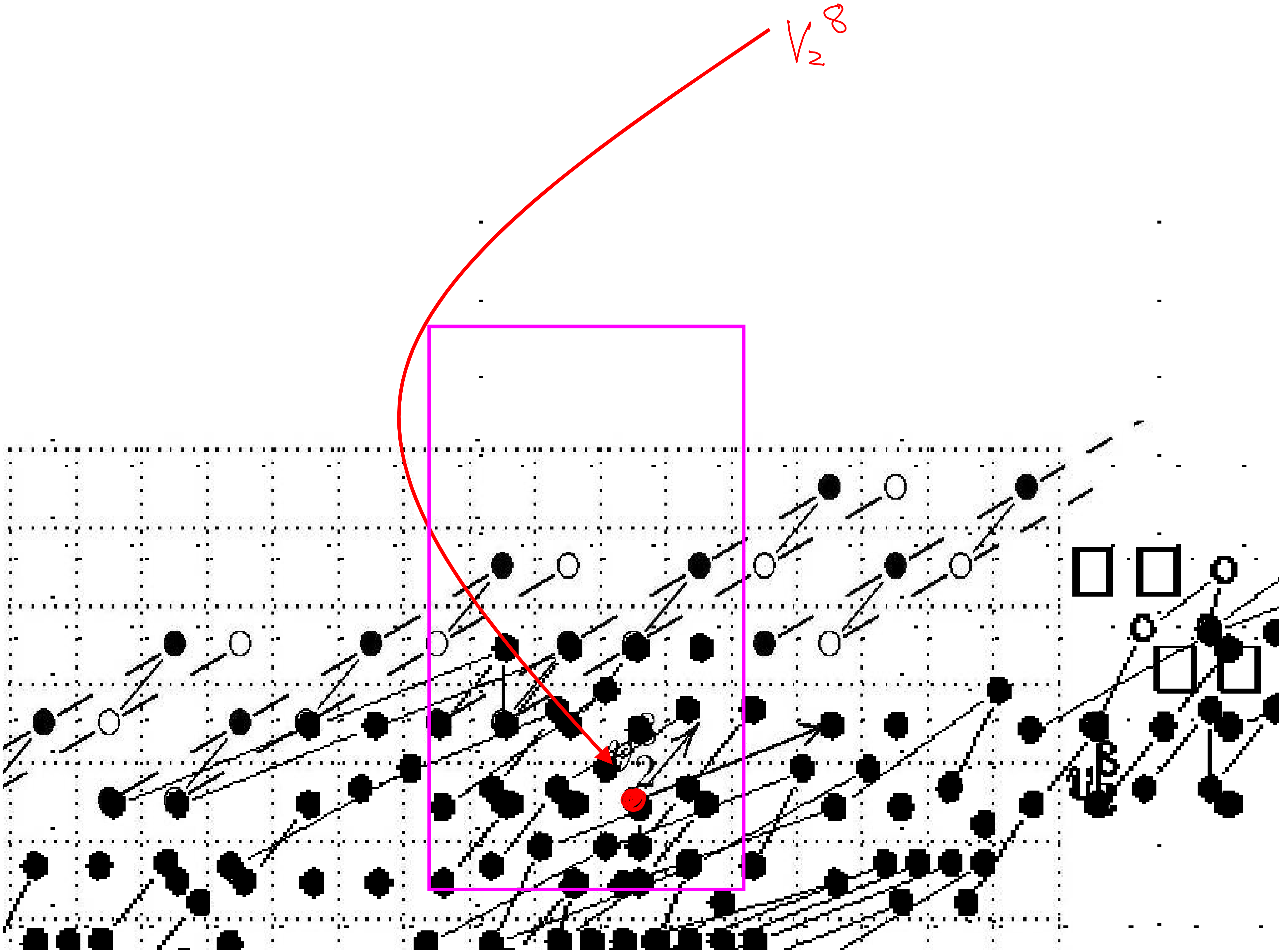
$\Rightarrow d_r(v_2^{32})$ detected on $bo_1^{\otimes j}$ $j \leq 3$

