Manifolds and Cobordism

Mark Behrens

Manifolds

Def: A d-dimensional manifold is a subset $M \subseteq \mathbb{R}^n$ which "locally looks like $\mathbb{R}^d$"

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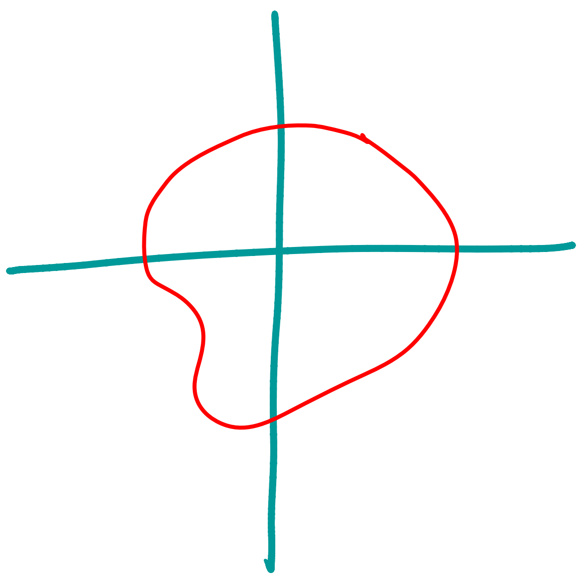
Example:



A 1-dim'l manifold in $\mathbb{R}^2$

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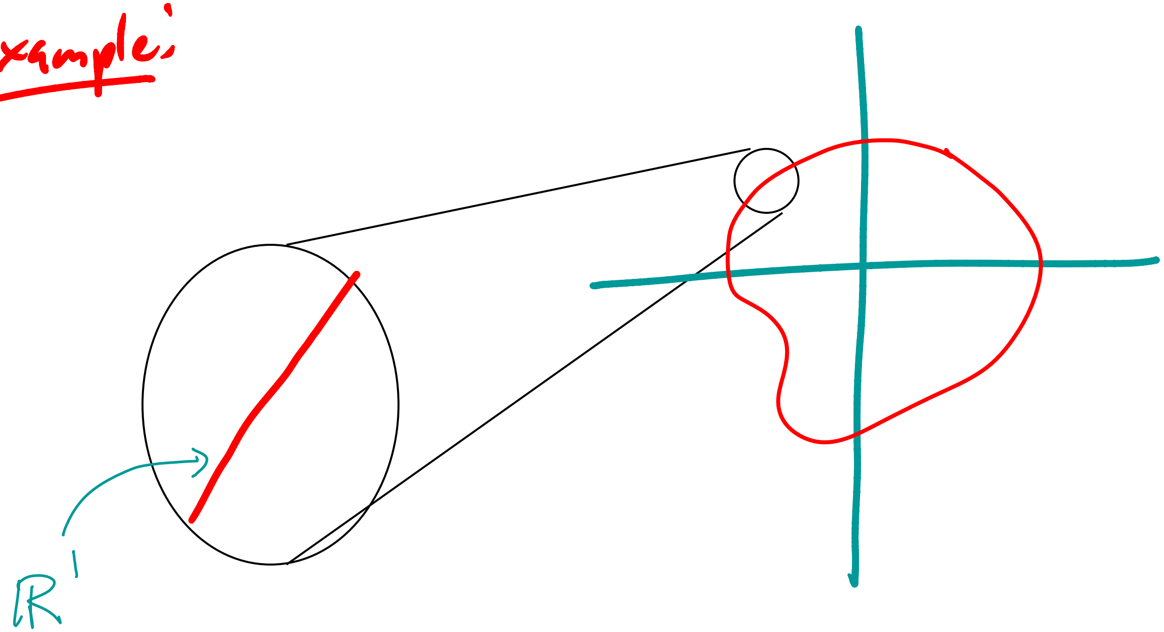

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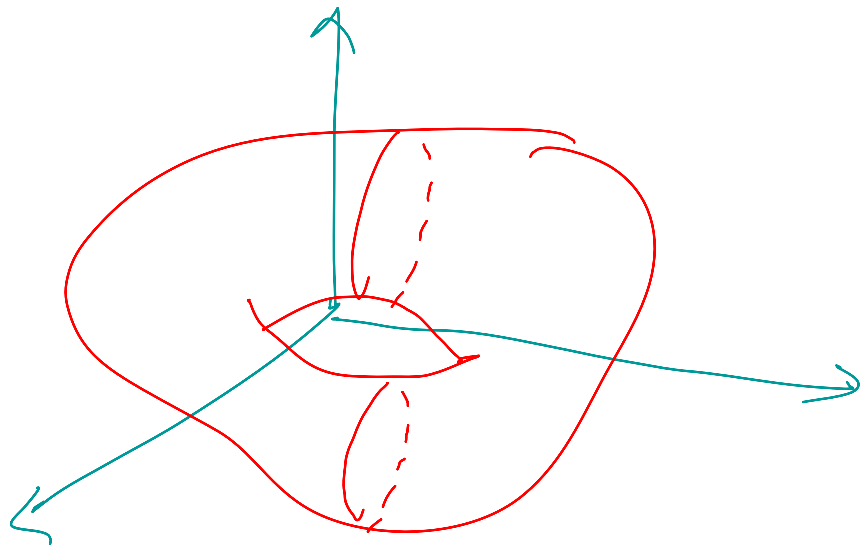

a 2-dim'l manifold in $\mathbb{R}^3$

Manifolds

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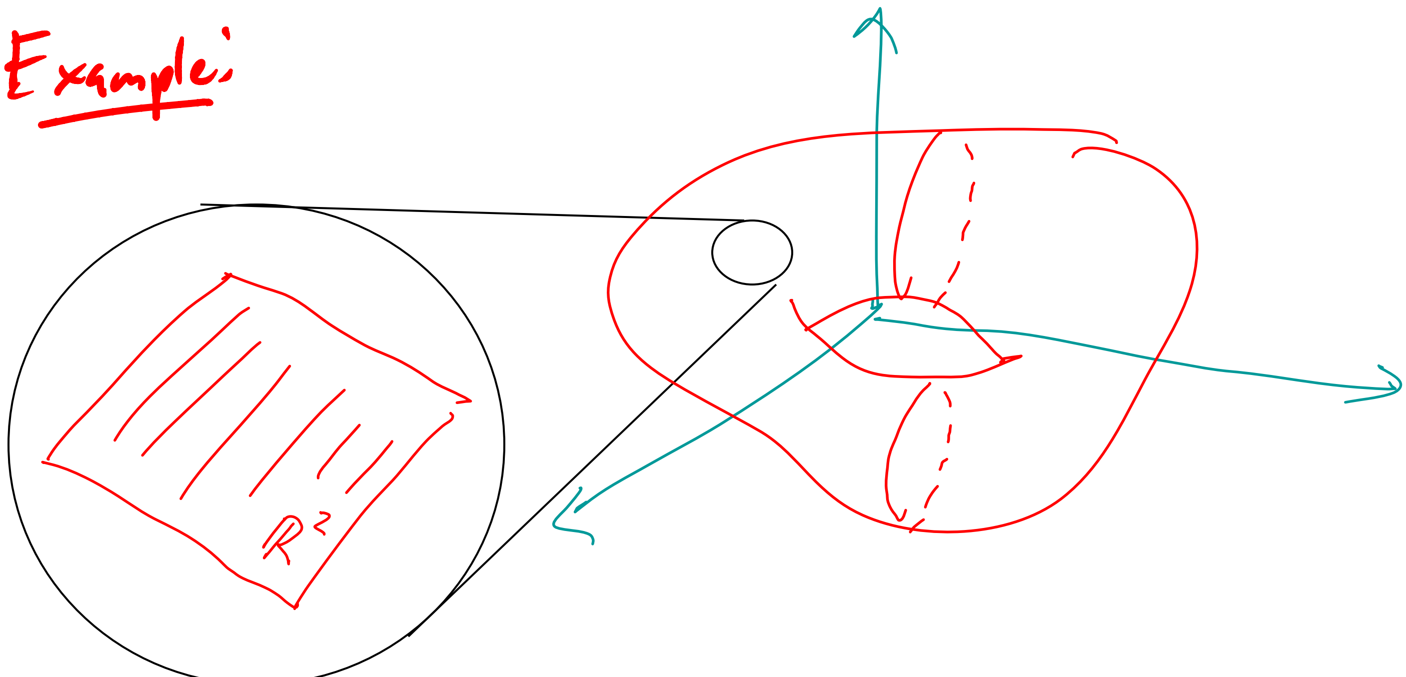

Manifolds

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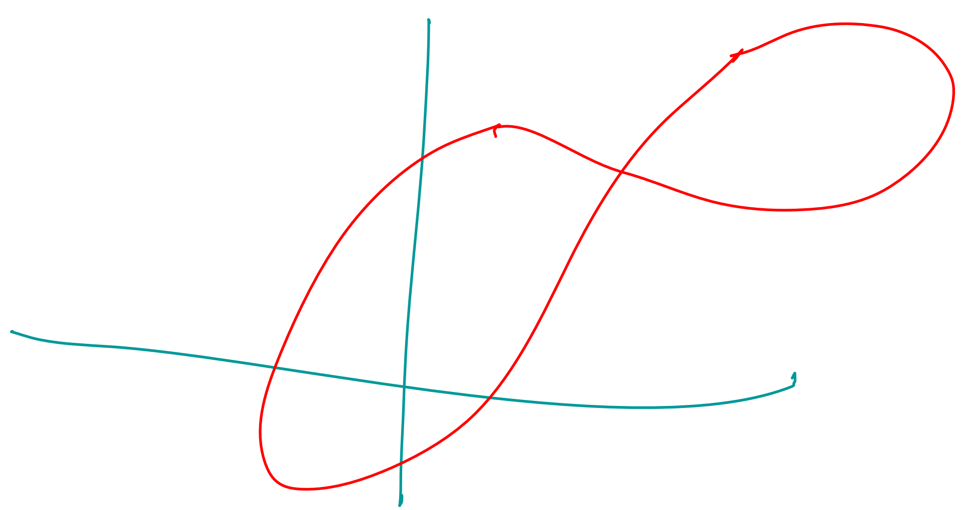
NON-EXAMPLE



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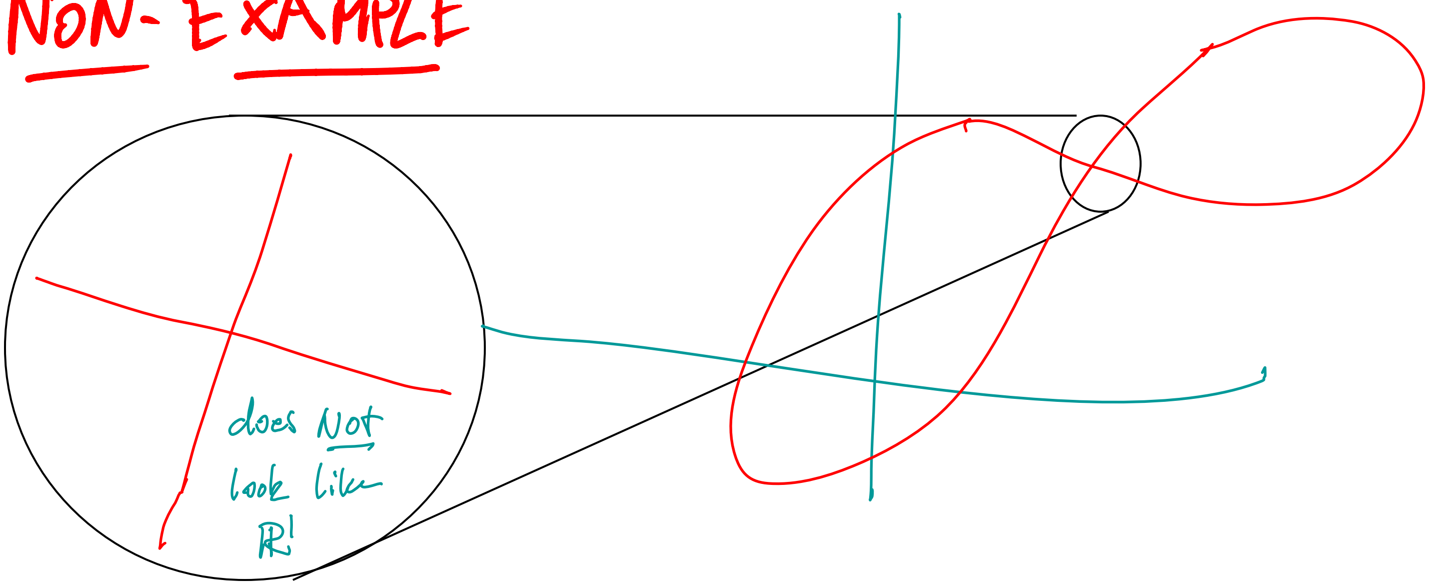
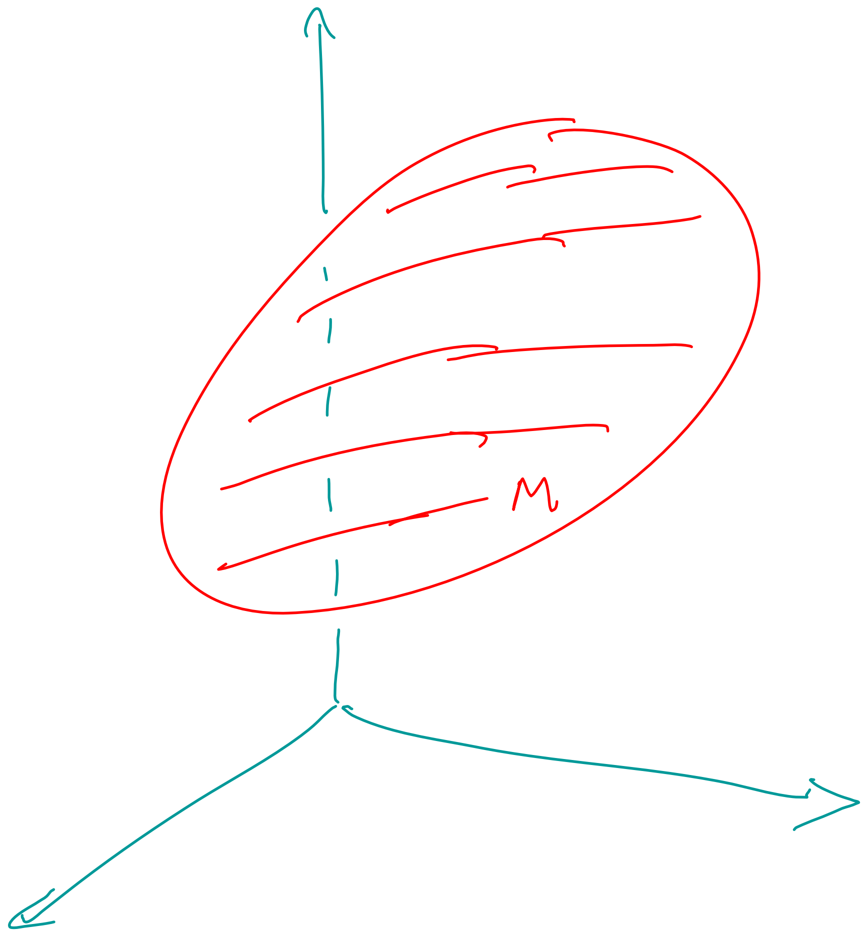


