Congruences amongst modular forms and the divided β family

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(MIT)

Chromatic Theory

$(\pi_*^s)_{(p)}$ = p-local stable htpy grps of spheres

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- Admits a filtration (chromatic filtration)
- k^{th} layer exhibits periodic behavior (v_k-periodicity)

$$|v_k| = 2(p^k - 1)$$

Chromatic Theory

$(\pi_*^S)_{(p)}$ = p-local stable htpy gps of spheres

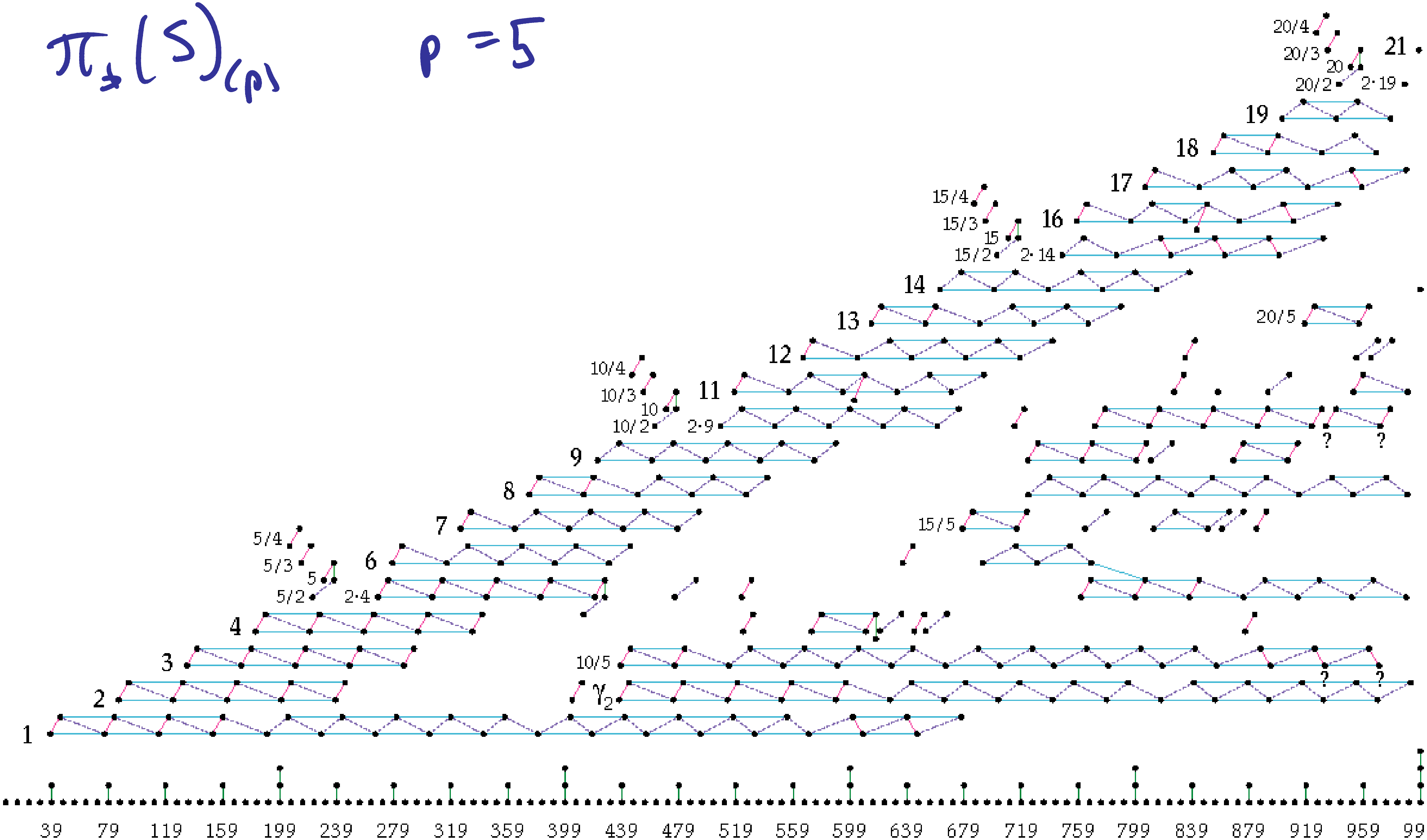
- Admits a filtration (chromatic filtration)

$$S_{(p)} \to \cdots \to S_{E(2)} \to S_{E(1)} \to S_{\mathbb{Q}}$$

- k^{th} layer exhibits periodic behavior (v_k-periodicity)

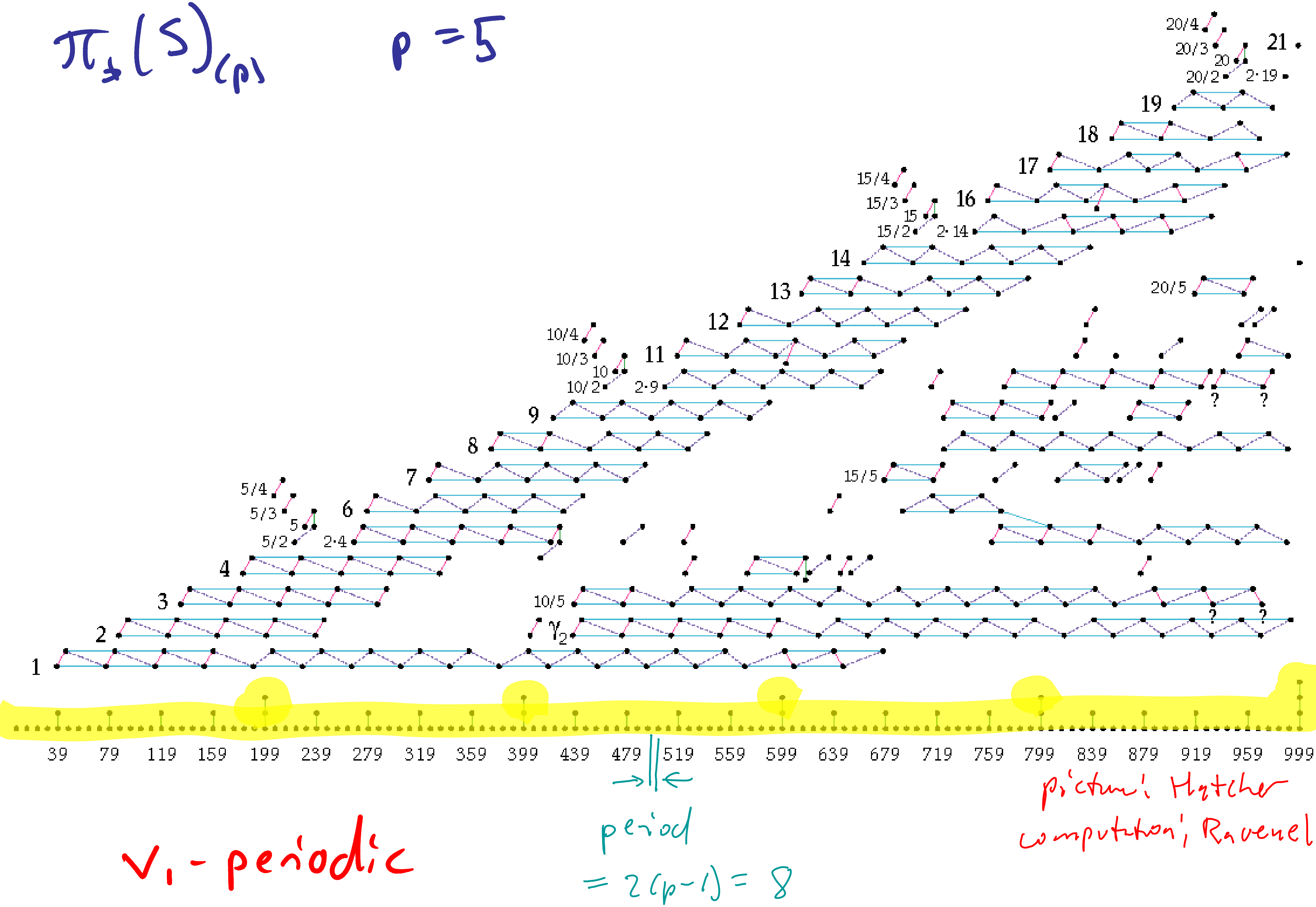
"k^{th} layer" $\quad M_k S \longrightarrow S_{E(k)} \longrightarrow S_{E(k-1)}$