algebraic tmf-resolution for $M^0_{1,4}$

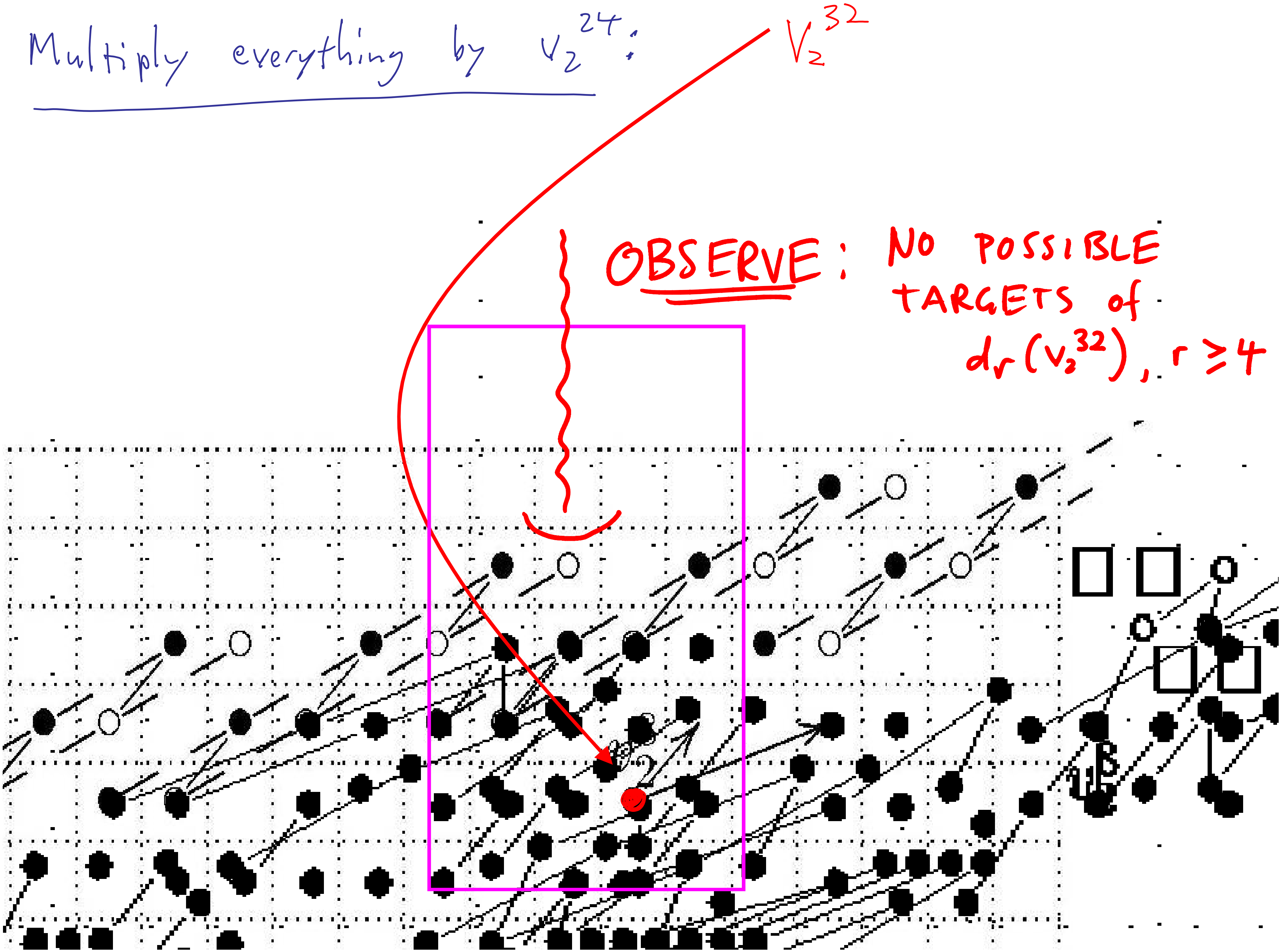
$$\bigoplus_{0\leq j\leq 3} \mathrm{Ext}_{A(2)}\left(bo_1^{\otimes j} \otimes M^0_{1,4}\right)$$