Manifolds with boundaries



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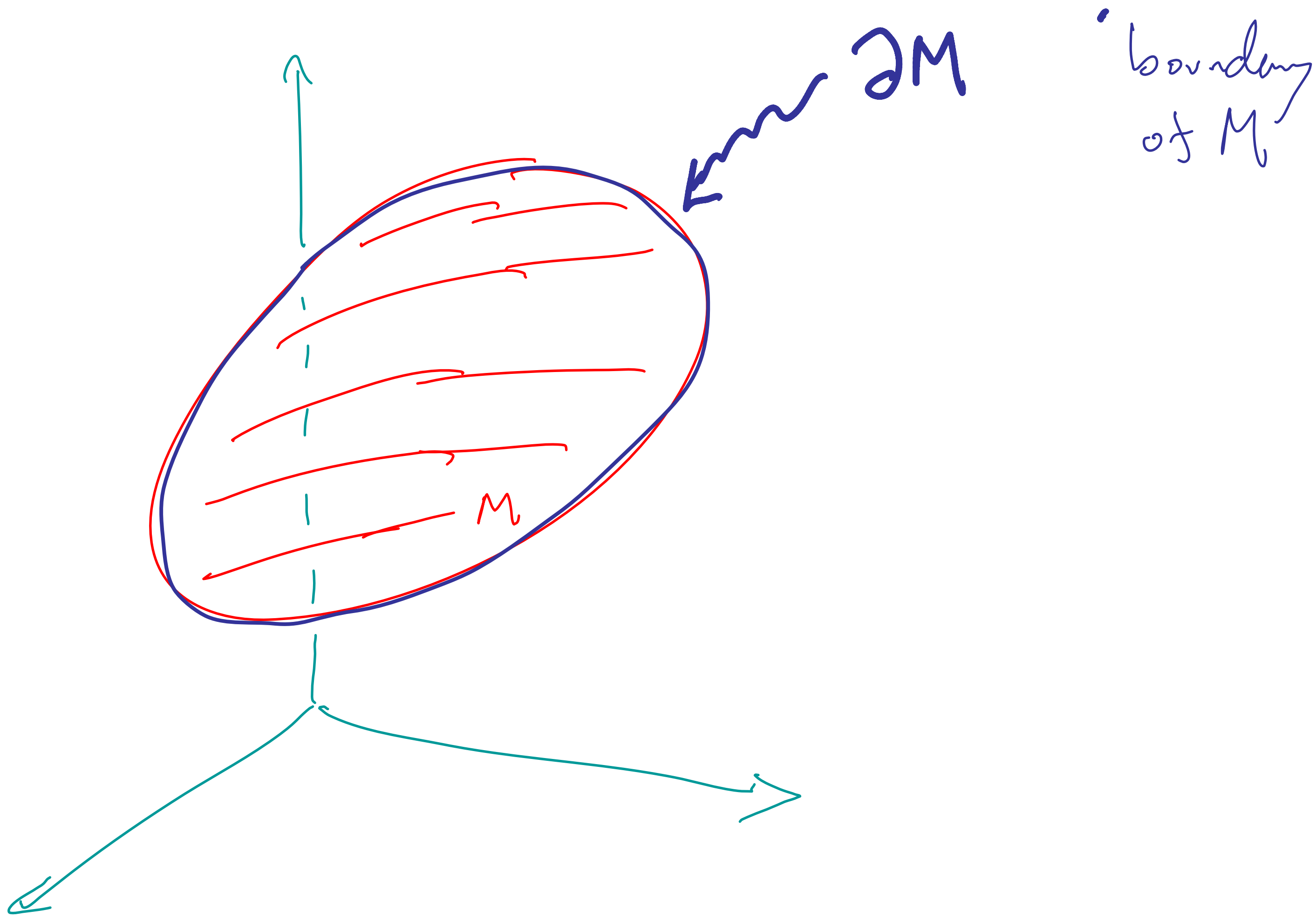
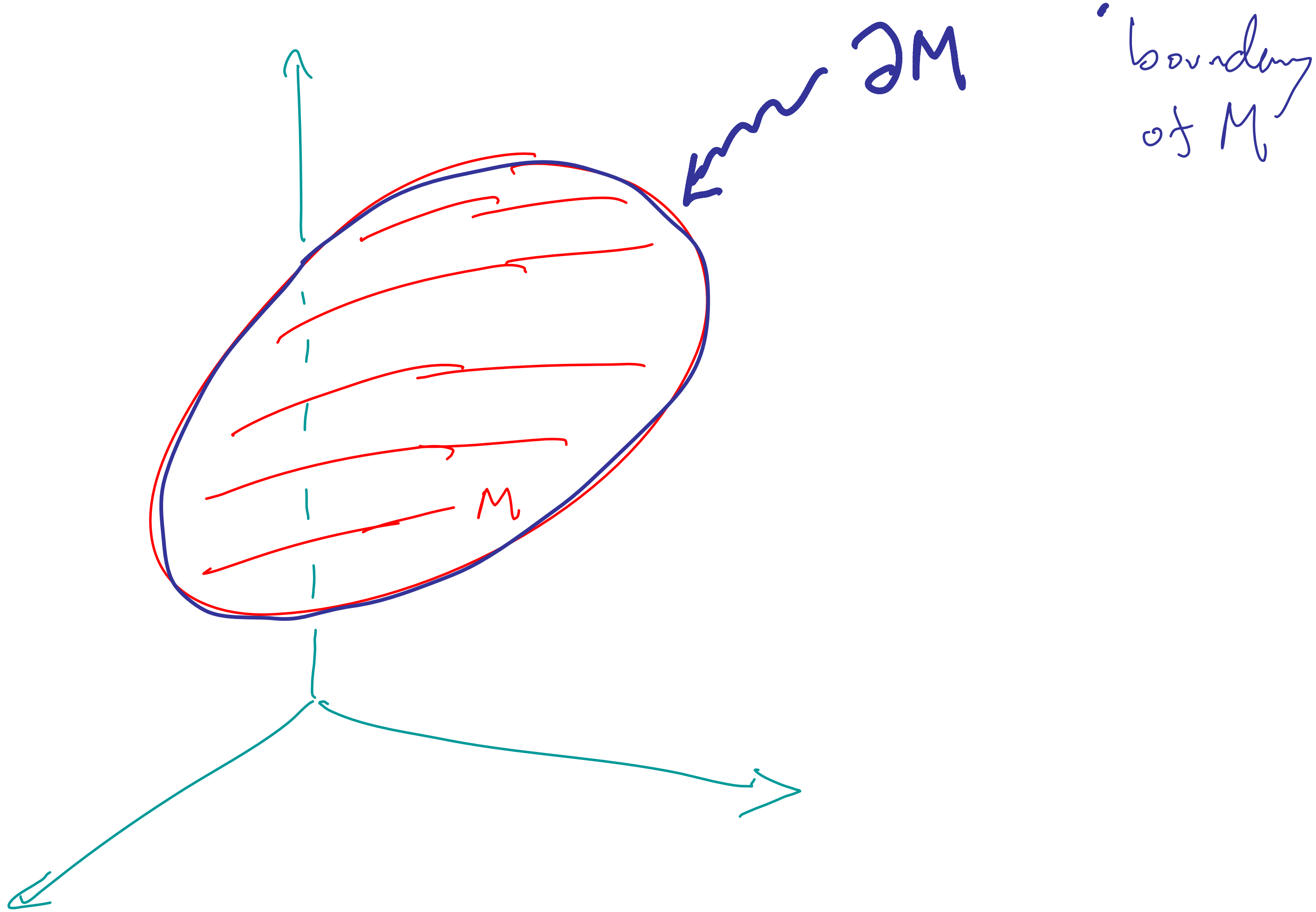
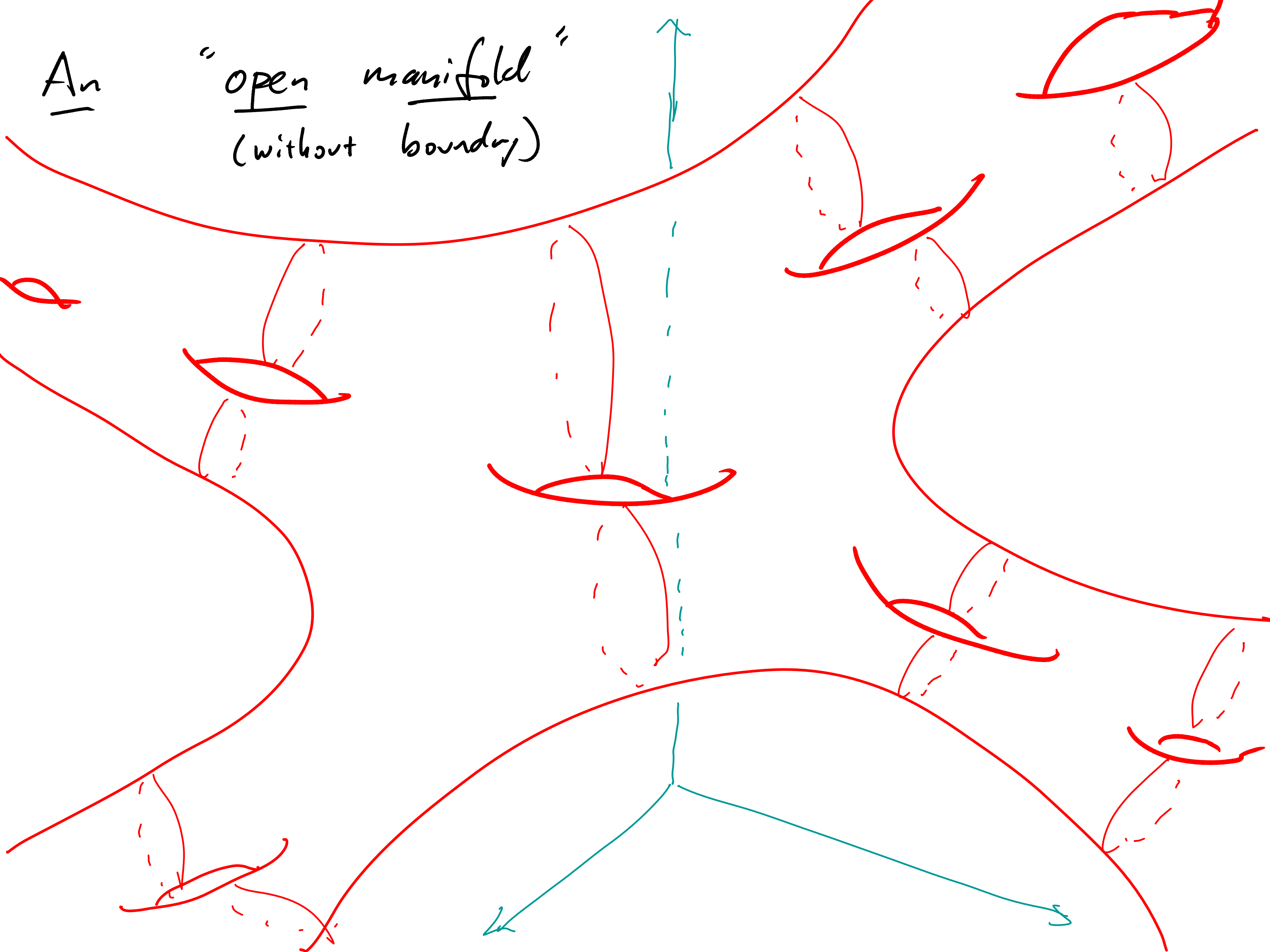


Manifolds with boundaries



Note! If M is d-dim'l, ∂M is $(d-1)$-dim'l

An "open manifold"
(without boundary)



Closed Manifolds

Def: A closed mfld M satisfies

- $\partial M = \emptyset$
- bounded

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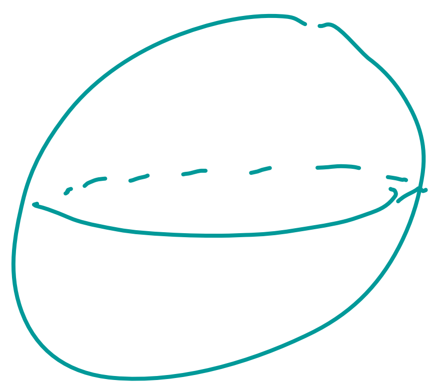
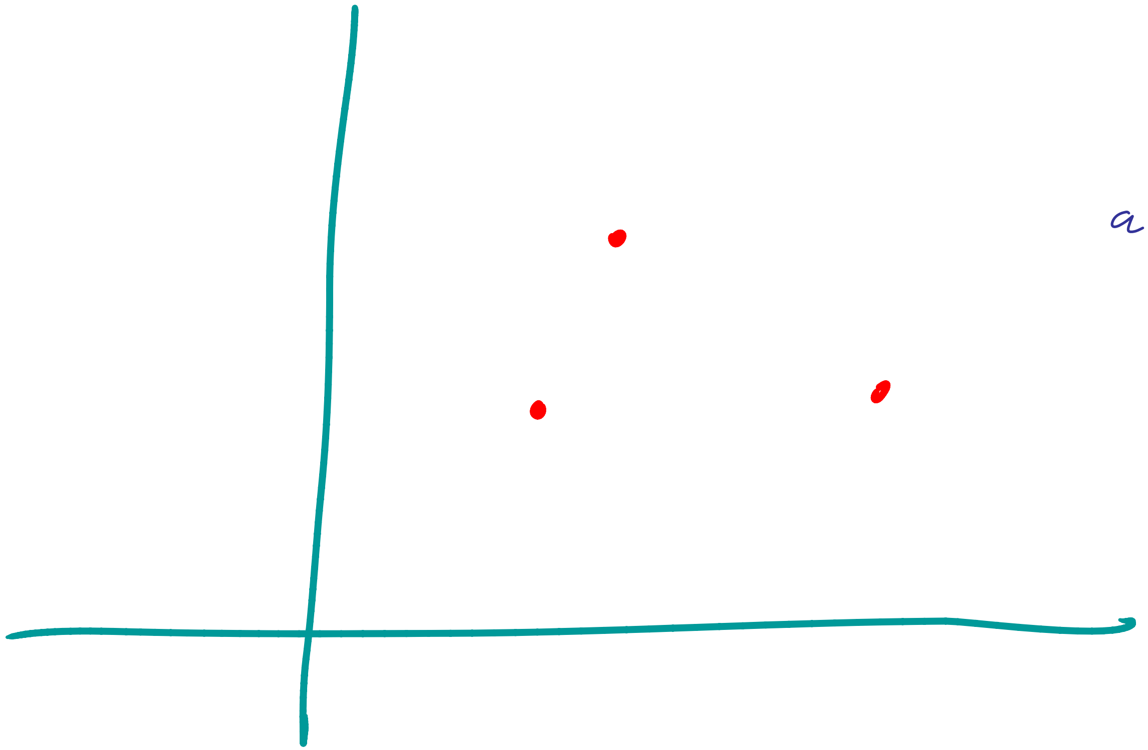


$S^2 =$ 2-sphere

0-dim'l manifolds

$\mathbb{R}^0$ = a point.



a 0-dim'l manifold in $\mathbb{R}^2$

0-dim'l manifolds

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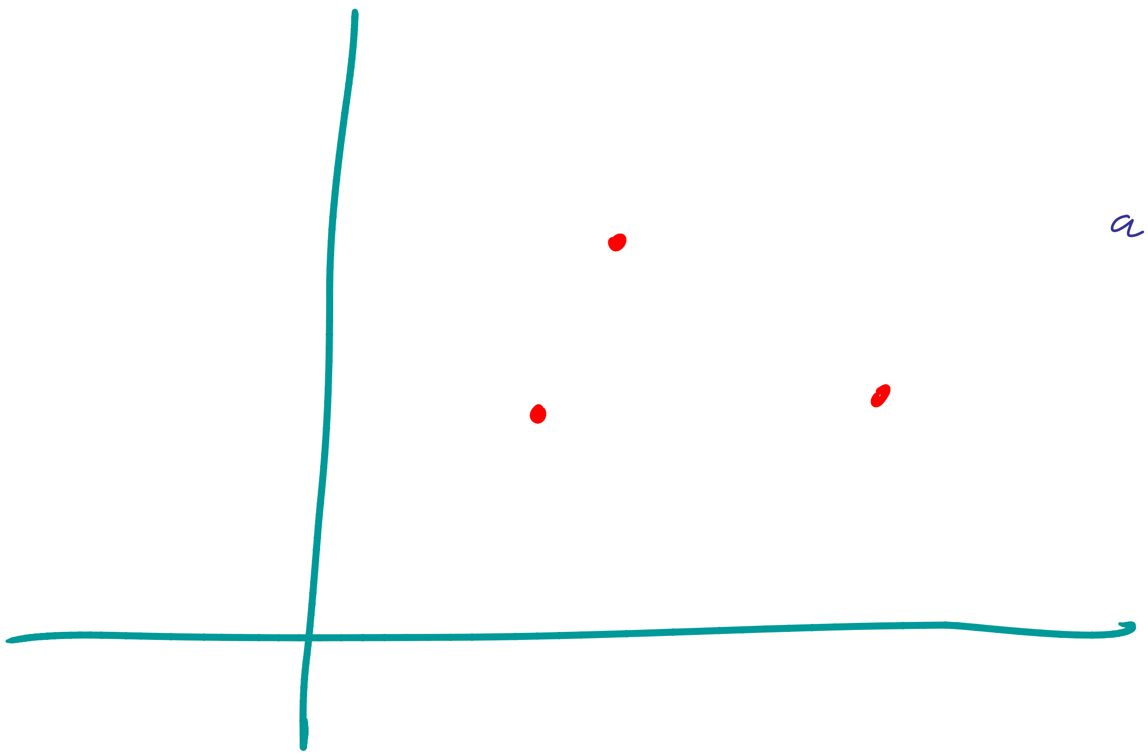


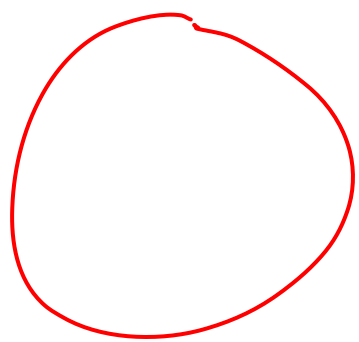
a 0-dim'l manifold in $\mathbb{R}^2$

Every 0-dim'l manifold is a disjoint union of points.

Connected 1-dim'l manifolds

Only two kinds:

$\mathbb{R}^1$ [not closed]

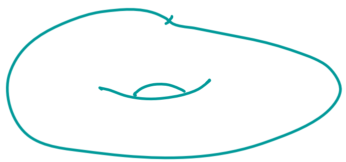


S^1 [closed]

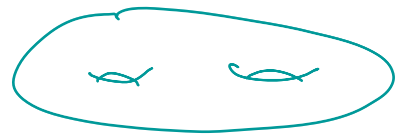
(circle)

Connected Closed 2-dim'l manifolds:

genus 0



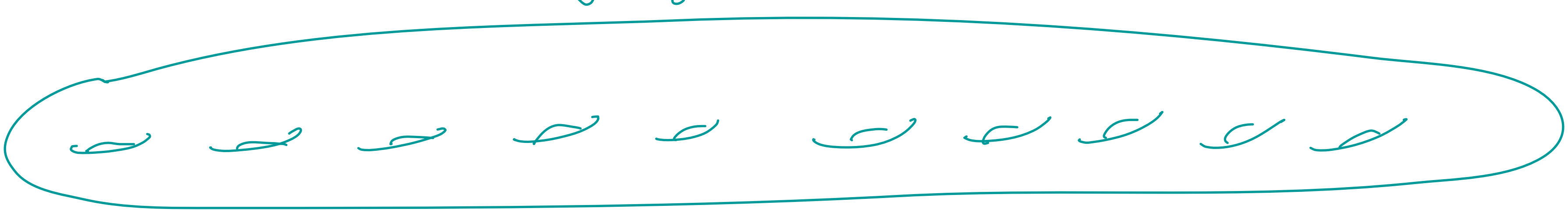
genus 1



genus 2

. . . .