$\pi_*(S)_{(p)}$ $p=5$



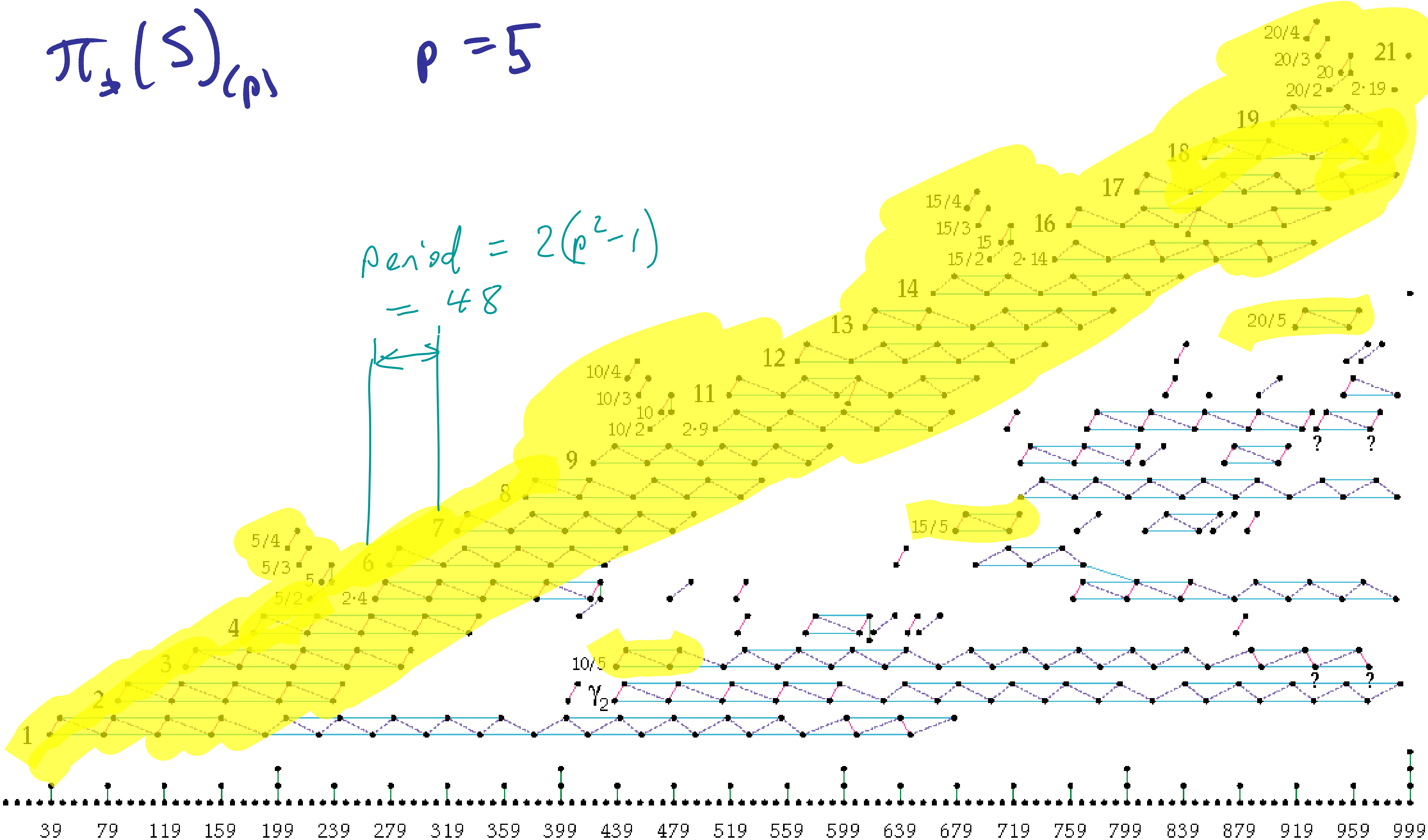
picture: Hatcher
computation: Ravenel

$\pi_*(S)_{(p)}$ $p=5$



v_1-periodic

$\pi_*(S)_{(p)}$ $\quad p = 5$

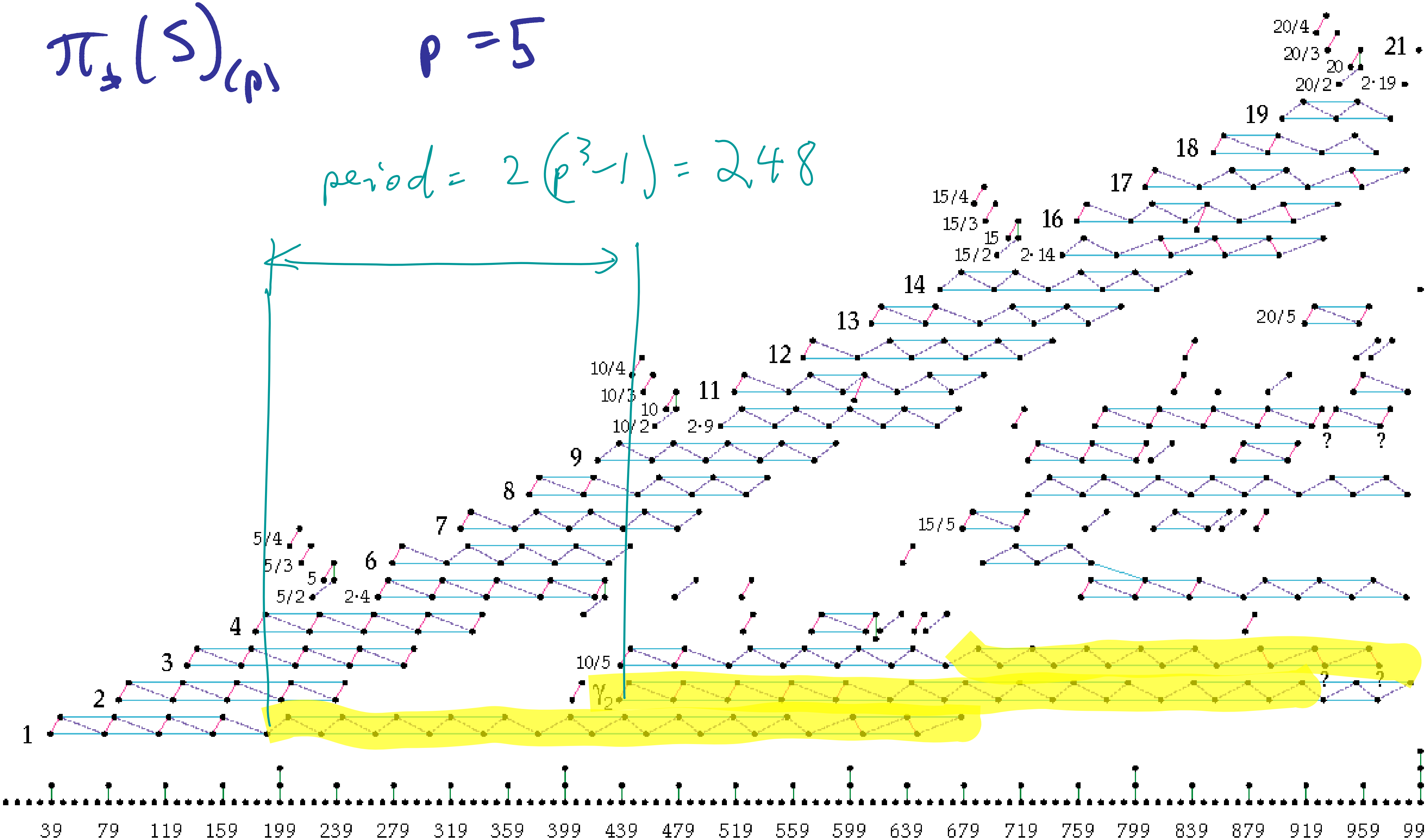


picture: Hatcher
computation: Ravenel

v_2-periodic

$\pi_*(S)_{(p)}$ $p=5$

period $= 2(p^3-1) = 248$



v_3-periodic (?)