ZOOM in on this area…


V_2^8

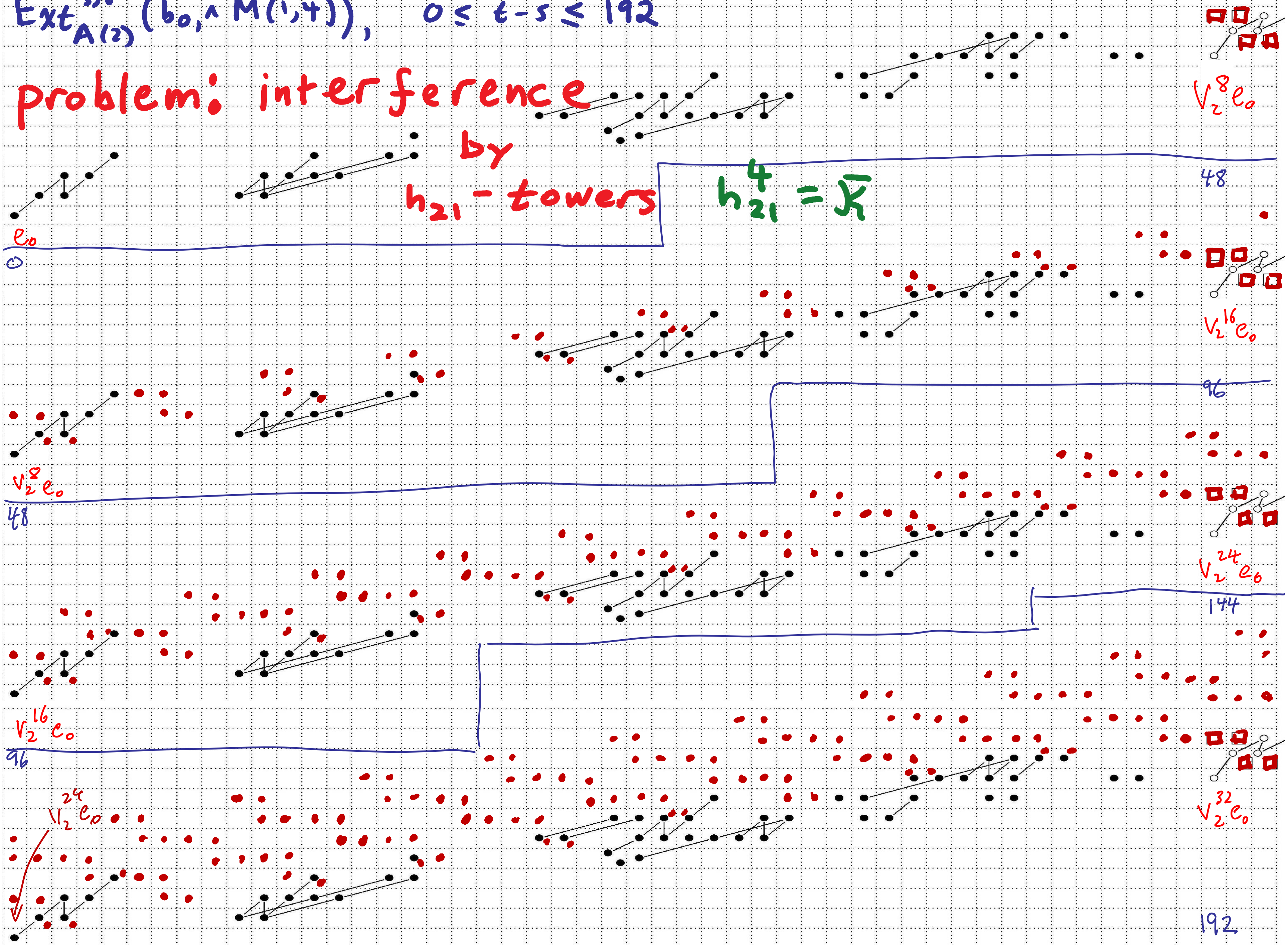
Multiply everything by v_2^{24}:



$\mathrm{Ext}_{A(2)}^{s,t}(bo_1 \wedge M(1,4))$, $0 \le t-s \le 192$

problem: interference by h_{21}-towers

$h_{21}^4 = \bar{\kappa}$

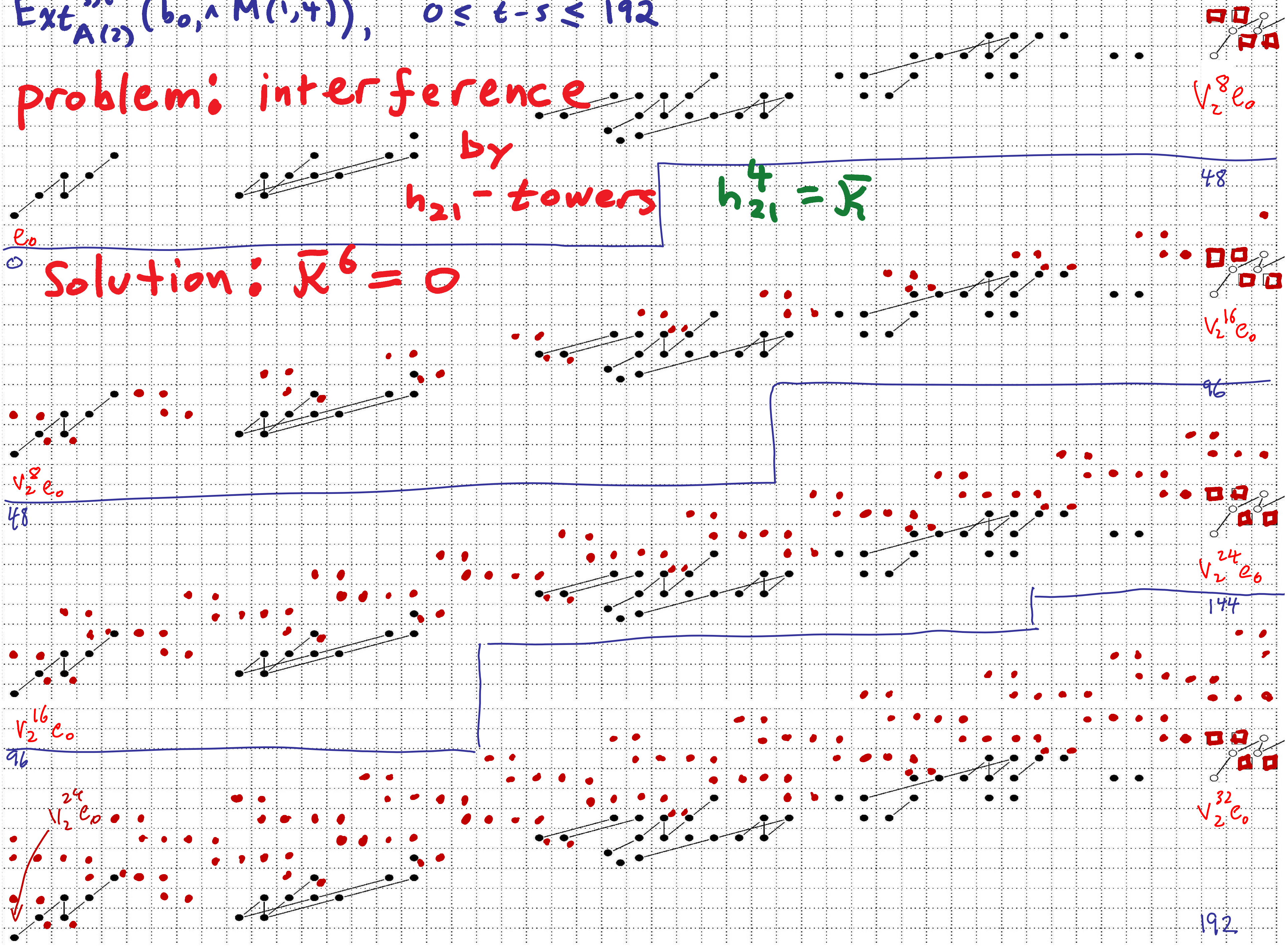


$\mathrm{Ext}^{s,t}_{A(2)}(bo_1 \wedge M(1,4))$, $0 \le t-s \le 192$

problem: interference by h_{21}-towers

$h_{21}^4 = \bar{\kappa}$

Solution: $\bar{\kappa}^6 = 0$