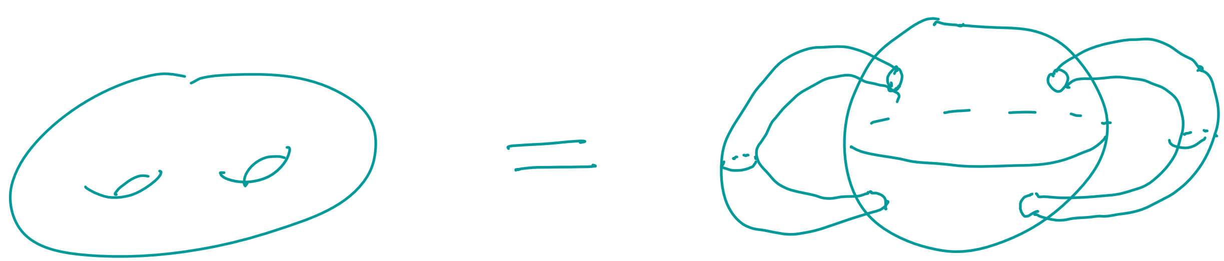
genus 10



Connected Closed 2-dim'l manifolds:

genus 0 genus 1 genus 2

genus = "# of handles"



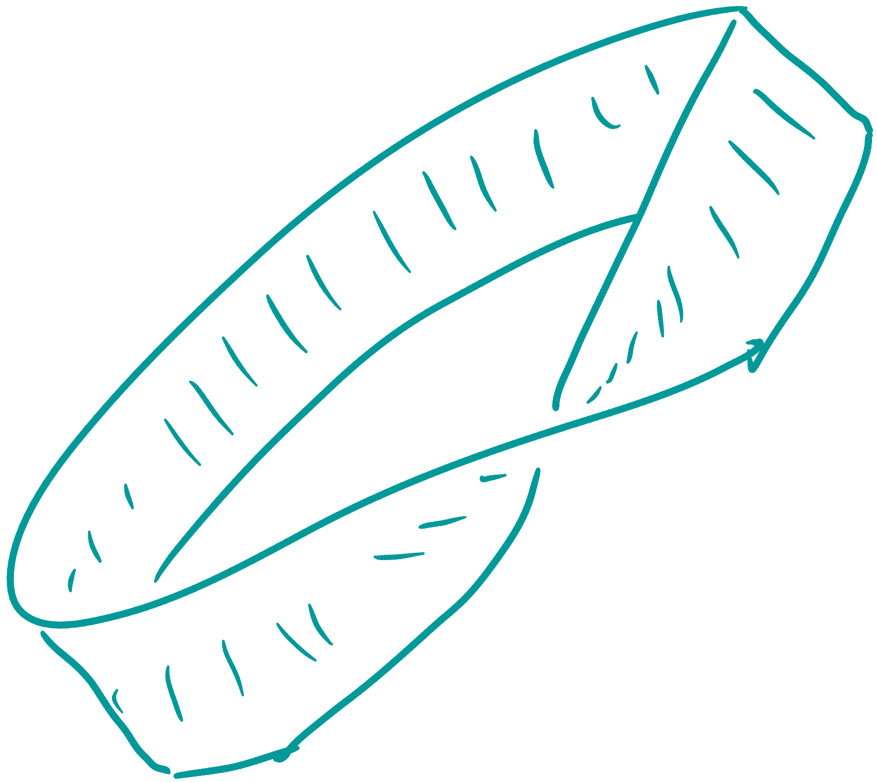
2 handles

Connected ORIENTABLE Closed 2-dim'l manifolds:

genus 0 genus 1 genus 2

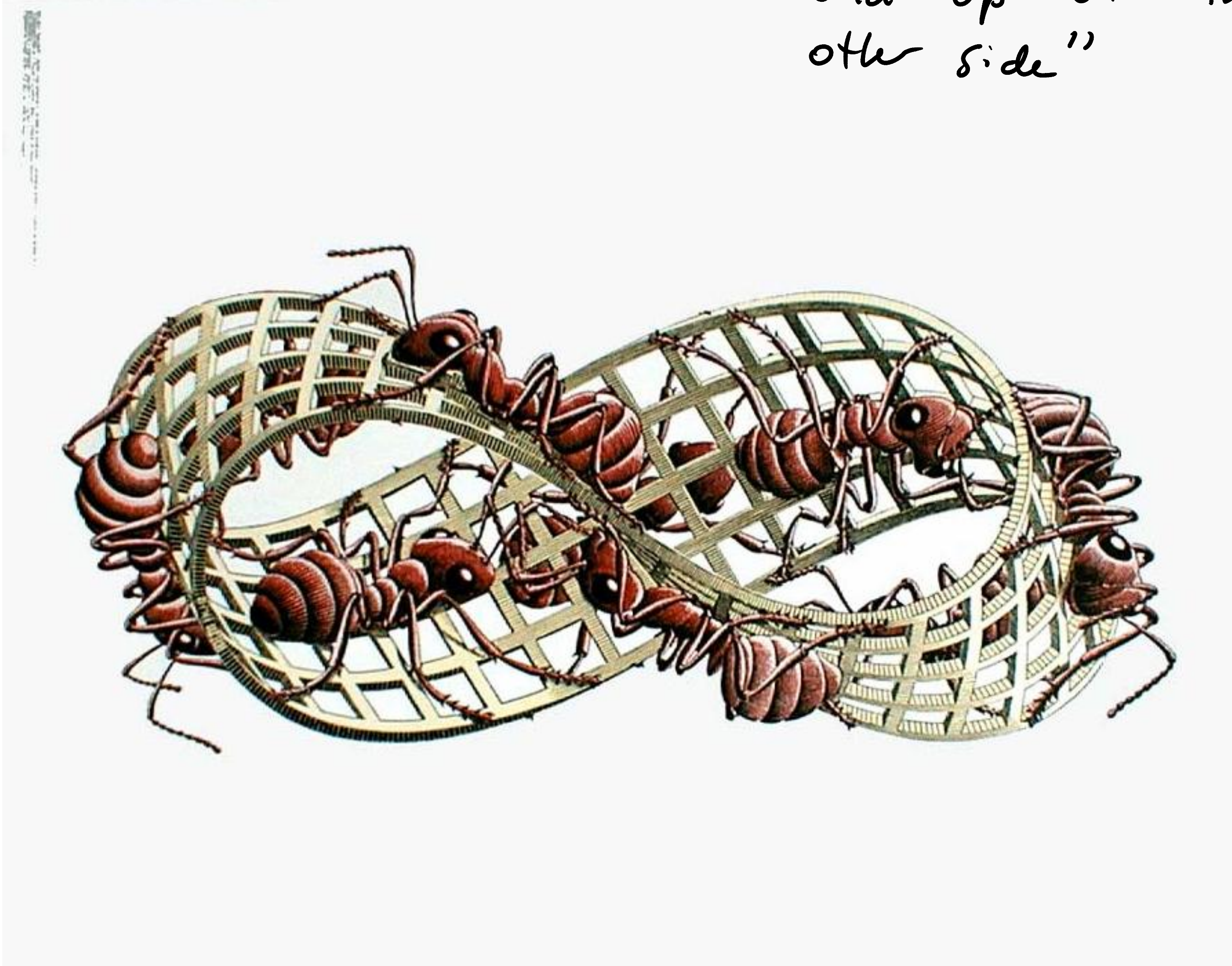
genus 10

The Möbius Strip



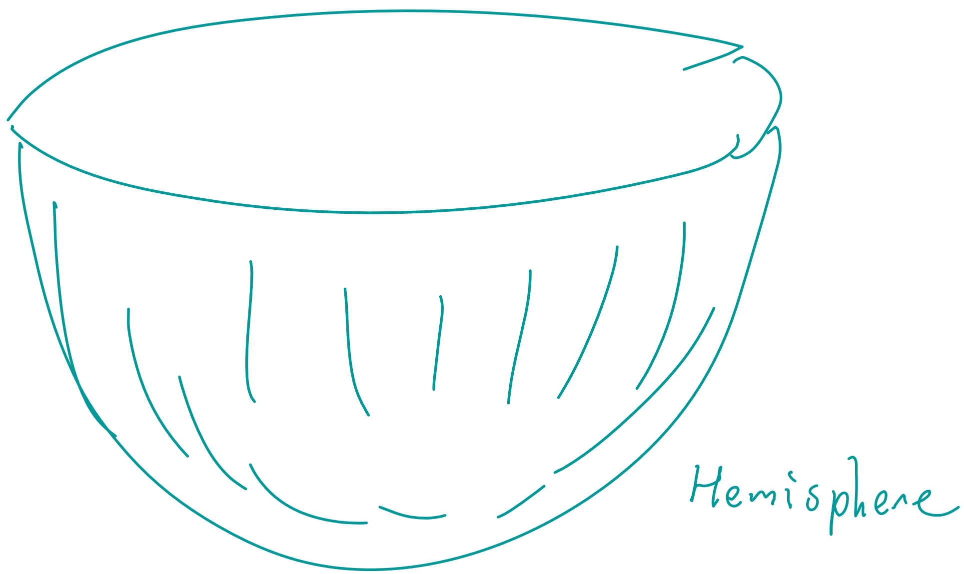
(Non-orientable manifold w/ boundary)

Non-orientable: "you travel along a loop and end up on the other side"

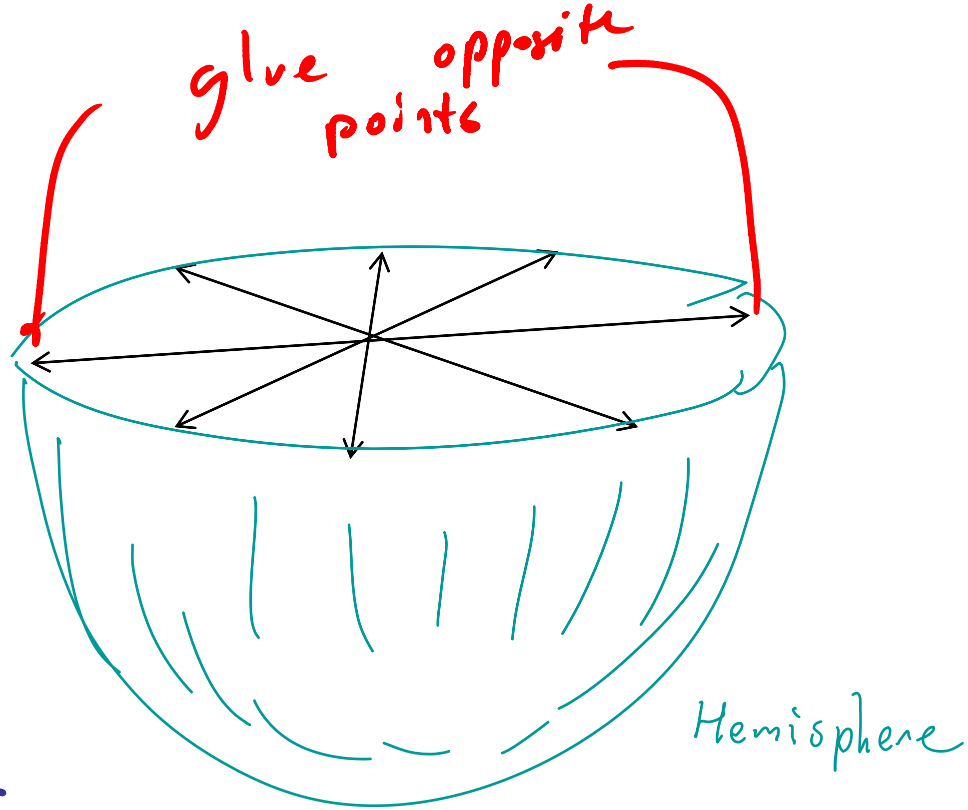


[M.C. Escher]

Projective space: A closed non-orientable surface

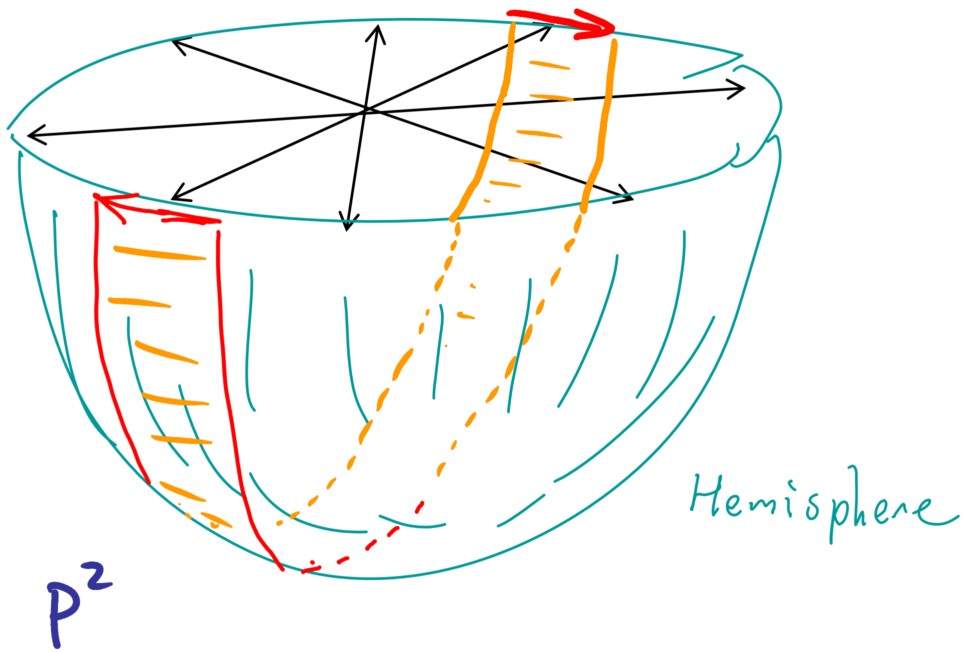


Projective space: A closed non-orientable surface



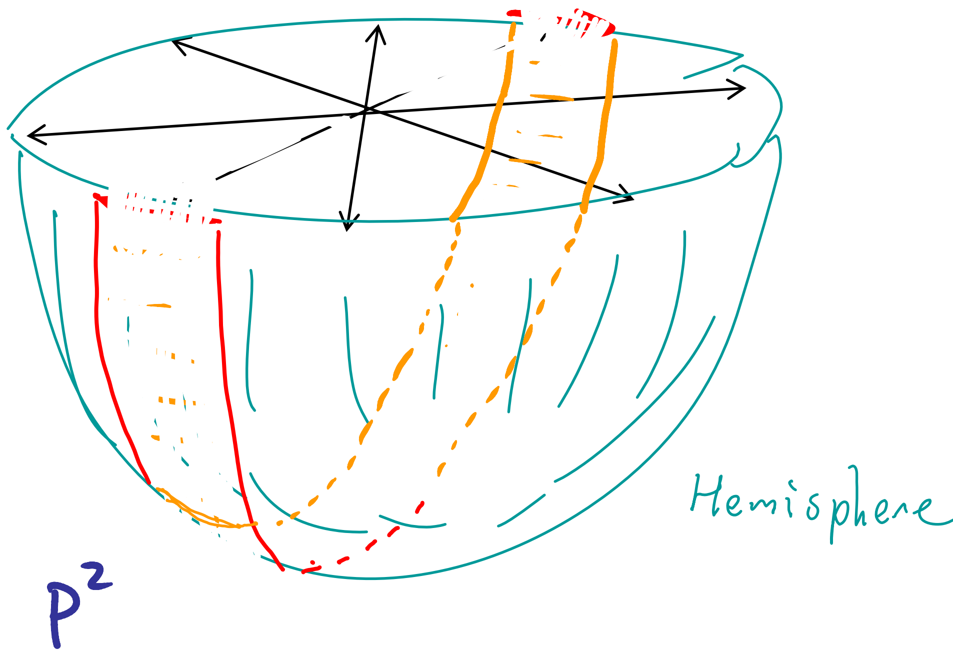
P^2

Projective space: A closed non-orientable surface



Non-orientable: contains a Mobius band.

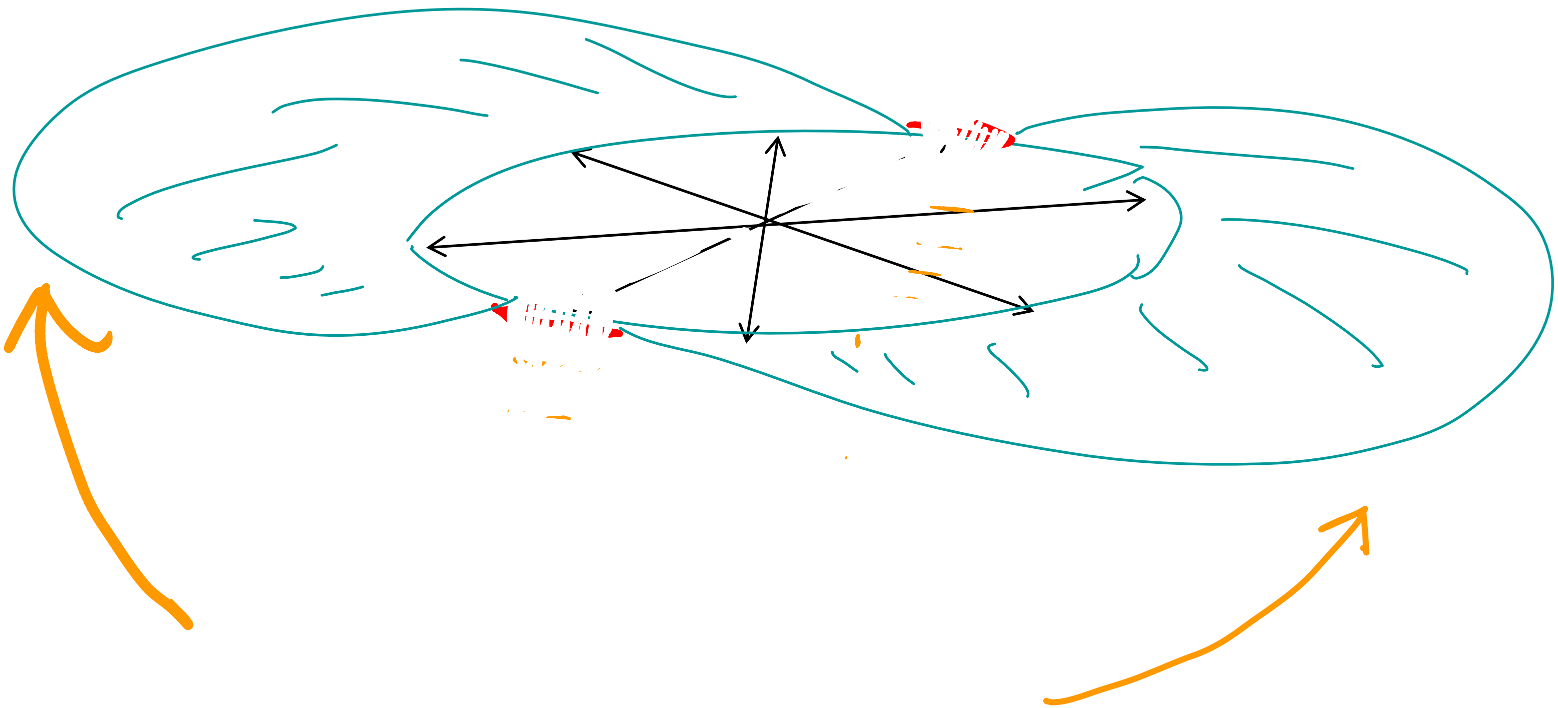
Projective space: A closed non-orientable surface



Cut out the möbius band and...