picture: Hatcher
computation: Ravenel

Greek letter elements

The most fundamental v_n-periodic elts are

the <u>GREEK</u> <u>LETTER</u> <u>ELTS</u>

Greek letter elements

The most fundamental v_n-periodic elts are the **GREEK LETTER ELTS**

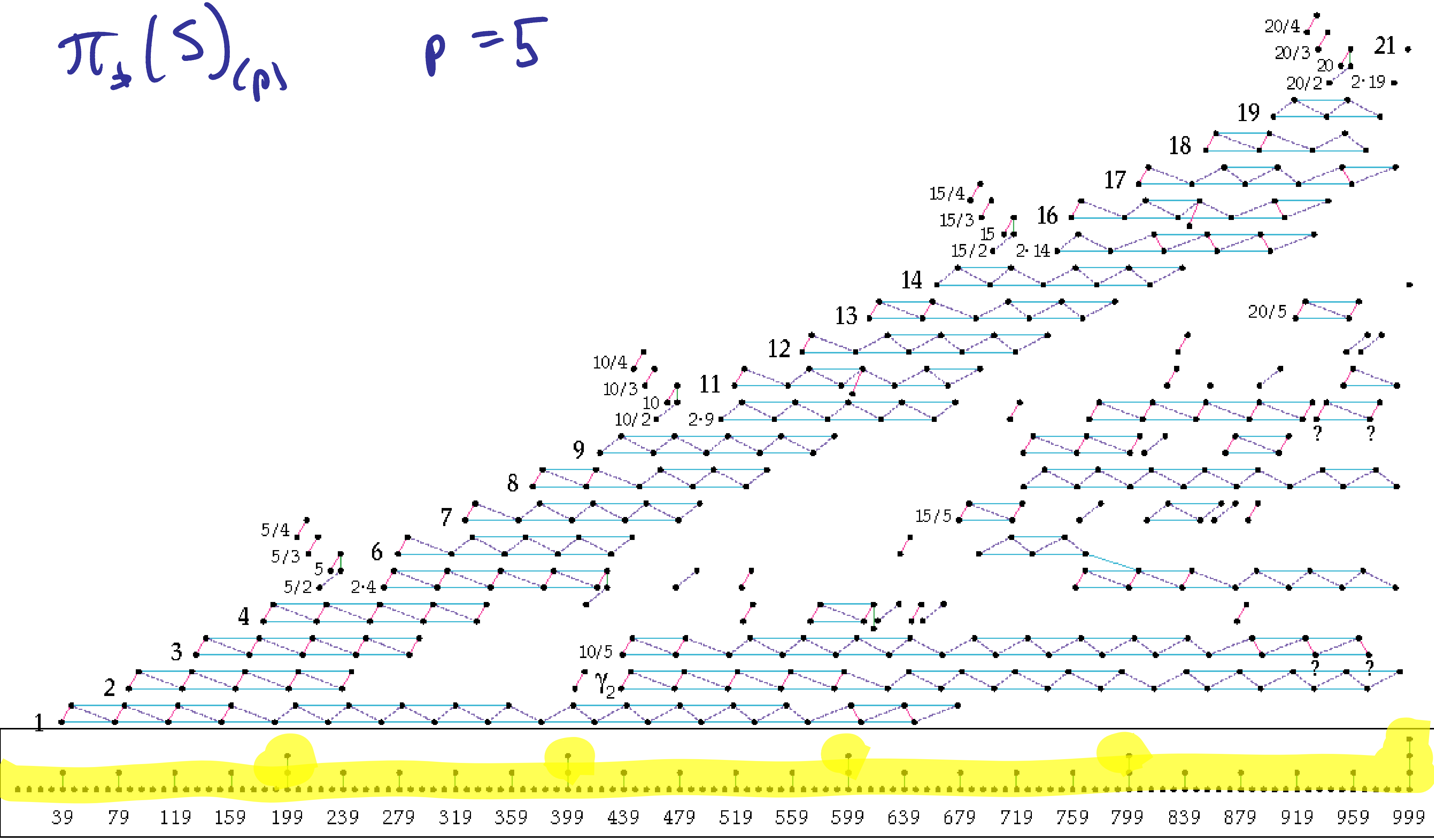
Notation

v_1-periodic: $\alpha_{i/j}$

v_2-periodic: $\beta_{i/j,k}$