Modifications for the case of $M^0_{3,8}$

- Only potential targets of $d_r(v_2^{32})$ come from bo_1^j for $0 \leq j \leq 6$ and $bo_1^j \otimes bo_2$ for $0 \leq j \leq 2$.

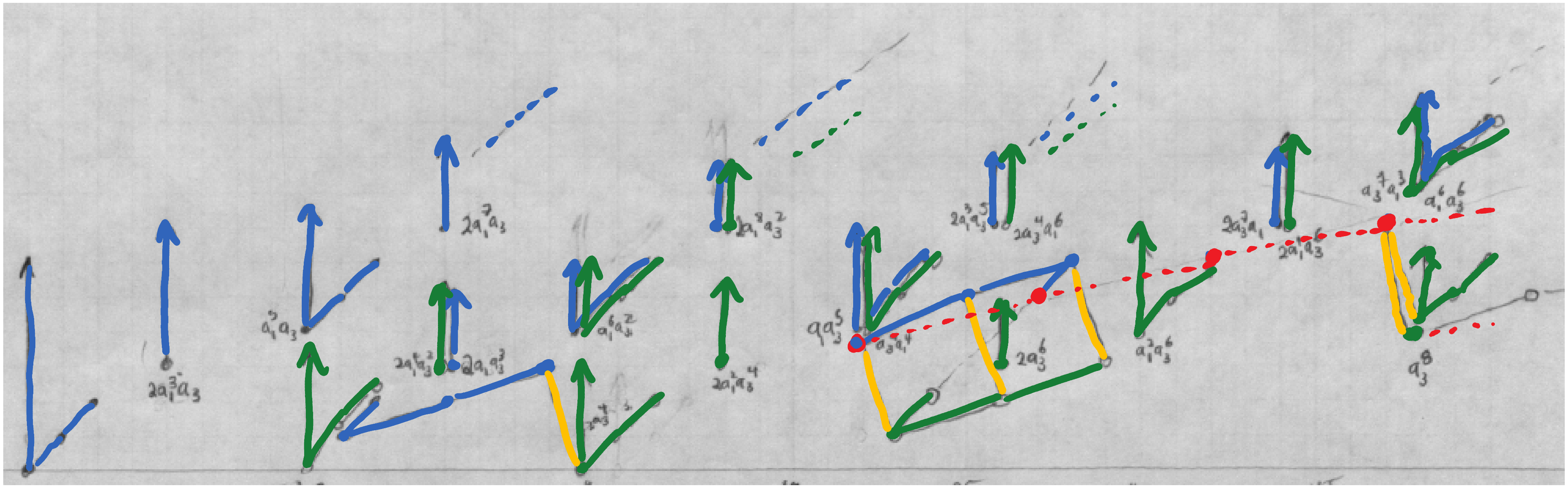
- Many potential contributions from $h_{2,1}^s$ for $s < 24$, which are not handled by $\bar{\kappa}^6 = 0$.