Non-orientable: contains a Mobius band.

Projective space: A closed non-orientable surface



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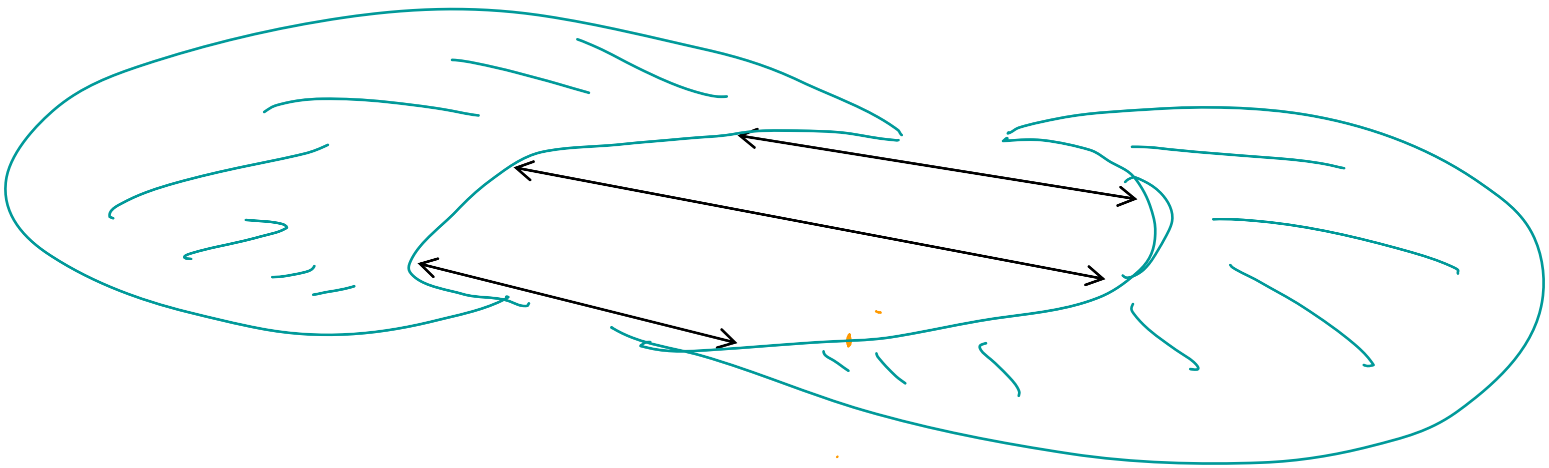

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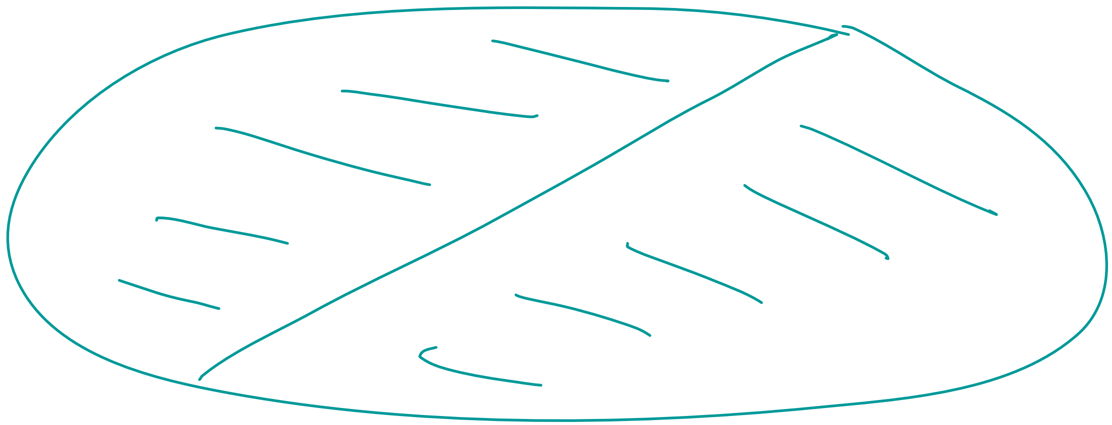
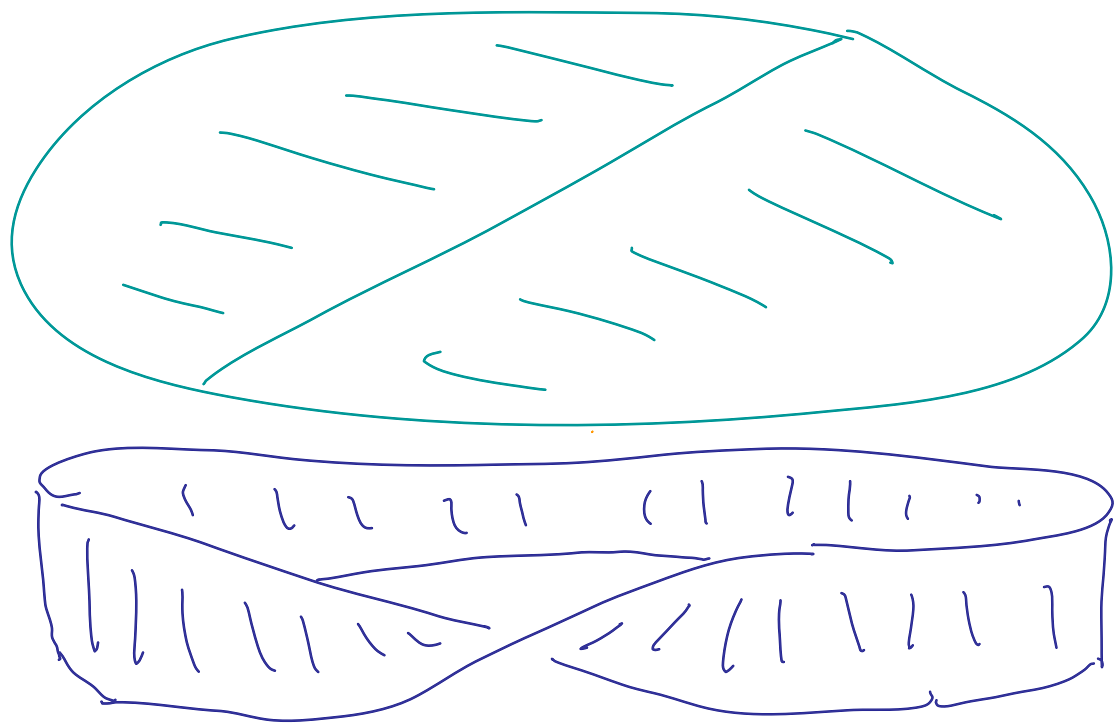


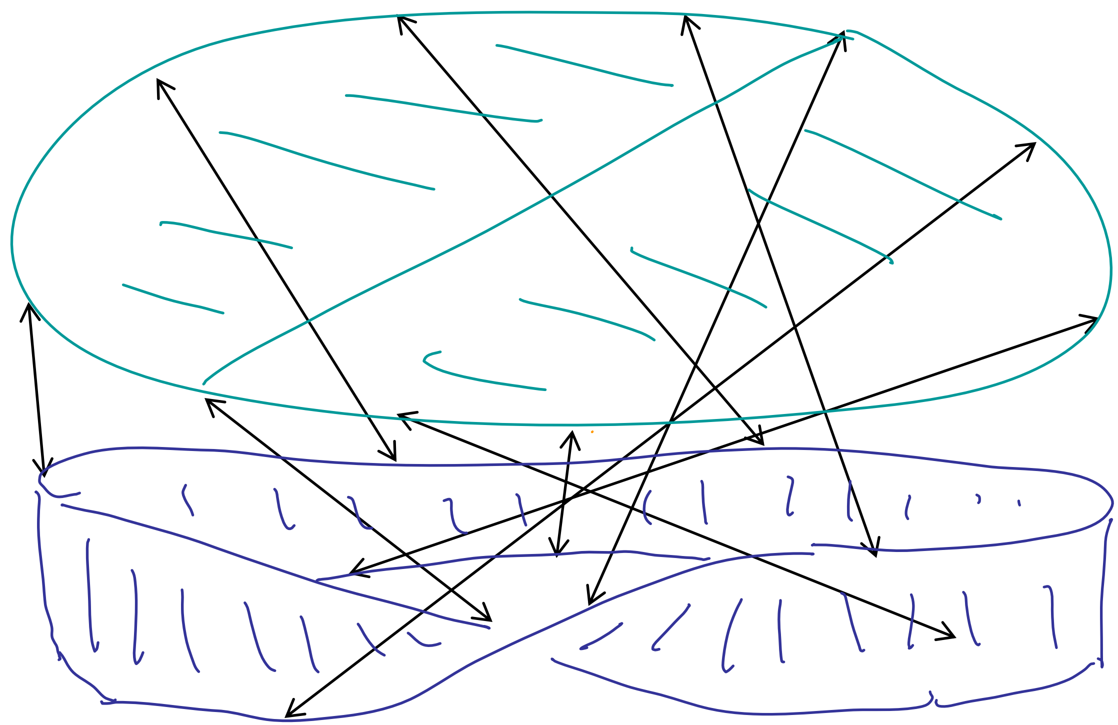
... and you get a disk.

Projective space: A closed non-orientable surface



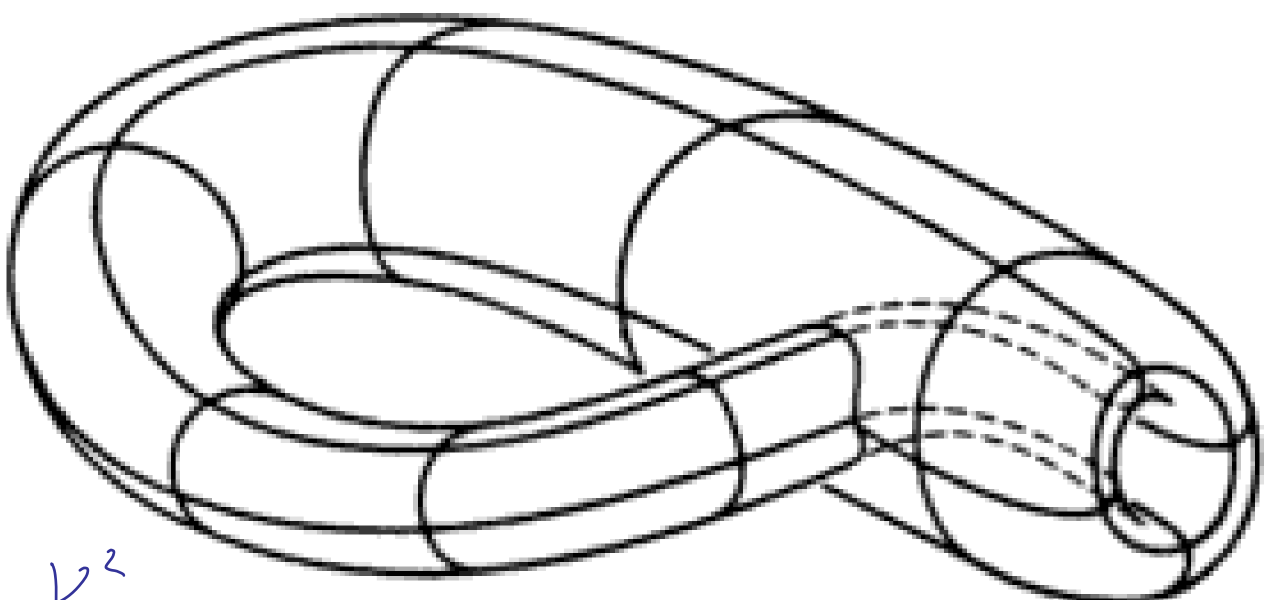
So: projective space is a möbius band with a disk glued along its boundry.

Projective space: A closed non-orientable surface



So: projective space is a möbius band with a disk glued along its boundary.

The Klein Bottle: another non orientable surface.



K^2

["The shape of space", J. Weeks]

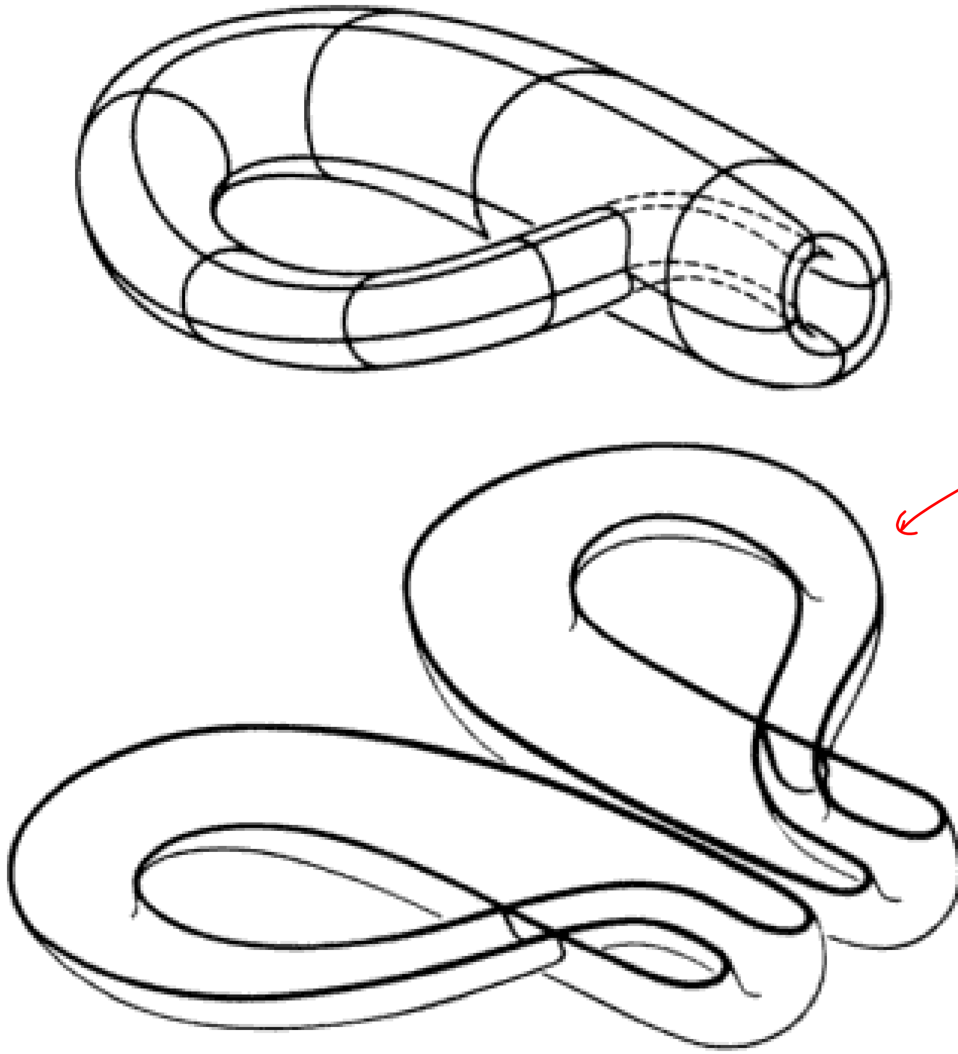
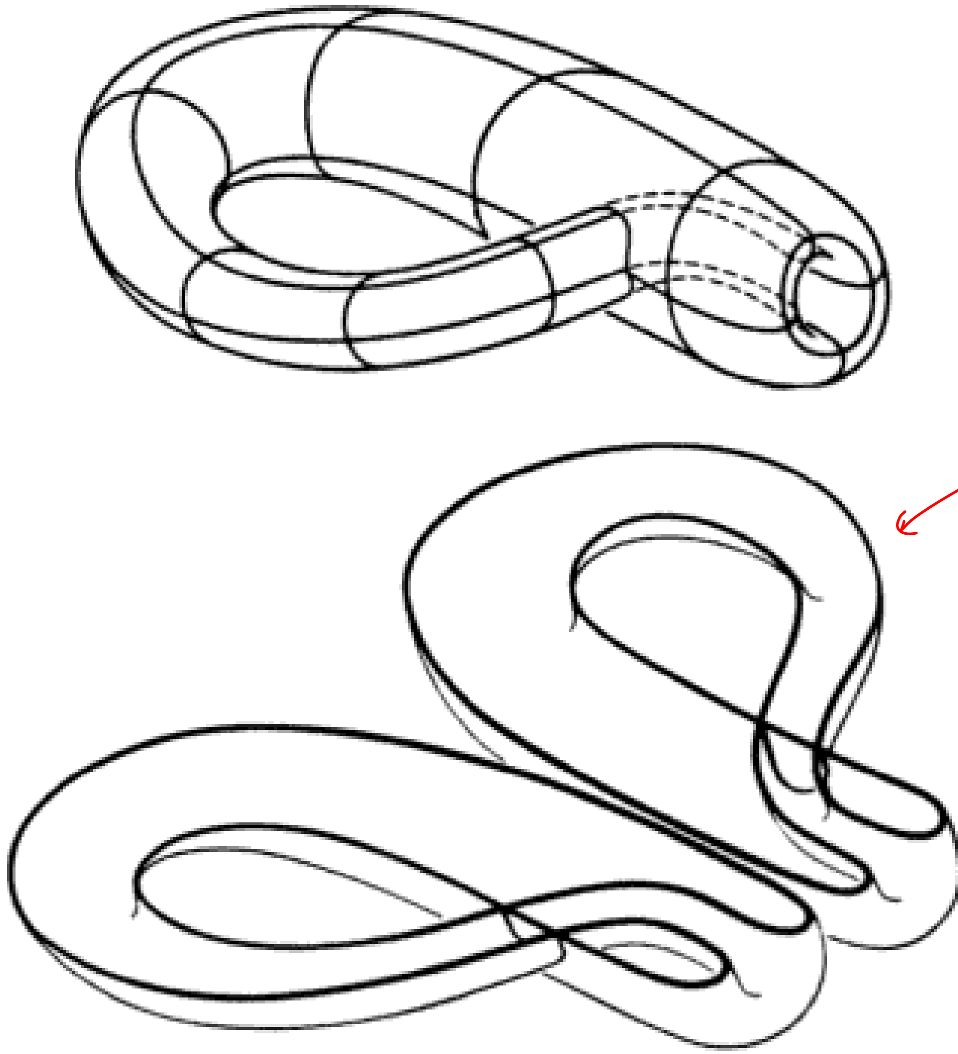


Figure 5.6 Cutting a Klein bottle in two.