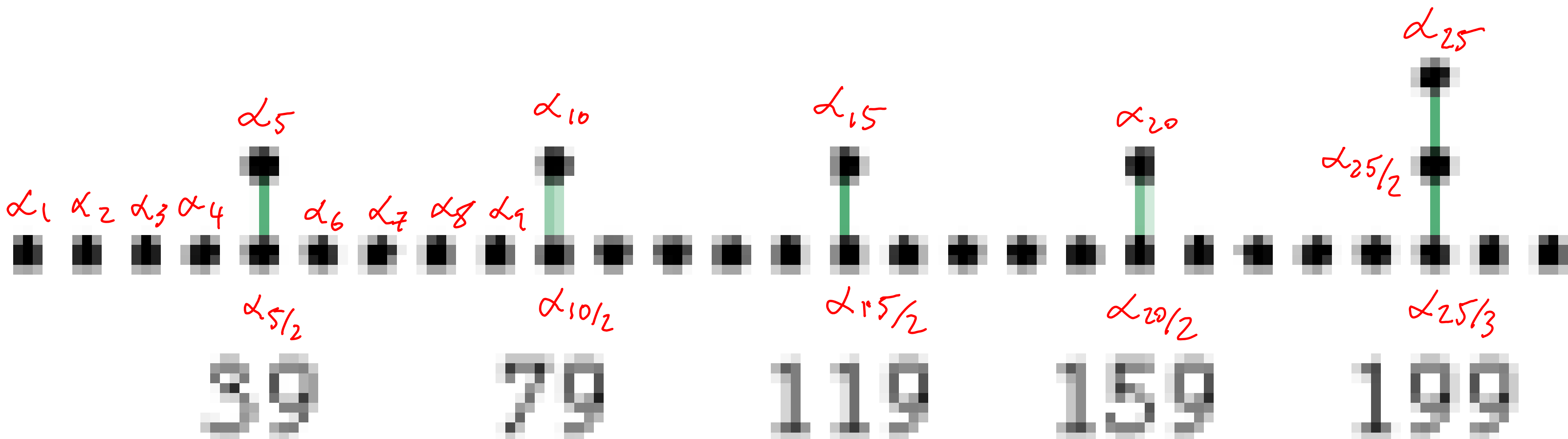
v_3-periodic: $\gamma_{i/j,k,l}$

$\pi_*(S)_{(p)}$ $p = 5$



v_1-periodic: α-family

Greek letter notation: $\alpha_{i/j} \in \left(\pi^S_{2(p-1)i-1}\right)$

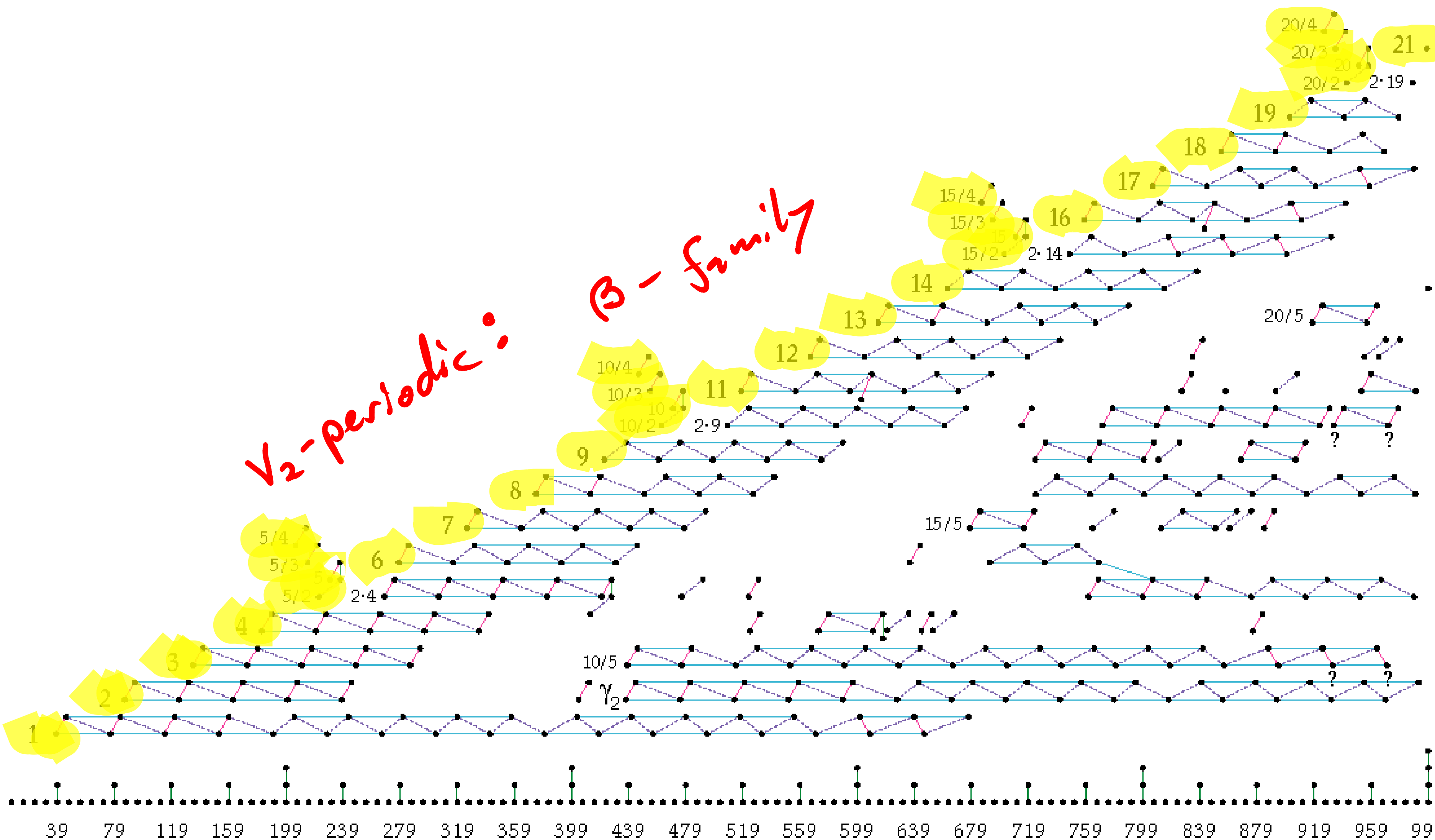


$\alpha_{i/j}$ is p^j – torsion

$j \le \nu_p(i)+1$

$(\alpha_i := \alpha_{i/1})$