- Use result of Davis-Mahowald-Rezk:

$$tmf \wedge tmf = \bigcup_n \Sigma^{8n}\, tmf \wedge \underline{bo}_n$$

In this decomposition, $\underline{bo}_2$ attaches nontrivially to $\underline{bo}_1$

In $tmf \wedge \underline{bo}_1 \vee \underline{bo}_2$, h_{21}-towers cancel!



Q: So why does the “dual” of tmf show up in π_*^s?

A: Gross-Hopkins duality:

$v_2^{-1}\pi_* M_{3,8}^0$ is self-dual

Homotopy carried by bottom cell is dual to homotopy carried on top cell.

Bottom cell carries $\pi_* tmf$ $\Rightarrow$ top cell carries $\pi_* tmf^\vee$

$$\pi_* M_{3,8}^0 \to \pi_*^s$$

Coker J

n = 0 mod 4

Stem	p = 2	p = 3	p = 5
4	0	0	0
8	ε	0	0
12	0	0	0
16	η4	0	0
20	κbar	β1^2	0
24	h4 ε η	0	0
28	ε κbar	0	0
32	q	0	0
36	t	β2 β1	0
40	κbar^2	β1^4	0
44	g2	0	0
48	e0 r	0	0
52	κbar q	β2^2	0
56	κbar t	0	0
60	κbar^3	0	0
64	η6	0	0
68	v2^8 k v^2	<α1,β3/2,β2>	0
72		β2^2 β1^2	0
76		0	β1^2
80	κbar^4	0	0
84		β5 β1	0
88	g2^2	0	0
92		β6/3 β1	0
96	η6 d1	0	0
100	κbar^5	β2 β5	0
104	v2^16 ε		0
108	η6 g2		0
112		β6/3 β1^3	0
116	2v2^16 κbar		0
120	(v2^16 η κbar)ν		0
124	v2^16 k^2		β2 β1
128	v2^16 q		0
132	(h2 h6^2)ν		0
136	<v2^16 k κbar,2,ν^2>		0
140			0
144	((v2^16 ηw)η/2)η	v2^9	0
148	v2^16 ε κbar		0
152			β1^4
156	<Δ^6 ν^2,2ν,η^2>		0
160			0

n = -2 mod 8 (including Kervaire Inv 1)

Stem	p = 2	p = 3	p = 5
6	ν^2	0	0
14	k	0	0
22	ε k	0	0
30	θ4	β1^3	0
38	y	β3/2	β1
46	w η	β2 β1^2	0
54	v2^8 ν^2	0	0
62	h5 n	β2^2 β1	0
70	<κbar w,ν,η>	0	0
78		β2^3	0
86		β6/2	β2
94		β5	0
102	v2^16 ν^2	β6/3 β1^2	0
110	v2^16 k		0
118	v2^16 η^2 κbar		0
126			0
134			β3
142	v2^16 η w		0
150	(v2^16 ε κbar)η^2	v2^9	0
158			0
166			0
174	beta32/8	β1^3	0
182	beta32/4	β3/2	β4
190		β2 β1^2	β1^5
198	v2^32 ν^2		0
206	k	β2^2 β1	β5/4
214	ε k		β5/3
222		β2^3	β5/2
230		β6/2	β5
238	w η	β5	β2 β1^4
246	v2^8 ν^2	β6/3 β1^2	0
254			0
262	<κbar w,ν,η>		0
270			0
278			β1
286			β3 β1^4
294	v2^16 ν^2	v2^18	0
302	v2^16 k		0
310	v2^16 η^2 κbar		0
318		β1^3	0

n = 2^k - 3 (where Θ_n^bp = 0 because of Kervaire class)

Stem	p = 2	p = 3	p = 5
1	0	0	0
5	0	0	0
13	0	β1 α1	0
29	0	β2 α1	0
61	0	β4 α1	0
125?	w κbar^4		0

🟪 = in tmf
🟨 = not in tmf, not known to be v2-periodic
🟧 = not in tmf, but v2-periodic
🟥 = Kervaire
🟦 = trivial