Möbius band

Möbius band

Figure 5.6 Cutting a Klein bottle in two.

<u>Conclusion</u>: A klein bottle is two mobius bands glued together

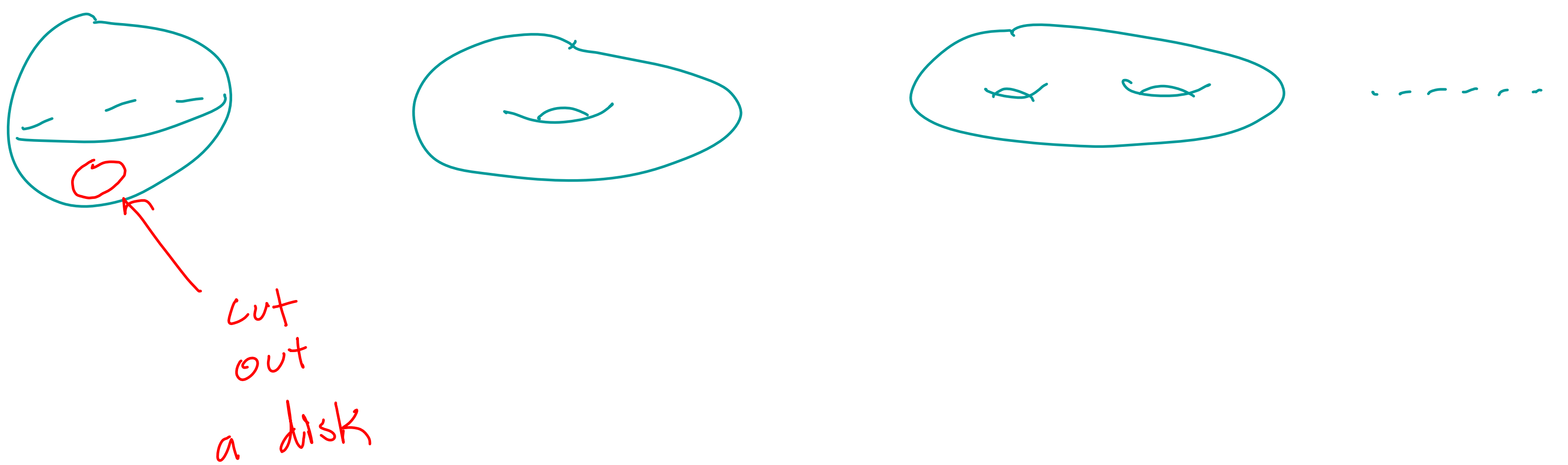
Connected Closed 2-dim'l manifolds:

oriented:

genus 0   genus 1   genus 2

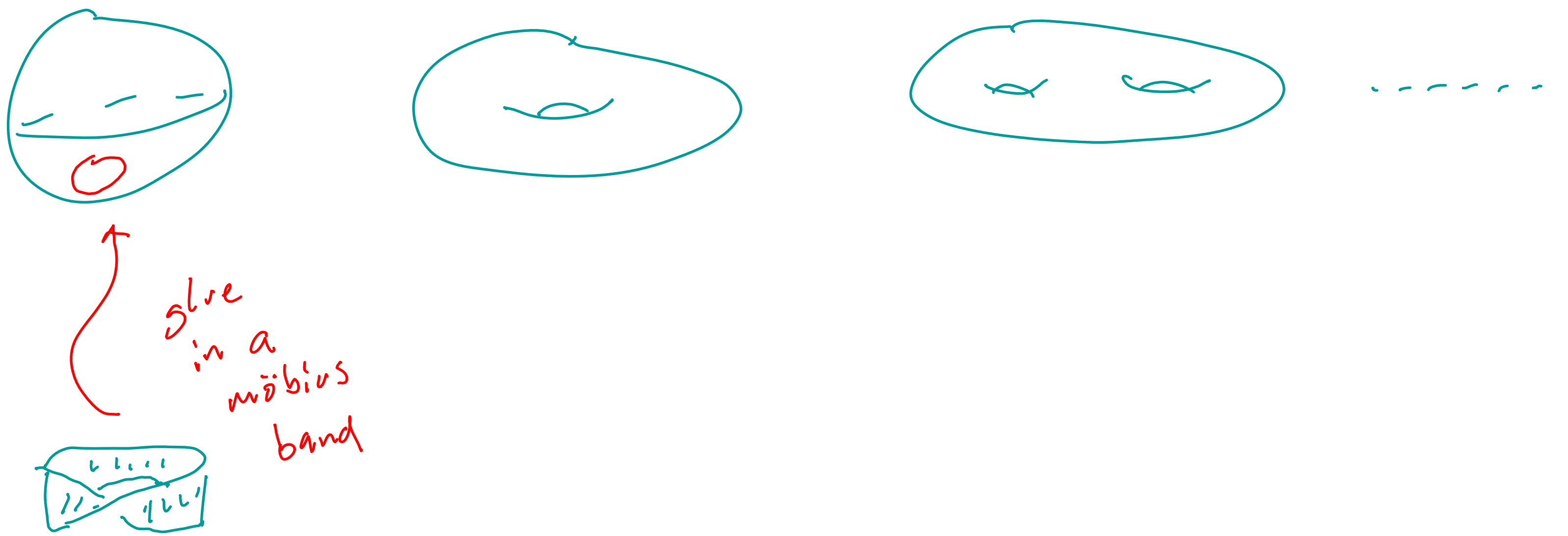
Connected Closed 2-dim'l manifolds:

oriented:



Connected Closed 2-dim'l manifolds:

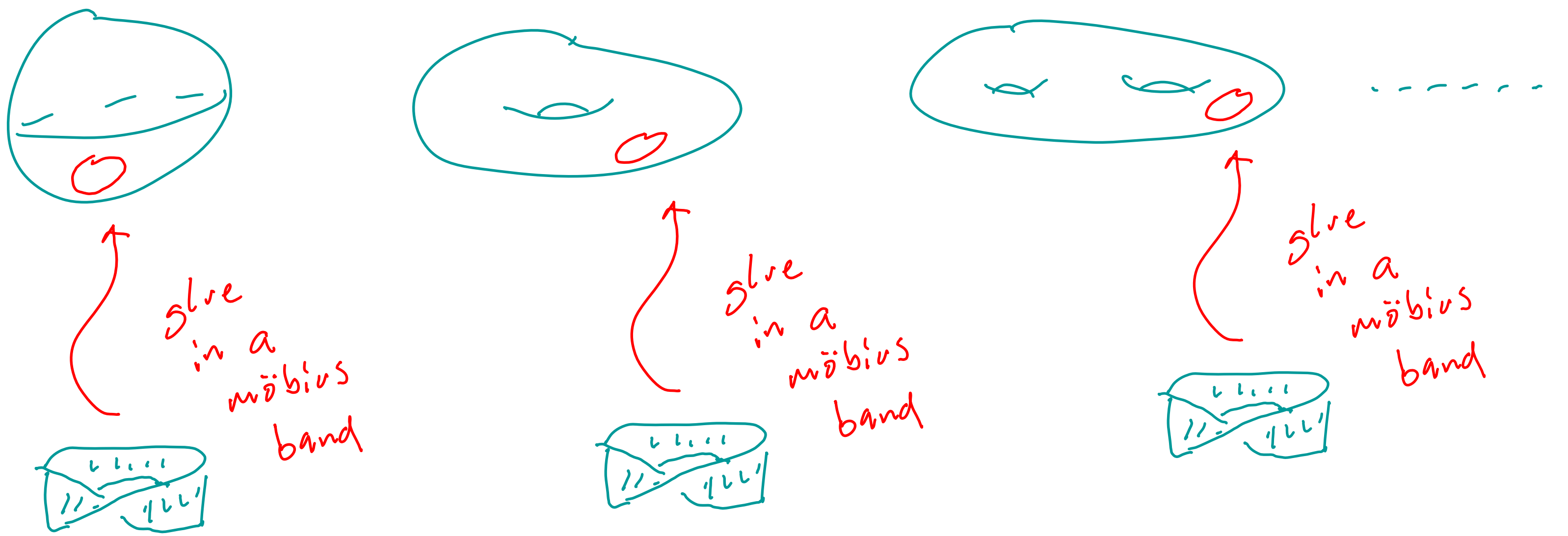
oriented:



P^2

Connected Closed 2-dim'l manifolds:

Non-oriented:



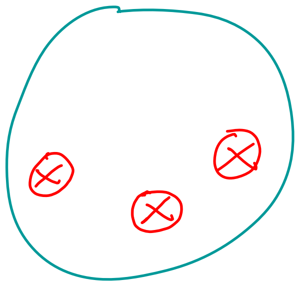
Connected Closed 2-dim'l manifolds:

Non-oriented:

K^2

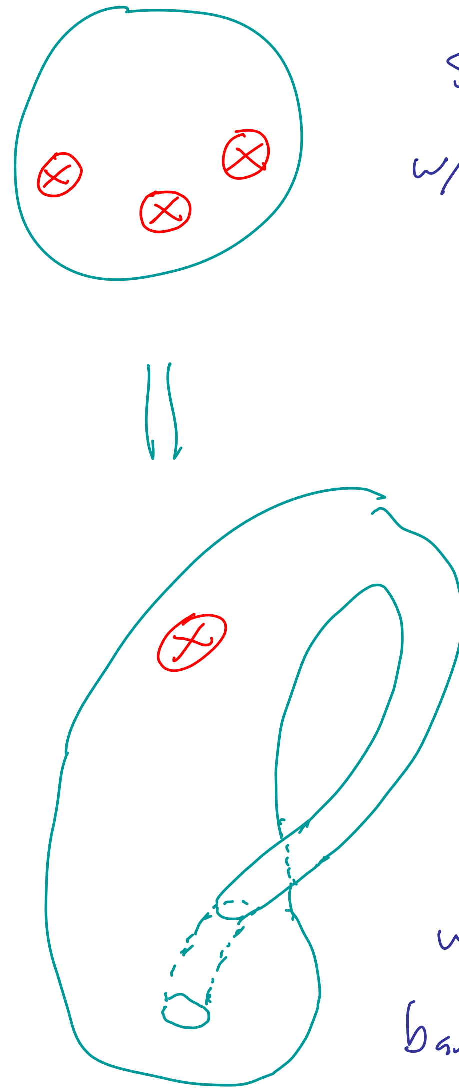
this gives a complete classification!

Question: What about



Sphere w/ 3 mobius bands glued in?

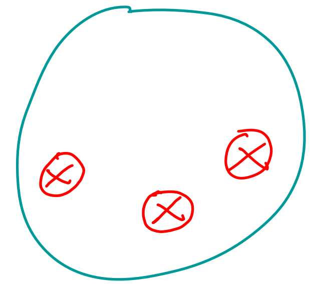
Question: What about



Sphere w/ 3 mobius bands glued in?

Klein bottle w/ 1 mobius band glued in

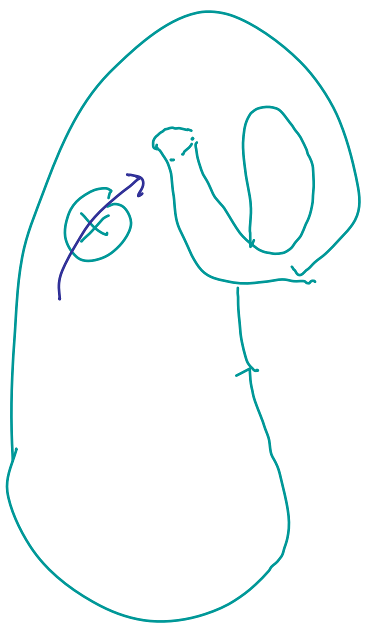
Question: What about



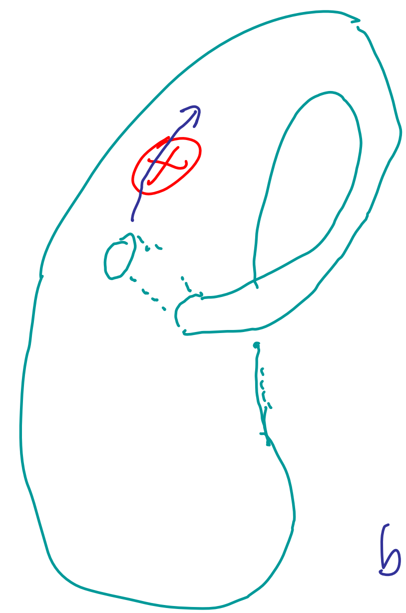
Sphere w/ 3 mobius bands glued in?

$\Downarrow$

now the handle is attached on the outside

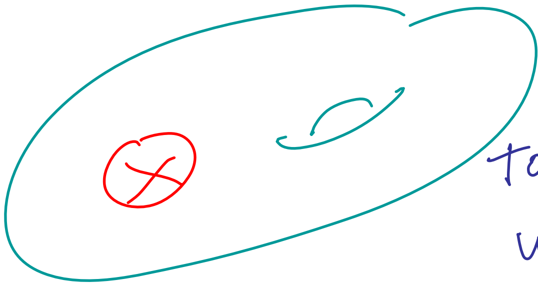


$=$

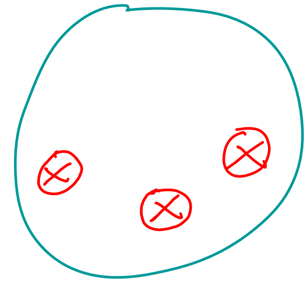


Klein bottle w/ 1 mobius band glued in

Question: What about

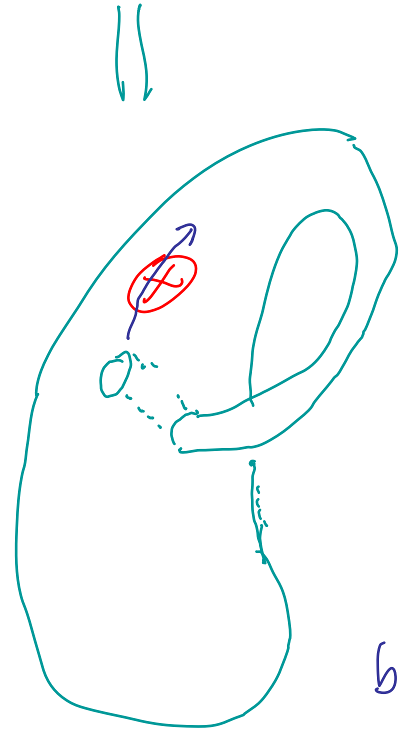


torus w/ 1 mobius band glued in



Sphere w/ 3 mobius bands glued in?

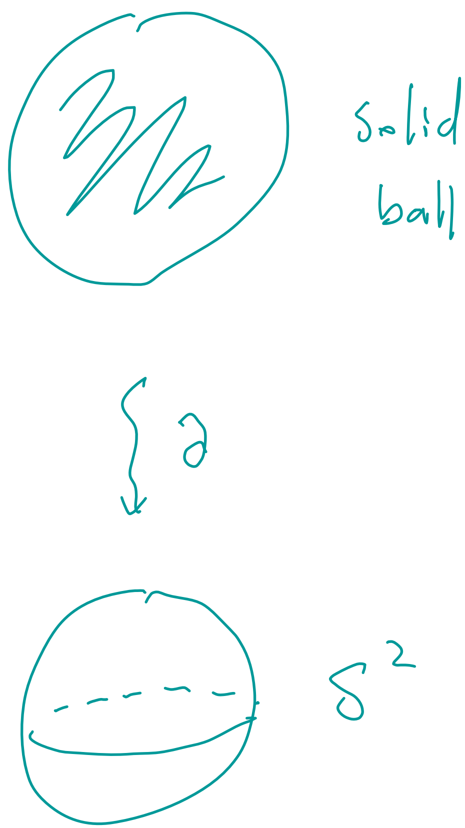
=



Klein bottle w/ 1 mobius band glued in

On to 3-mflds...