$\pi_*(S)_{(p)}$ $p=5$



v_2-periodic: β-family
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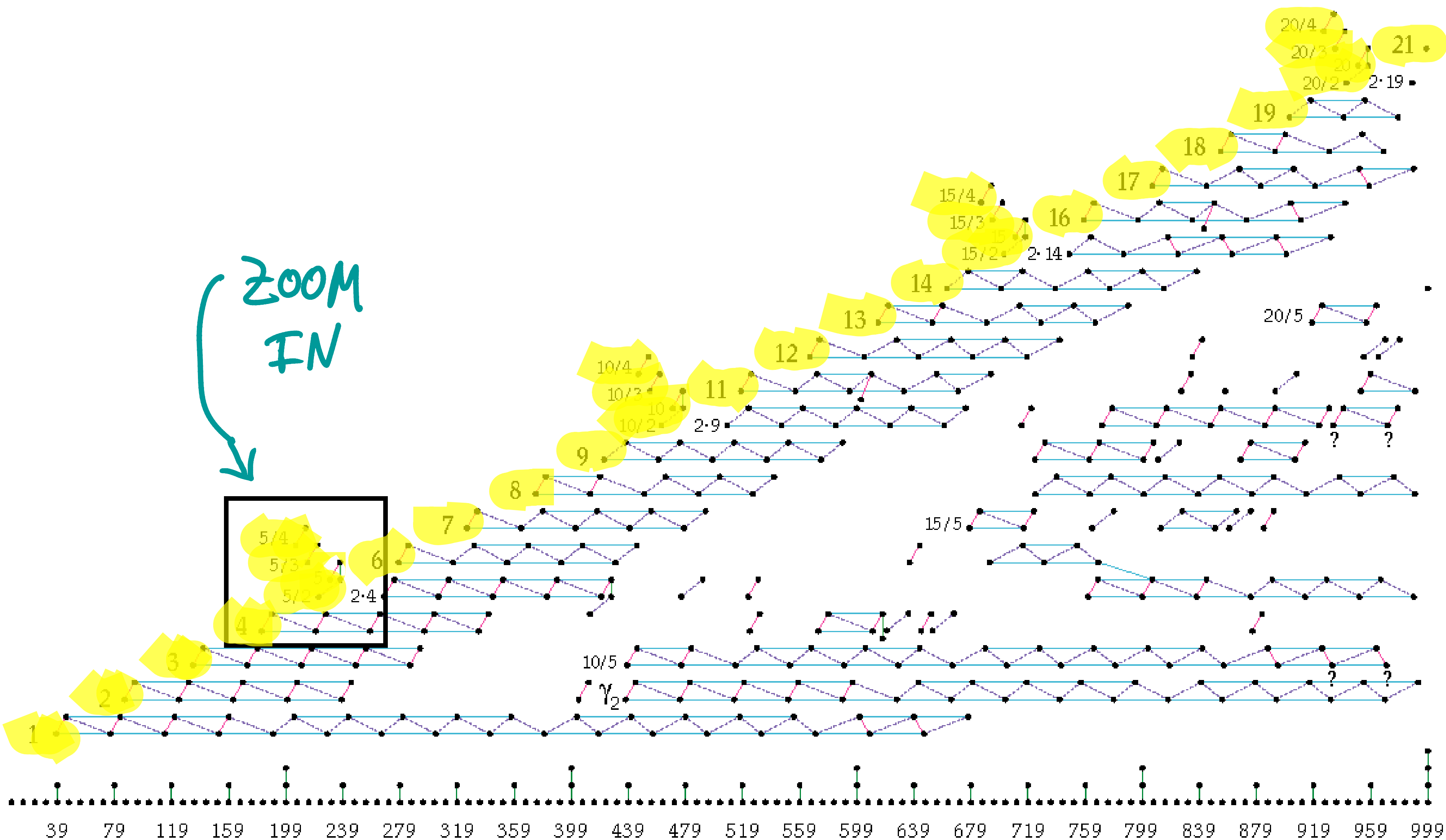
$\pi_*(S)_{(p)}$ $p=5$


ZOOM IN
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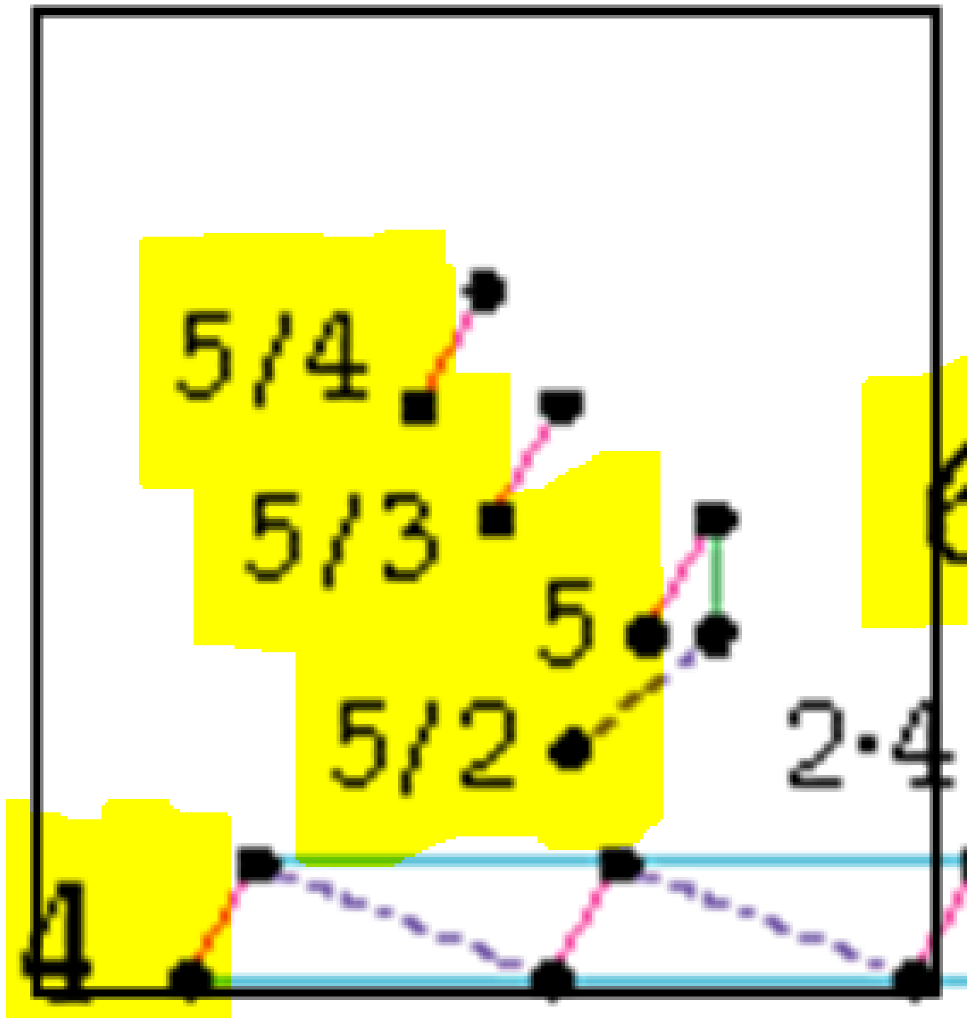
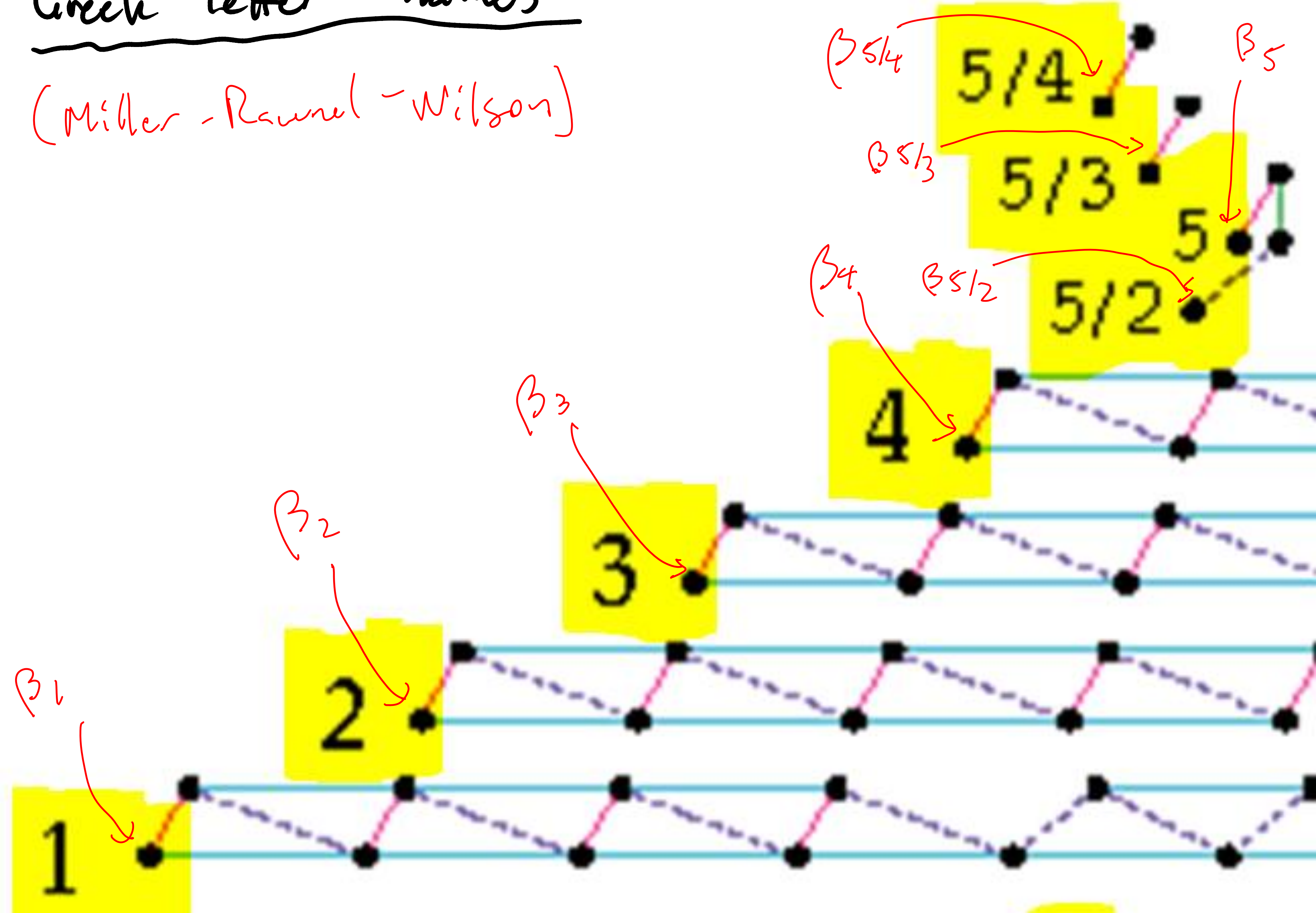
v_1-torsion in y_2-family



"5/4" $\xrightarrow{v_1}$ "5/3" $\xrightarrow{v_1}$ "5/2" $\xrightarrow{v_1}$ "5" $\xrightarrow{v_1}$ 0

"Greek letter names"

(Miller-Rawnsley-Wilson)



β-family notation

$$\beta_{i/j,k} \in \left(\pi^S_{2(p^2-1)i-2(p-1)j-2}\right)_{(p)}$$

p^k-torsion

Conversion

$\beta_{i/j,1} =: \beta_{i/j}$

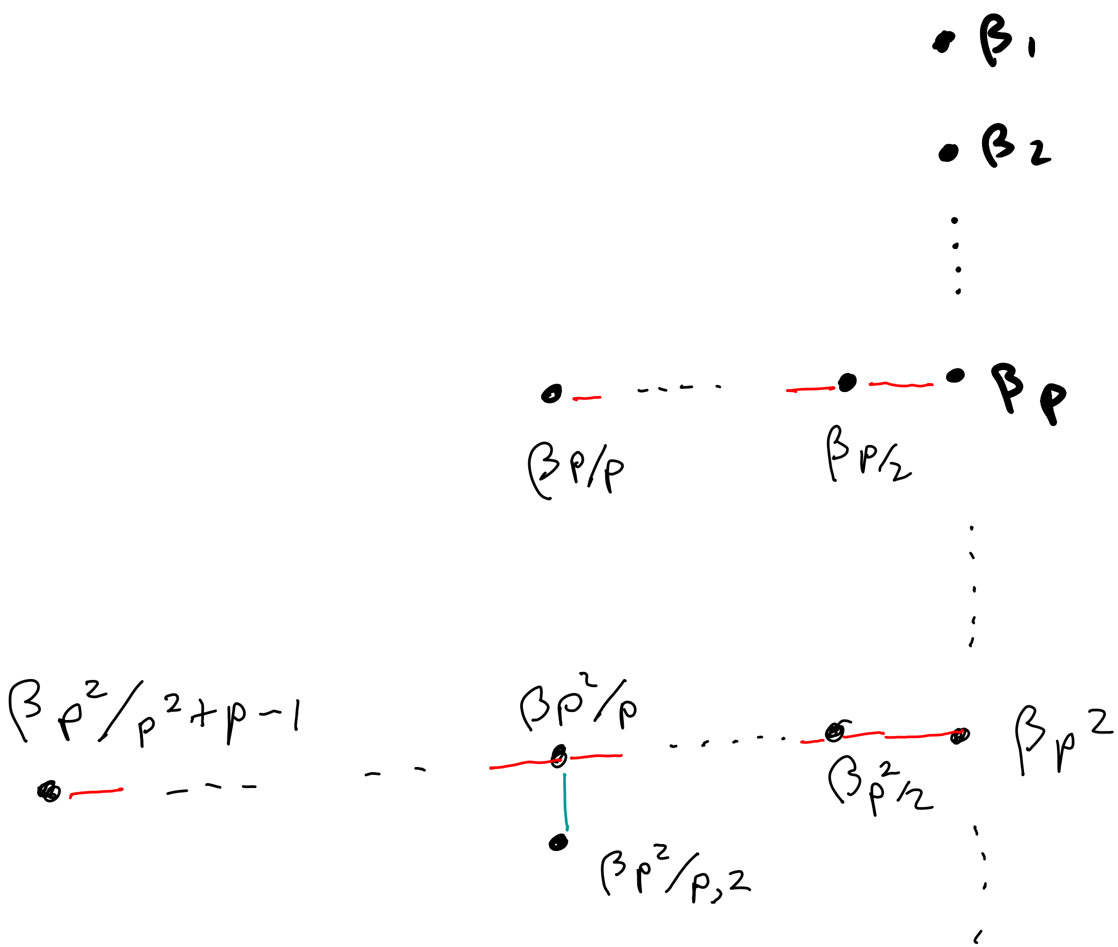
$\beta_{i/1} =: \beta_i$

$$v_2 \beta_{i/j,k} = \beta_{i+1/j,k}$$

$$v_1 \beta_{i/j,k} = \beta_{i/j-1,k}$$

$$p\, \beta_{i/j,k} = \beta_{i/j,k-1}$$

Description of β family



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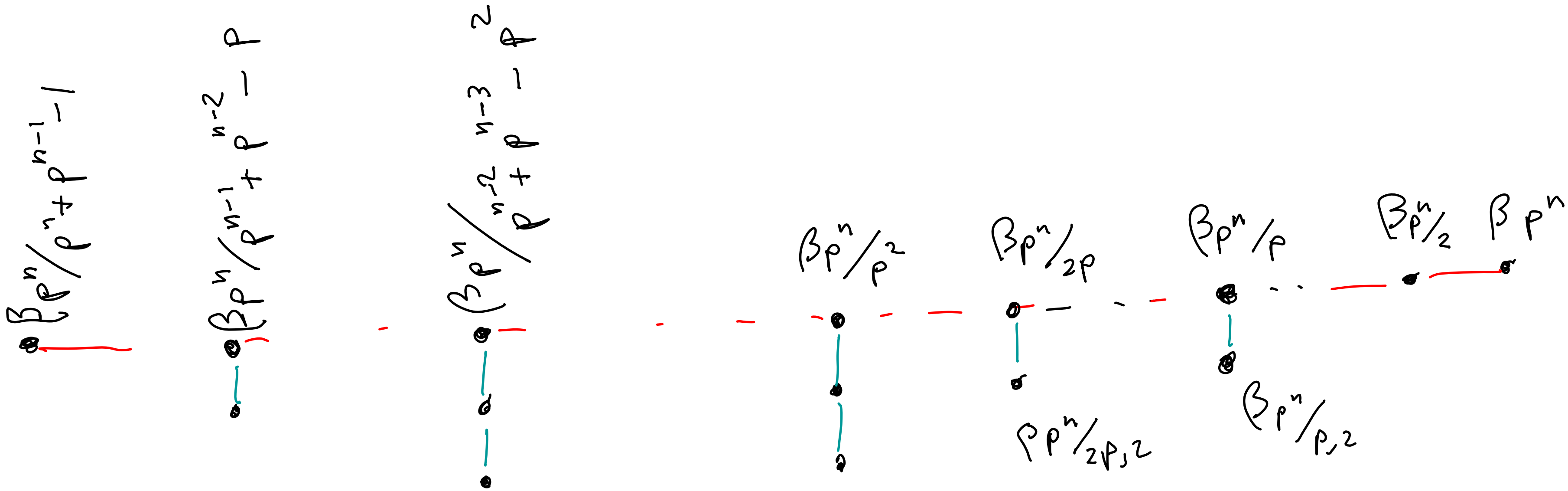