Some 3-manifolds w/ boundry



On to 3-mflds...

Some 3-manifolds without boundry ?

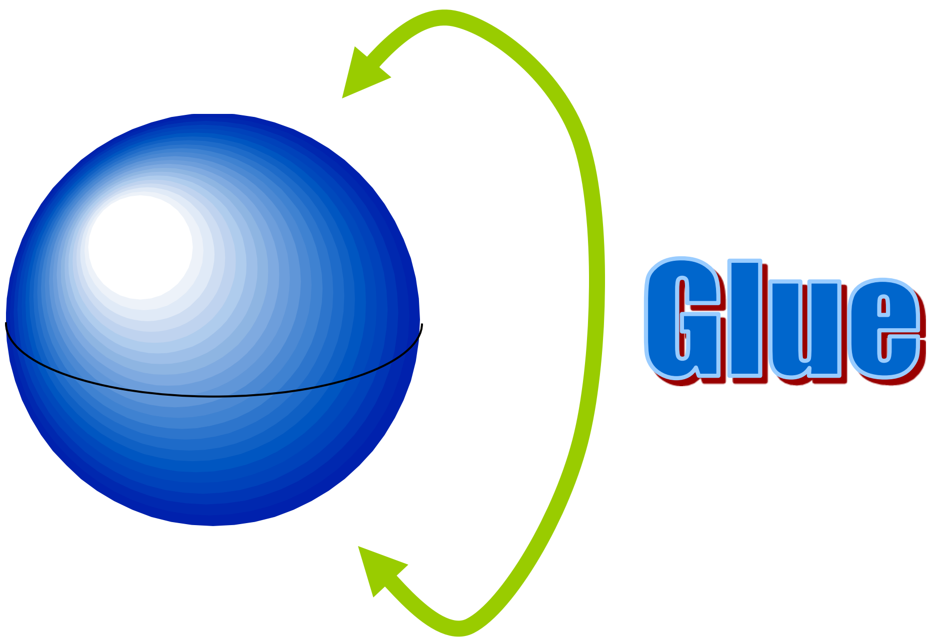
(closed 3-mflds)

$$S^3 \hookrightarrow \mathbb{R}^4$$

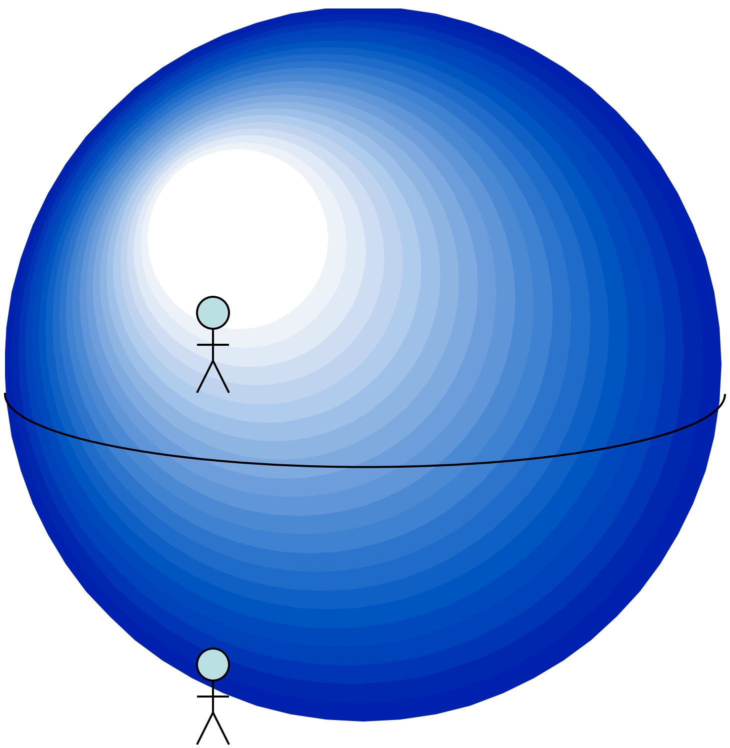
$$\|$$

$$\{(x,y,z,w) \in \mathbb{R}^4 : x^2+y^2+z^2+w^2=1\}$$

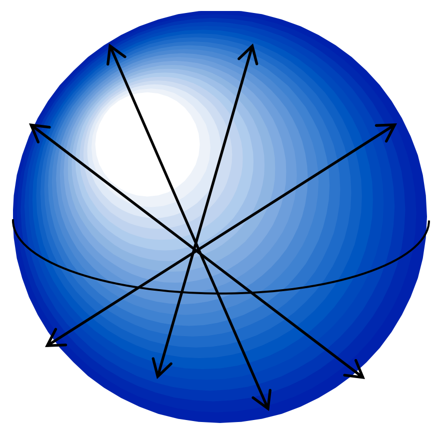
S^3 is obtained by taking a solid ball and gluing the opposite hemispheres together:



You can think of S^3 this way: If you are flying around in S^3, and fly through the surface in the northern hemisphere, you reemerge in the southern hemisphere.

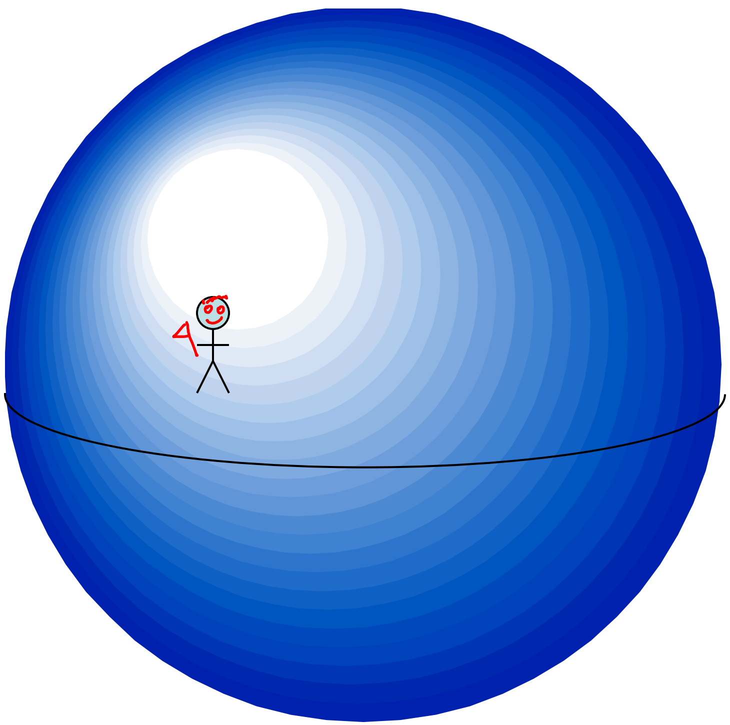


P^3 is obtained by taking a solid ball and gluing antipodal points together:



Glue

You can think of P^3 this way: If you are flying around in P^3, and fly through the surface in the northern hemisphere, you reemerge in the southern hemisphere, but flipped backwards.



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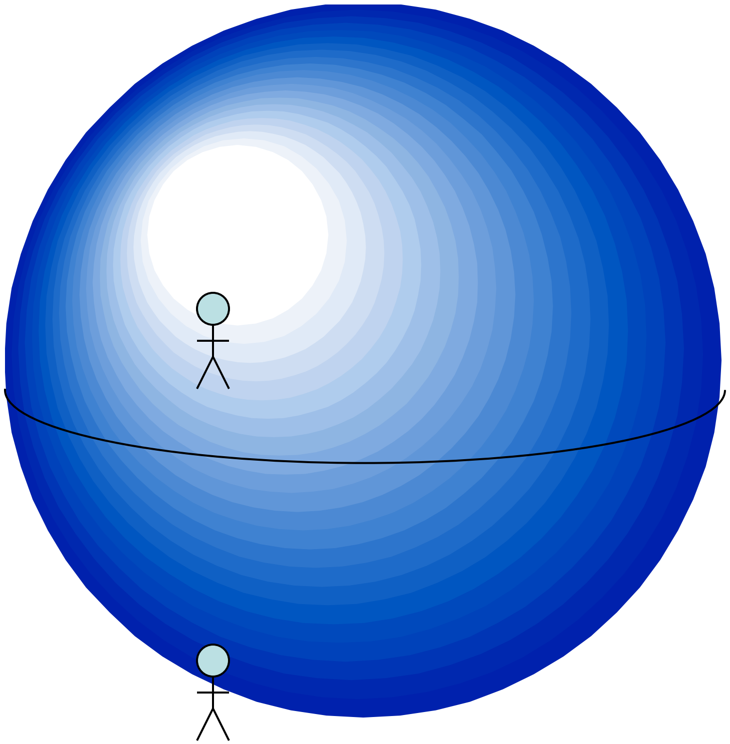

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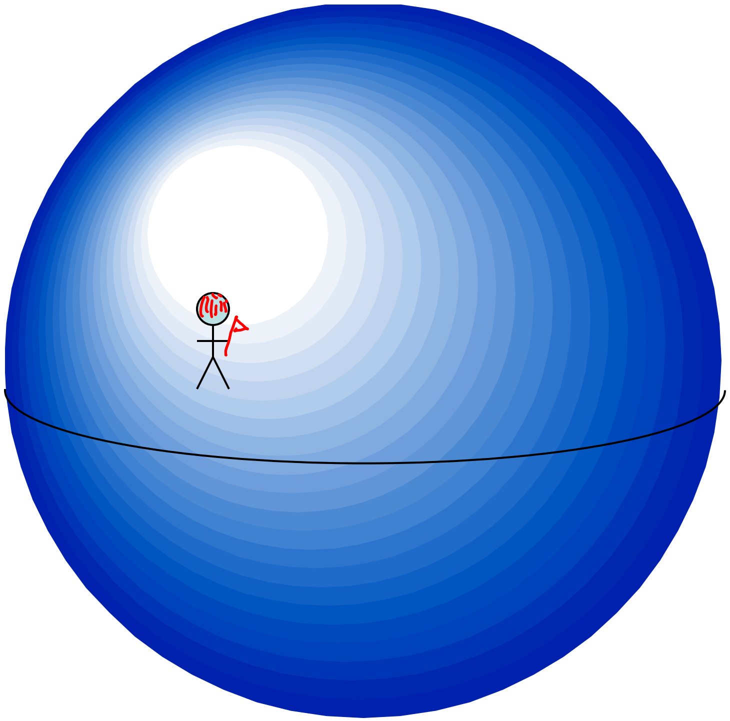


Classification of 3-mflds

...

Classification of 3-mflds . . .

Really Hard!

Classification of n-manifolds

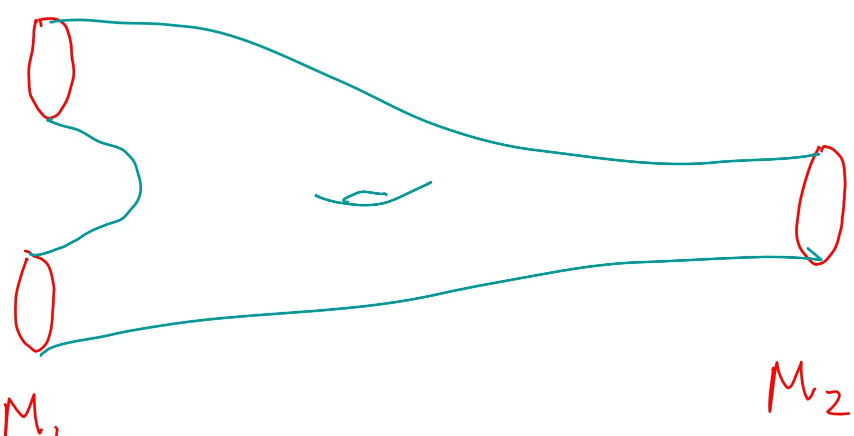
- Classification of 3-manifolds: “Thurston’s geometrization conjecture”. This was essentially proved by Perelman in 2003 – as a special case, the 100 year old “Poincare conjecture” was proven in dimension 3.
- Classification of 4-manifolds: MUCH HARDER
- Classification of 5-manifolds and higher: still hard, but “easier” than dimensions 3 and 4.
- Theorem: for $n > 3$, there is no ALGORITHM for determining if two n-manifolds are the same!

Simpler Task: Manifolds up to COBORDISM

Def Let M_1 and M_2 be n-manifolds.

We say M_1 and M_2 are cobordant if there exists an $(n+1)$-manifold W:

$$\partial W = M_1 \amalg M_2$$



Example:

$S^1 \amalg S^1$ is cobordant to S^1

Cobordism Groups

$$\Omega_n = \frac{\{\text{closed } n\text{-manifolds}\}}{M_1 \sim M_2 \text{ if cobordant}}$$

Note: Ω_n is an abelian gp.

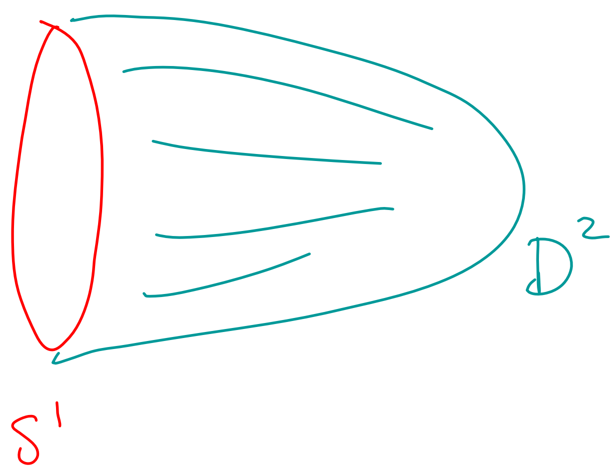
$$[M_1] + [M_2] = [M_1 \amalg M_2]$$

$$[\emptyset] = 0$$

Null cobordisms:

$[M] = 0$ in Ω_n iff $M \sim \emptyset$

i.e. $\exists\, W$ s.t. $\partial W = M$



e.g. $[S^1] = 0$

in Ω_1

$S^1 = \partial D^2$

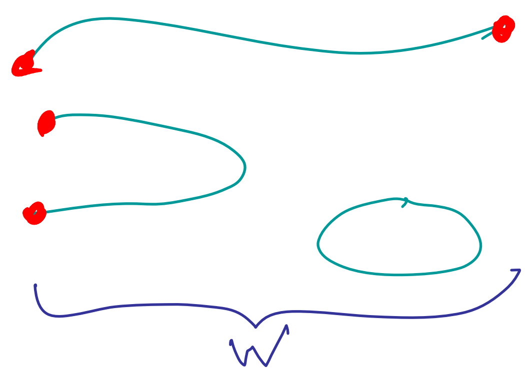
Computation of Ω_0

$$\left\{ \begin{matrix} \text{closed} \\ 0\text{-manifolds} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{finite collections} \\ \text{of points} \end{matrix} \right\}$$

$$\partial \left(\begin{matrix} \text{1-manifold} \\ \text{w/ bdry} \end{matrix} \right) = \text{even \# of points}$$

e.g.