Relationship to Bernoulli #'s

B_n = n^{th} Bernoulli number

n	0	1	2	4	6	8	10	12	14	16	18	20
✓ B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$	$-\frac{3617}{510}$	$\frac{43867}{798}$	$-\frac{174611}{330}$

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Thm (Adams) $p > 2$

$$\alpha_{i/j} \text{ exists} \iff p^j \mid \operatorname{denom}\left(\frac{B_n}{n}\right)$$

$n = (p-1)i$

Adams' Thm: Gives a relationship

p-local v_1-periodic homotopy $\iff$ p-local arithmetic properties of Bernoulli #'s

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global objects which simultaneously encode p-primary homotopy for every prime p!

OUR GOAL: Give a relationship

p-local v_2-periodic homotopy (β-family) $\Longleftrightarrow$ p-local arithmetic properties of Modular Forms

global objects which simultaneously encode p-primary homotopy for every prime p!

"pf" of Adam's thm:

"pf" of Adam's thm: (Disclaimer: Revisionist History!)

1) v_1-periodic homotopy $\iff$ $\pi_* M_1 S$

"pf" of Adams's thm: (Disclaimer: Revisionist History!)

1) v_1-periodic homotopy $\Longleftrightarrow$ $\pi_* M_1 S$

2) $J \longrightarrow KU_p \xrightarrow{\psi^\ell - 1} KU_p$ $\quad \langle \ell \rangle = \mathbb{Z}_p^\times$

$$M_1 S \xrightarrow{\simeq} M_1 J$$

3) $\qquad M_1 J \longrightarrow M_1 KU \xrightarrow[\psi^{\ell}-1]{} M_1 KU$

<u>induces</u> SES

$$0 \longrightarrow \pi_{2k} M_1 J \longrightarrow \mathbb{Q}/\mathbb{Z}_{(p)} \xrightarrow[\ell^k - 1]{} \mathbb{Q}/\mathbb{Z}_{(p)} \longrightarrow 0$$

3) $M_1 J \longrightarrow M_1 KU \xrightarrow[\psi^\ell - 1]{} M_1 KU$

<u>induces</u> SES

$$0 \longrightarrow \pi_{2k} M_1 J \longrightarrow \mathbb{Q}/\mathbb{Z}_{(p)} \xrightarrow[\ell^k - 1]{} \mathbb{Q}/\mathbb{Z}_{(p)} \longrightarrow 0$$

4) <u>Thm</u> (Lipshitz - Sylvester)

$$(\ell^k - 1)\frac{B_k}{k} \in \mathbb{Z}_{(p)}$$

[with p-adic valuation 0 if $(p-1) \mid k$]

Program: Emulate this

KU $\rightsquigarrow$ TMF

J $\rightsquigarrow$ $\mathbb{Q}(\ell)$

numbers $\rightsquigarrow$ modular forms

denominators of Bernoulli #'s $\rightsquigarrow$ congruences between modular forms

Remainder of this talk:

- Modular forms
- Statement of main result
- TMF + $Q(\ell)$
- Outline of proof of main result

Modular forms:

C = elliptic curve over k (char$(k) \neq p$)

$$T_\ell C = \varprojlim_i C(\bar{k})[\ell^i]$$

$$T_\ell C \cong \mathbb{Z}_\ell \times \mathbb{Z}_\ell$$

Modular forms:

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$$\cong$$

$$\mathbb{Z}_\ell \times \mathbb{Z}_\ell$$

Level structure

$$\eta : \mathbb{Z}_\ell \times \mathbb{Z}_\ell \xrightarrow{\cong} T_\ell C$$

Modular forms:

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Level structure $\eta : \mathbb{Z}_\ell \times \mathbb{Z}_\ell \xrightarrow{\cong} T_\ell C$

$K \subset GL_2(\mathbb{Z}_\ell)$

$GL_2(\mathbb{Z}_\ell) \circlearrowright$ level structures

K-level structure: $[\eta]_K$ K-orbit

$\mathcal{M}(k)$ = moduli of pairs $(C, [\eta]_k) / \mathbb{Z}[1/\ell]$

C = elliptic curve

$[\eta]_k$ = k-level structure

$$\begin{array}{c} \omega \\ \downarrow \\ \mathcal{M}(k) \end{array}$$

line bundle

$$\omega|_C = T_e^* C$$

Important examples

$K = K_0 := GL_2(\mathbb{Z}_\ell)$

$\mathcal{M}(K_0) = \mathcal{M}_{ell}[1/\ell] =$ moduli of elliptic curves C

$K = K_0(\ell) := \left\{ A \in GL_2(\mathbb{Z}_\ell) \;\middle|\; A \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod \ell \right\}$

$\mathcal{M}(K_0(\ell)) =$ moduli of elliptic curves w/ $\Gamma_0(\ell)$ - structure:

(C, H) $\quad H \leq C$ cyclic of order ℓ

Modular forms

weight n modular forms level K / R :

$$M_n(K)_R := \text{sections} \begin{pmatrix} \omega^{\otimes n} \\ \downarrow \\ \mathcal{M}(K)_R \end{pmatrix}$$

q-expansion

Modular forms have "q-expansions"

[Hypotheses on K omitted]

$$M_n(K)_R \hookrightarrow R[[q]]$$

$$f \longmapsto f(q)$$

q-expansions have interesting arithmetic properties:

Examples: Eisenstein Series

$E_4 = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$

$E_6 = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n,$

$E_8 = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n$

$E_{10} = 1 - 264 \sum_{n=1}^{\infty} \sigma_9(n) q^n$

$E_{12} = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n$

$E_{14} = 1 - 24 \sum_{n=1}^{\infty} \sigma_{13}(n) q^n.$

Note:

$E_n = 1 - \frac{2n}{B_n} q + \cdots$

$E_n \in M_n(K_0)_{\mathbb{Q}}$

(See: a course in arithmetic)

$n = (p^2-1)i$

Main Thm:

$p > 3$

$\langle \ell \rangle = \mathbb{Z}_p^\times$

$$f \in M_n(K_0)_{\mathbb{Z}}$$

$\beta_{i/j,k}$ exists $\iff$ (a) $f(q) \not\equiv h(q) \mod p$ for any $h \in M_{<n}(K_0)_{\mathbb{Z}}$