∂W = 4-points



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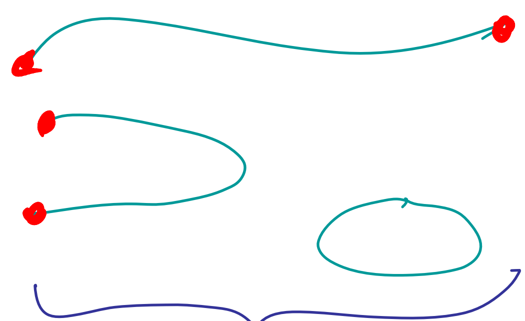


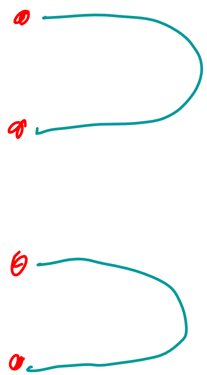
W

Thus

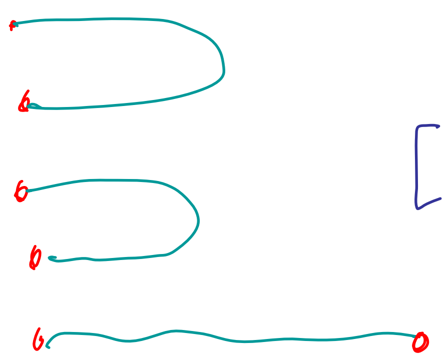
$[\text{point}] \neq 0$

in Ω_0

Computation of Ω_0

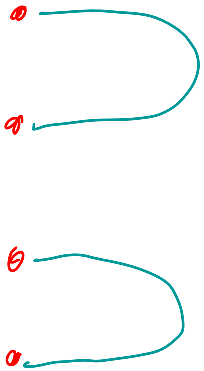


$$[4 \text{ points}] \simeq [\emptyset]$$



$$[5 \text{ points}] \simeq [1 \text{ point}]$$

Computation of Ω_0

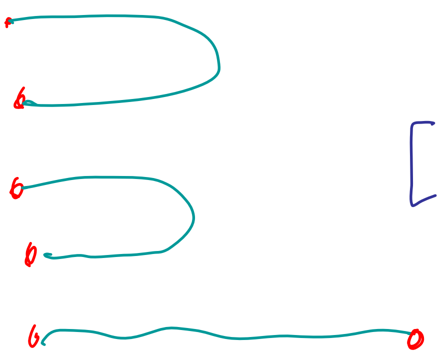


$[4 \text{ points}] \simeq [\phi]$

[Even # points]

$\simeq$

$[\phi]$



$[5 \text{ points}] \simeq [1 \text{ point}]$

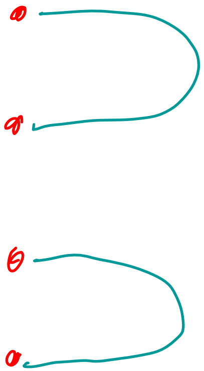
[odd # points]

$\simeq$

[1 point]

Computation of Ω_0

$$\boxed{\Omega_0 \cong \mathbb{Z}/2}$$

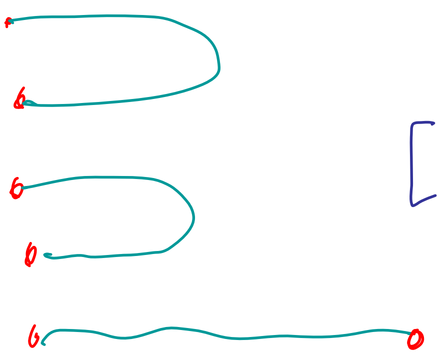


[4 points] $\simeq$ [$\emptyset$]

[Even # points]

||

[$\emptyset$]



[5 points] $\simeq$ [1 point]

[odd # points]

||

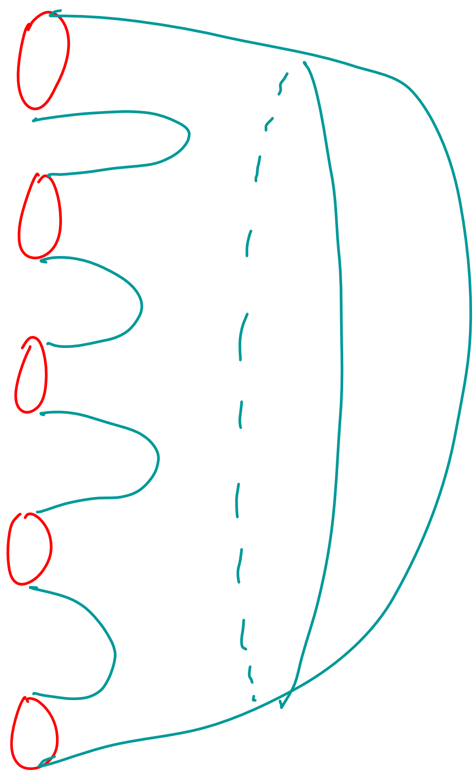
[1 point]

Computation of Ω_1

<u>Recall</u>: every closed 1-manifold is a disjoint union of circles

Computation of Ω_1

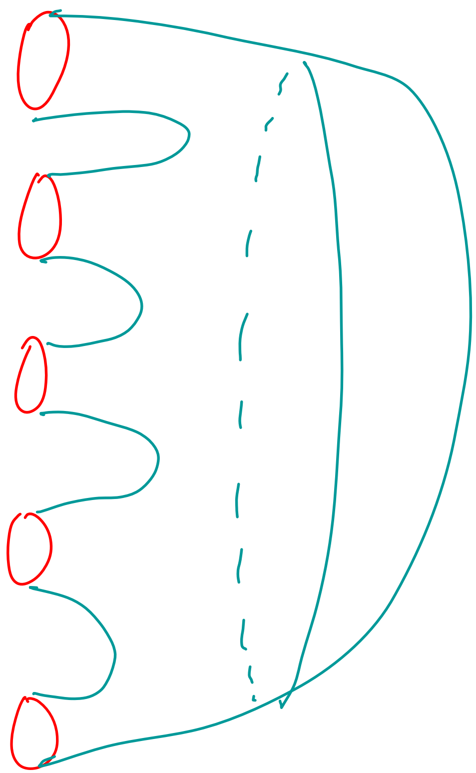
Recall: every closed 1-manifold is a disjoint union of circles



NULL-COBORDANT!

Computation of Ω_1

Recall: every closed 1-manifold is a disjoint union of circles



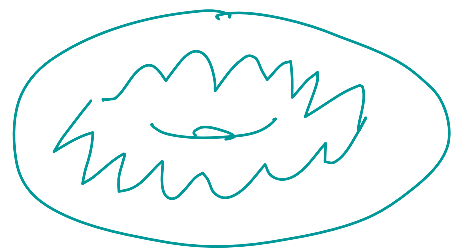
NULL-COBORDANT!

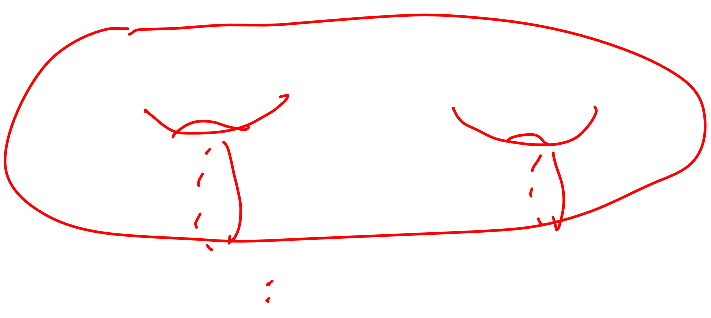
$$\boxed{\Omega_1 = 0}$$

Trivial group

Computation of Ω_2

 $= \partial$  [solid ball]

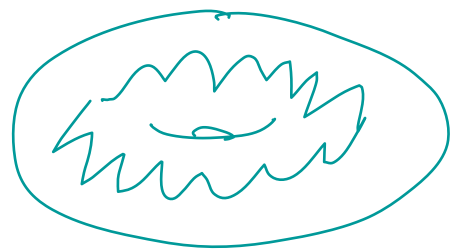
 $= \partial$  [solid Torus]

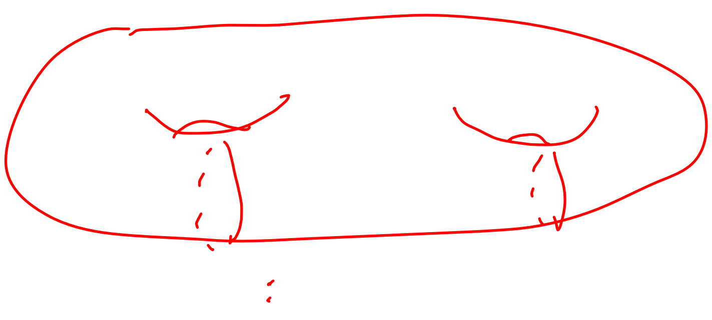
 $= \partial$ 

Computation of Ω_2

All orientable 2-mflds are NULL COBORDANT

 $= \partial$  [solid ball]

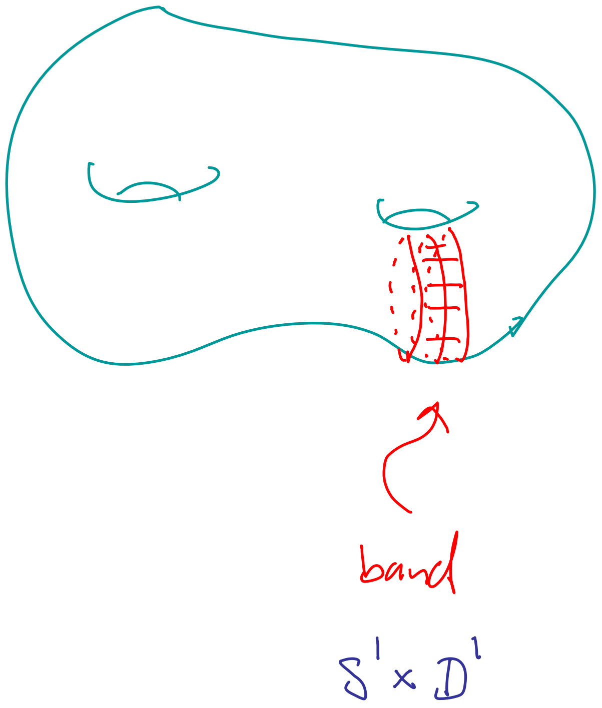
 $= \partial$  [solid Torus]

 $= \partial$ 

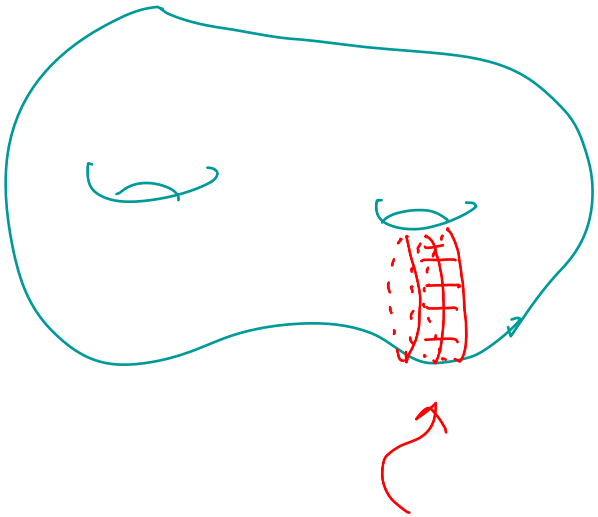
Theorem:

P^2 is Not null cobordant,

Surgery: A way to make cobordisms.

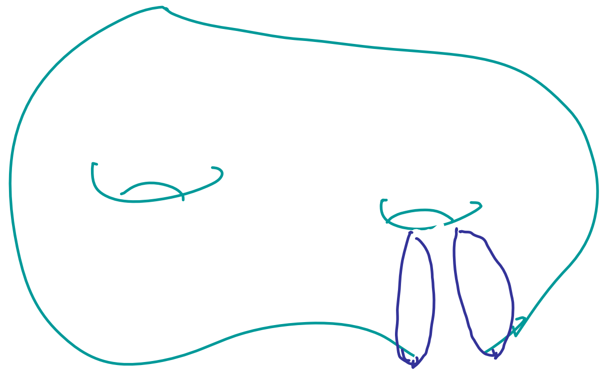


Surgery: A way to make cobordisms.

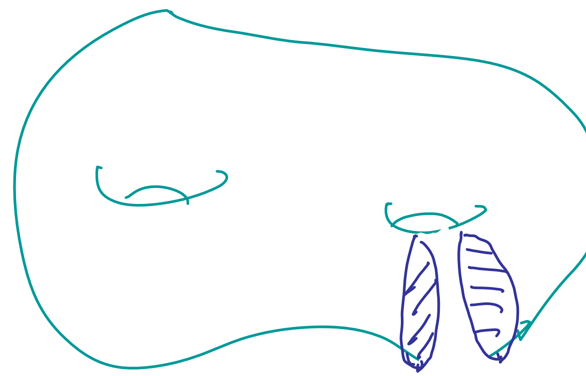


band

$S^1 \times D^1$

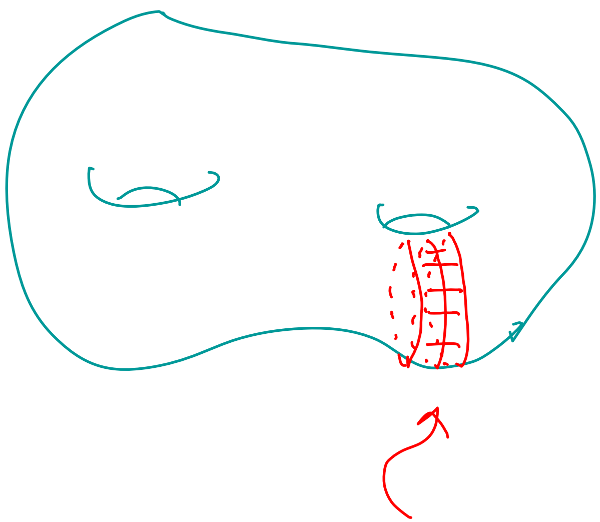


Remove the band



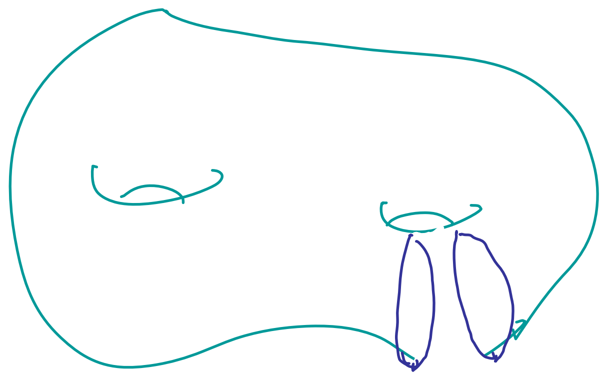
Glue in $D^2 \times S^0$

Surgery: A way to make cobordisms.



band

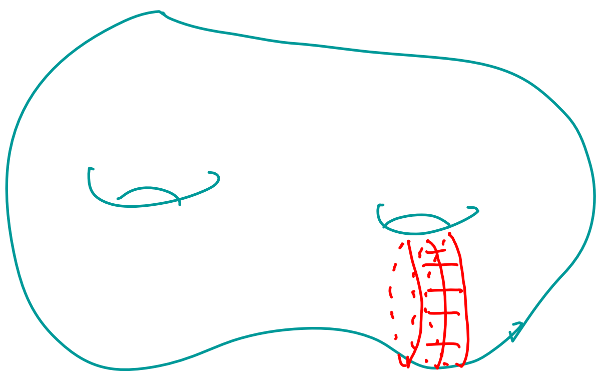
$S^1 \times D^1$



Remove the band

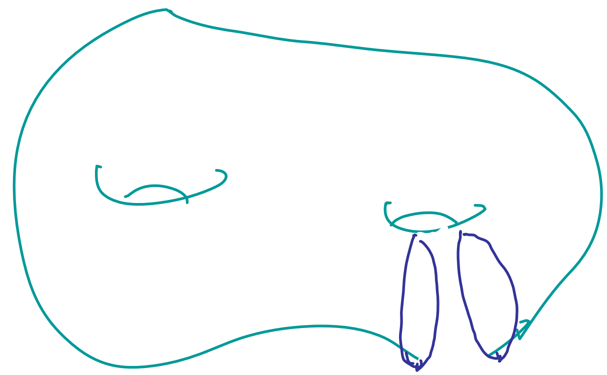
Glue in $D^2 \times S^0$

Surgery: A way to make cobordisms.