(in M_2S)

(b) There exists $g \in M_{n-(p-1)j}(K_0(\ell))_{\mathbb{Z}}$

s.t.

$$f(q^\ell) - f(q) \equiv g(q) \mod p^k$$

Picture

$$M_{n-(p-1)j}(K_0(\ell)) \ni g(q) \rightsquigarrow f(q^\ell) - f(q) \in M_n(K_0(\ell))$$

congruent mod p^k

$$f(q^\ell) - f(q) \longleftarrow f \in M_n(K_0)_{\mathbb{Z}}$$

Not congruent to form of lower weight

Topological Modular forms: [Goerss - Hopkins - Miller]

$\mathcal{O}_{ell}$ = sheaf of E_∞-ring spectra on $(\mathcal{M}_{ell})_{et}$

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$\mathcal{O}_{ell}$ = sheaf of E_∞-ring spectra on $(\mathcal{M}_{ell})_{et}$

$K \subset GL_2(\mathbb{Z}_\ell)$

$$\begin{array}{ccc} \mathcal{M}(K) & & (C, [\eta]_K) \\ \downarrow & & \downarrow \\ \mathcal{M}_{ell} & & C \end{array}$$

étale open

$\Longrightarrow \quad TMF(K) := \mathcal{O}_{ell}(\mathcal{M}(K))$

Relationship to modular forms

$p > 3$

$$\pi_{2n}\big(TMF(k)\big)_{(p)} = M_n(K)_{\mathbb{Z}_{(p)}}[\Delta^{-1}]$$

Δ = discriminant $\in M_{24}(K)$

$Q(\ell)$ spectrum

$K \triangleleft GL_2(\mathbb{Z}_\ell)$

$\Rightarrow$ $\mathcal{M}(K)$

$\downarrow$ Galois $GL_2(\mathbb{Z}_\ell)/K$

$\mathcal{M}_{ell}[1/\ell]$

$\Rightarrow$ $TMF(K) \circlearrowleft GL_2(\mathbb{Z}_\ell)/K$

$$V := \varinjlim_{K \triangleleft GL_2(\mathbb{Z}_\ell)} TMF(K)$$

$\circlearrowleft$

$GL_2(\mathbb{Z}_\ell)$

Fact: action extends to $GL_2(\mathbb{Q}_\ell)$

using "quasi-isogenies"

Sheaf condition on $\mathcal{O}_{ell}$

$$\Rightarrow \qquad TMF(K) \simeq V^{hK}$$

<u>Defn:</u>

$$Q(\ell) := V^{hGL_2(\mathbb{Q}_\ell)}$$

Cosimplicial Resolution

$GL_2(\mathbb{Q}_\ell) \circlearrowright \mathcal{B}$ Building for $GL_2(\mathbb{Q}_\ell)$

(2-dim'l simpl'l complex)

Contractibility of $\mathcal{B}$ + simplicial decomposition

$\Longrightarrow$

$$Q(\ell) \simeq \mathrm{Tot}\left(TMF(K_0) \rightrightarrows \begin{matrix} TMF(K_0(\ell)) \\ \times \\ TMF(K_0) \end{matrix} \substack{\rightarrow \\ \rightarrow \\ \rightarrow} TMF(K_0(\ell)) \right)$$

Relationship to v_2-periodicity

$$S \longrightarrow Q(\ell)$$

$\mathbb{S}_2$ = Morava stabilizer gp

There is a subgp $\Gamma \subset \mathbb{S}_2$ and comparison of ANSS's [Morava change of rings]

$$\begin{array}{ccc} H^*\left(\Gamma, (E_2)_*/(p^\infty, u_1^\infty)\right)^{Gal} & \Longrightarrow & \pi_* M_2 Q(\ell) \\ \uparrow & & \uparrow \\ H^*_{cont}\left(\mathbb{S}_2, (E_2)_*/(p^\infty, u_1^\infty)\right)^{Gal} & \Longrightarrow & \pi_* M_2 S \end{array}$$

Proof of Main Thm:

$$
\begin{array}{ccc}
H^*\left(I, (E_2)_*/(p^\infty, u_1^\infty)\right)^{Gal} & \Longrightarrow & \pi_* M_2 \mathbb{Q}(\ell) \\
\uparrow & & \uparrow \\
H^*_{cont}\left(\mathbb{S}_2, (E_2)_*/(p^\infty, u_1^\infty)\right)^{Gal} & \Longrightarrow & \pi_* M_2 S
\end{array}
$$

$\beta_{i/j,k}$ live in H^0

Thm: (B-Lawson)

$\langle \ell \rangle = \mathbb{Z}_p^\times$, $p > 2$

$$\Gamma \underset{\text{dense}}{\hookrightarrow} \mathbb{S}_2$$

profinite

Proof of Main Thm:

$$H^*\left(\mathcal{I}, (E_2)_*/(p^\infty, u_1^\infty)\right)^{Gal} \Longrightarrow \pi_* M_2 \mathbb{Q}(\ell)$$

$$\uparrow \qquad\qquad\qquad\qquad \uparrow$$

iso on H^0 by density

$$H^*_{cont}\left(\mathbb{S}_2, (E_2)_*/(p^\infty, u_1^\infty)\right)^{Gal} \Longrightarrow \pi_* M_2 S$$

$\beta_{i/j,k}$ live in H^0

Proof of Main Thm:

$\beta_{i/j,k}$ lie in $H^0(\Gamma)$

$$H^*\left(\Gamma, (E_2)_*/(p^\infty, u_1^\infty)\right)^{Gal} \Longrightarrow \pi_* M_2 Q(\ell)$$

$$\uparrow \text{ iso on } H^0 \text{ by density} \qquad\qquad \uparrow$$

$$H^*_{cont}\left(\mathbb{S}_2, (E_2)_*/(p^\infty, u_1^\infty)\right)^{Gal} \Longrightarrow \pi_* M_2 S$$

$\beta_{i/j,k}$ live in H^0

Building decomposition:

$$H^0(\mathcal{I}) = \text{equalizer}\left(\pi_* M_2TMF(K_0) \underset{d_1}{\overset{d_0}{\rightrightarrows}} \begin{array}{c} \pi_* M_2TMF(K_0(\ell)) \\ \oplus \\ \pi_* M_2TMF(K_0) \end{array} \right)$$

$$d_0 : \underset{\text{weight } n}{f(q)} \longmapsto \left(\ell^n f(q^\ell),\ \ell^n f(q) \right)$$

$$d_1 : f(q) \longmapsto \left(f(q),\ f(q) \right)$$

Final ingredient: (using results of Serre + Swinnerton-Dyer)

$$\pi_{2n} M_2 TMF(K) = \lim_{\substack{\longrightarrow \\ j,k}} M_n(K)_{\mathbb{Z}} \Big/ \begin{array}{l}\text{Modular forms} \\ \text{congruent mod } p^k \\ \text{to a form} \\ \text{of weight} \\ n - j(p-1)\end{array}$$

Comments / Questions:

- Shows complicated β-pattern reflects a p-local part of a global phenomenon in modular forms
- Can you use this to give NEW computation of β-pattern (Miller-Ravenel-Wilson) ⟹ (new? congruences)
- Relationship to G. Laures' "f-invariant"?