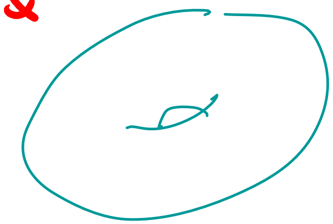
band

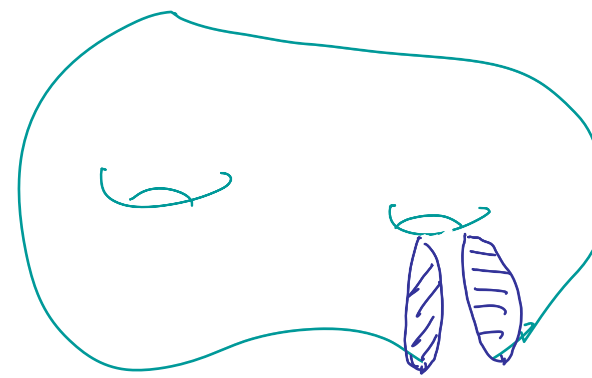
$S^1 \times D^1$



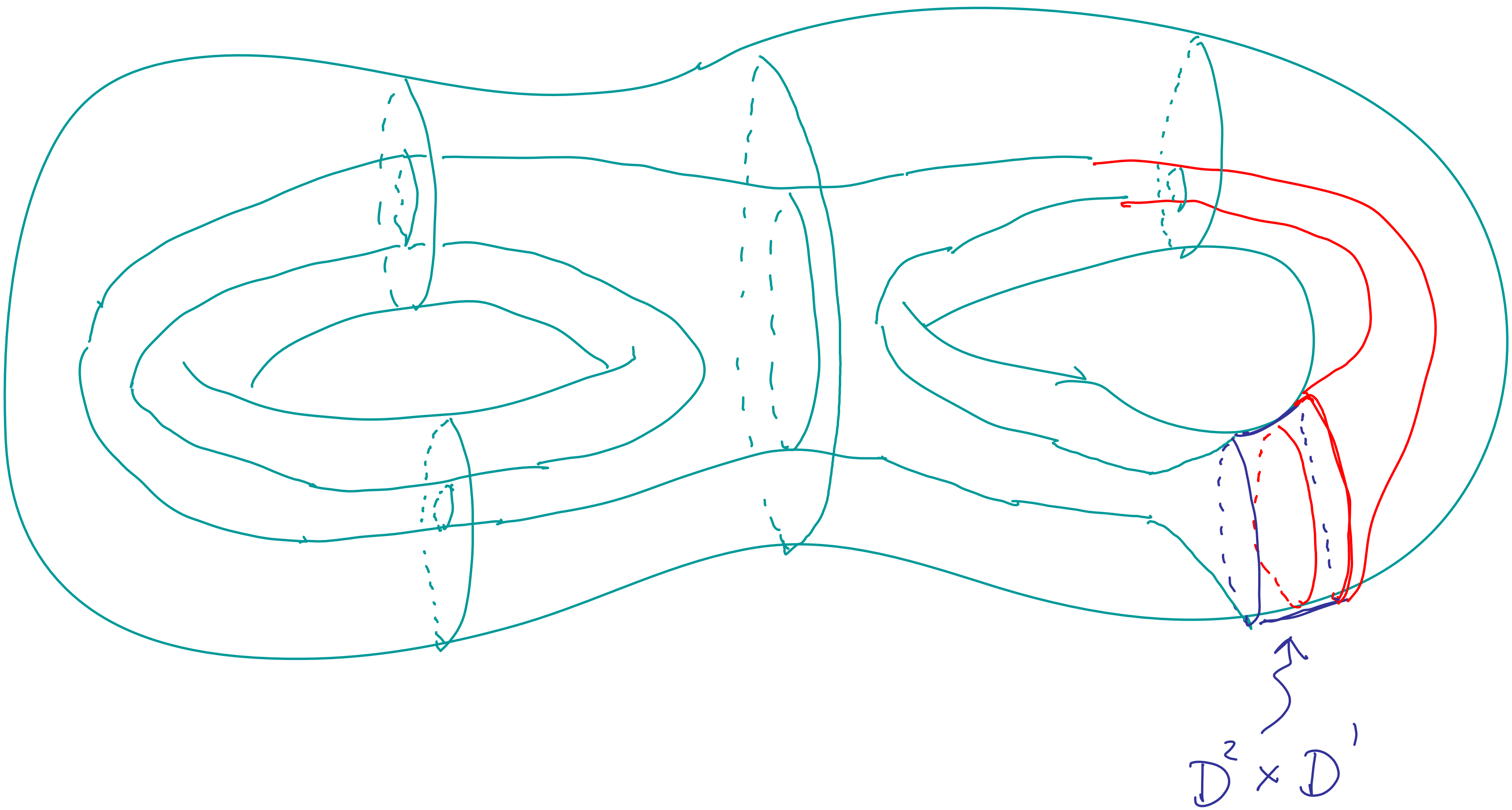
Remove the band

The genus is reduced by 1



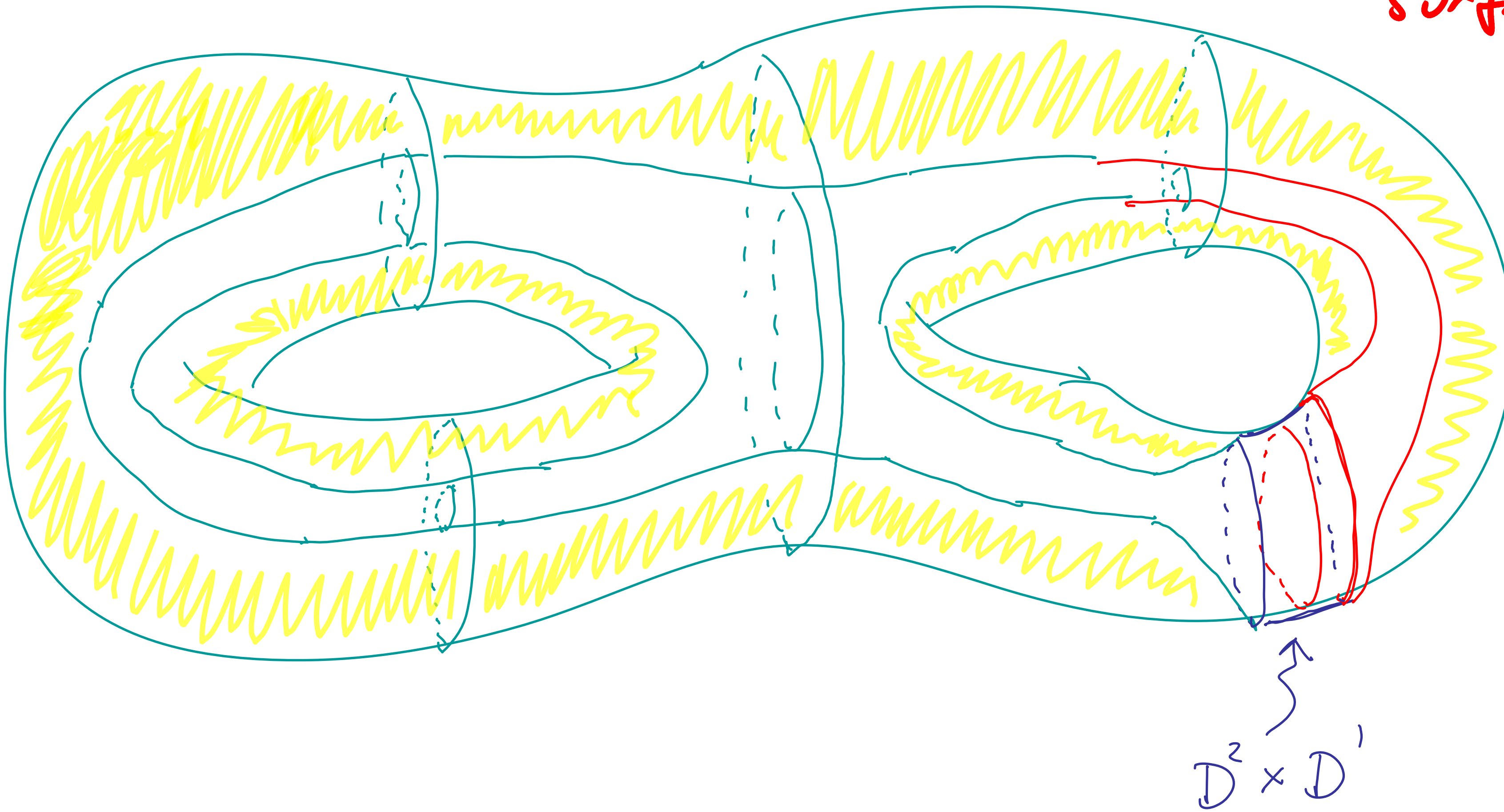


Glue in $D^2 \times S^0$



$$\partial(D^2 \times D^1) = S^1 \times D^1 \cup S^0 \times D^2$$

Solid region between the surfaces is a cobordism between the outer surface + inner surface



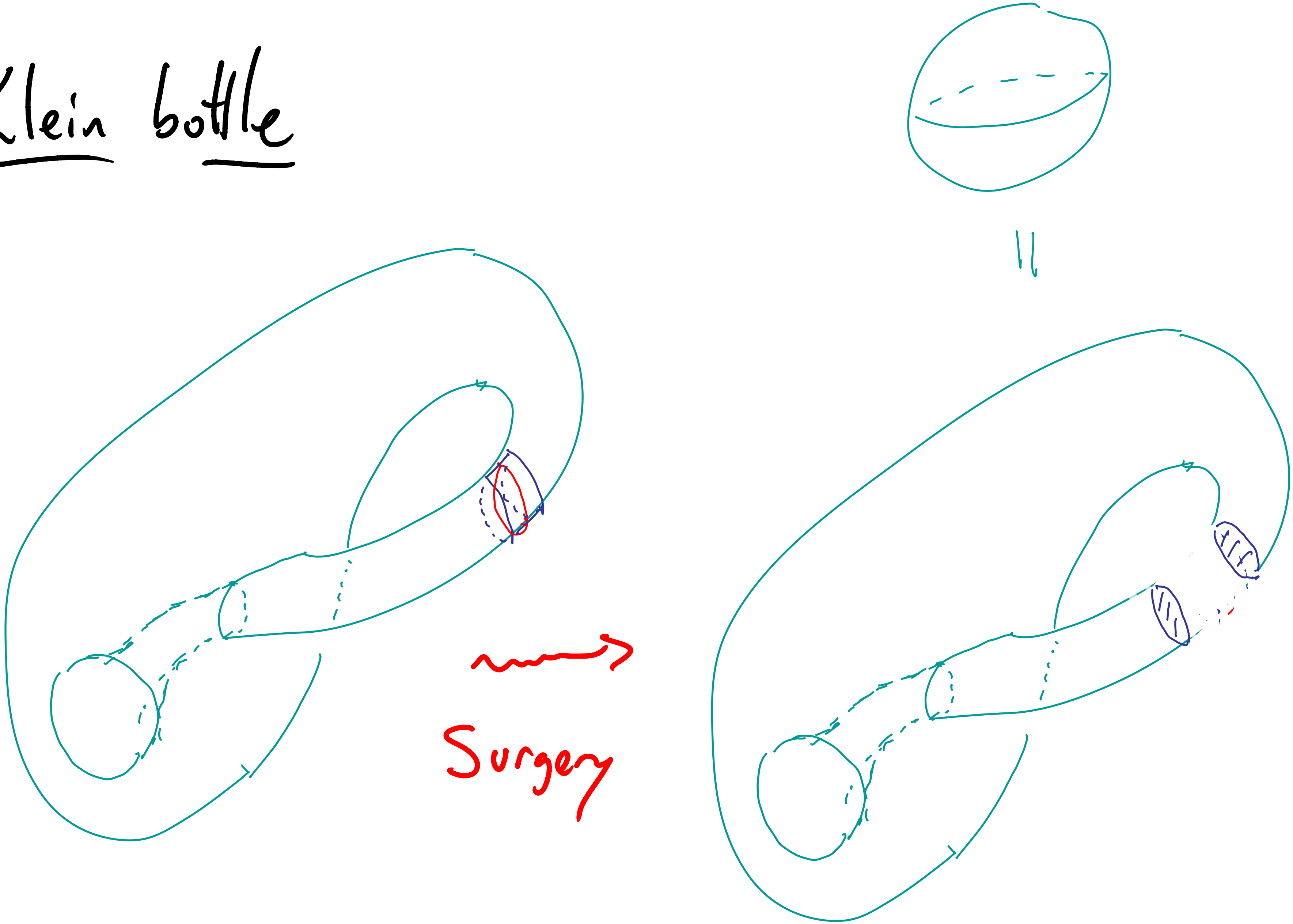
$$\partial(D^2 \times D^1) = S^1 \times D^1 \cup S^0 \times D^2$$

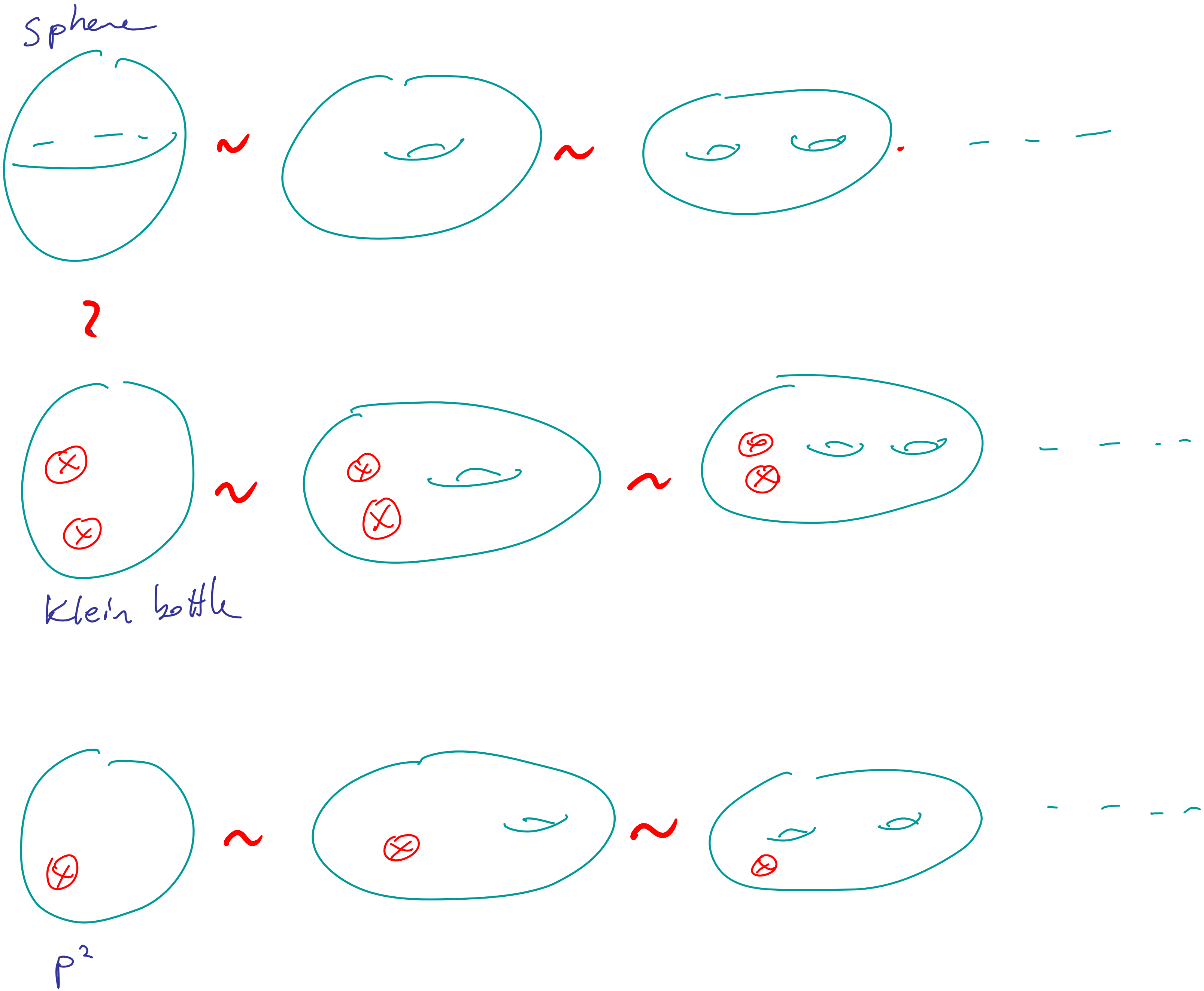
or add

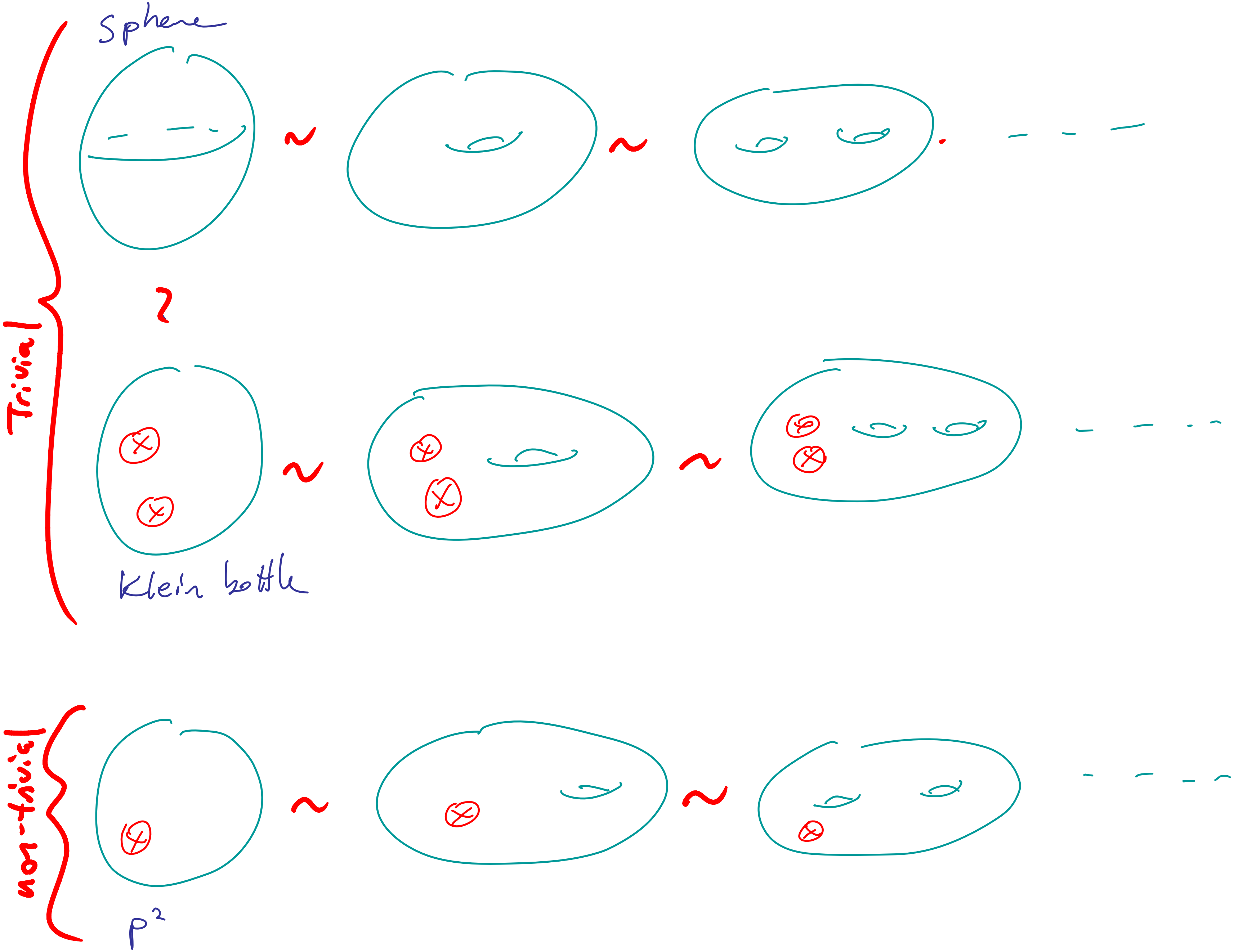
Surgery allows us to remove or add handles in a way that does not change cobordism class!

"Who did you say did your bypass surgery?"

Klein bottle




Sphere
Klein bottle
P^2


Sphere
Trivial
Klein bottle
non-trivial
P^2

So:

$$\Omega_2 \cong \mathbb{Z}/2$$

Theorem: (Thom)

$|x_i| = i$

as a graded ring

$$\Omega_* \cong \mathbb{Z}/2\,[x_i \mid i \neq 2^i - 1]$$

all dimensions

Theorem: (Thom)

$$\Omega_* \cong \mathbb{Z}/2\left[x_i \mid i \neq 2^k - 1\right]$$

0	1	2	3	4	5	6	7	8
1		x_2		x_2^2	x_5	x_2^3	$x_5 x_2$	x_2^4
				x_4		$x_4 x_2$		$x_4 x_2^2$
						x_6		x_4^2
								$x_6 x_2$
								x_8