v_2^{32} periodicity at the prime 2

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Joint with:
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$\pi_*(S)$ = stable htpy gps of spheres

Chromatic theory

p = prime

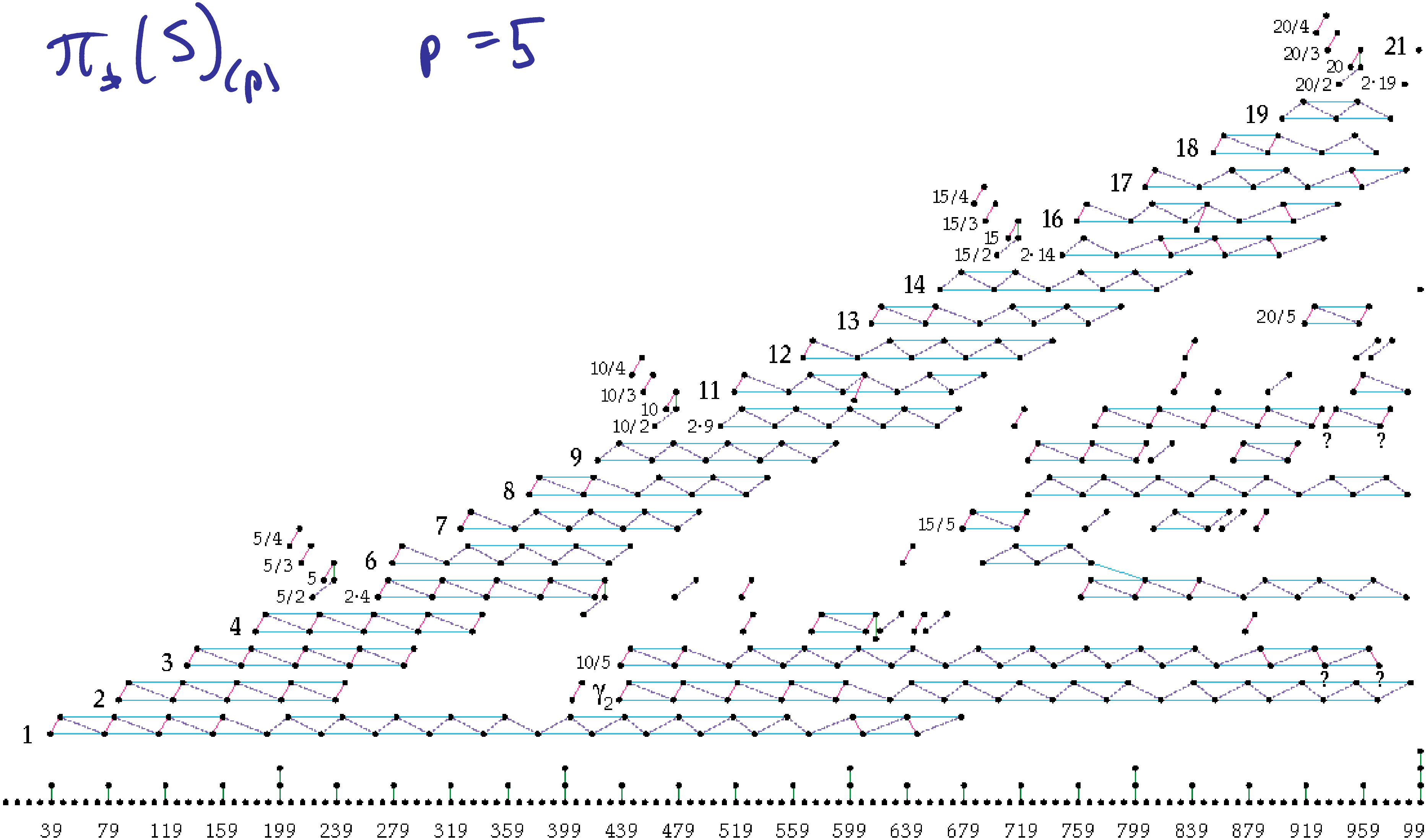
$\pi_*(S)_{(p)}$ carries a filtration

n^{th} layer $\longleftrightarrow$ "v_n-periodic"

$$|v_n| = 2(p^n - 1)$$

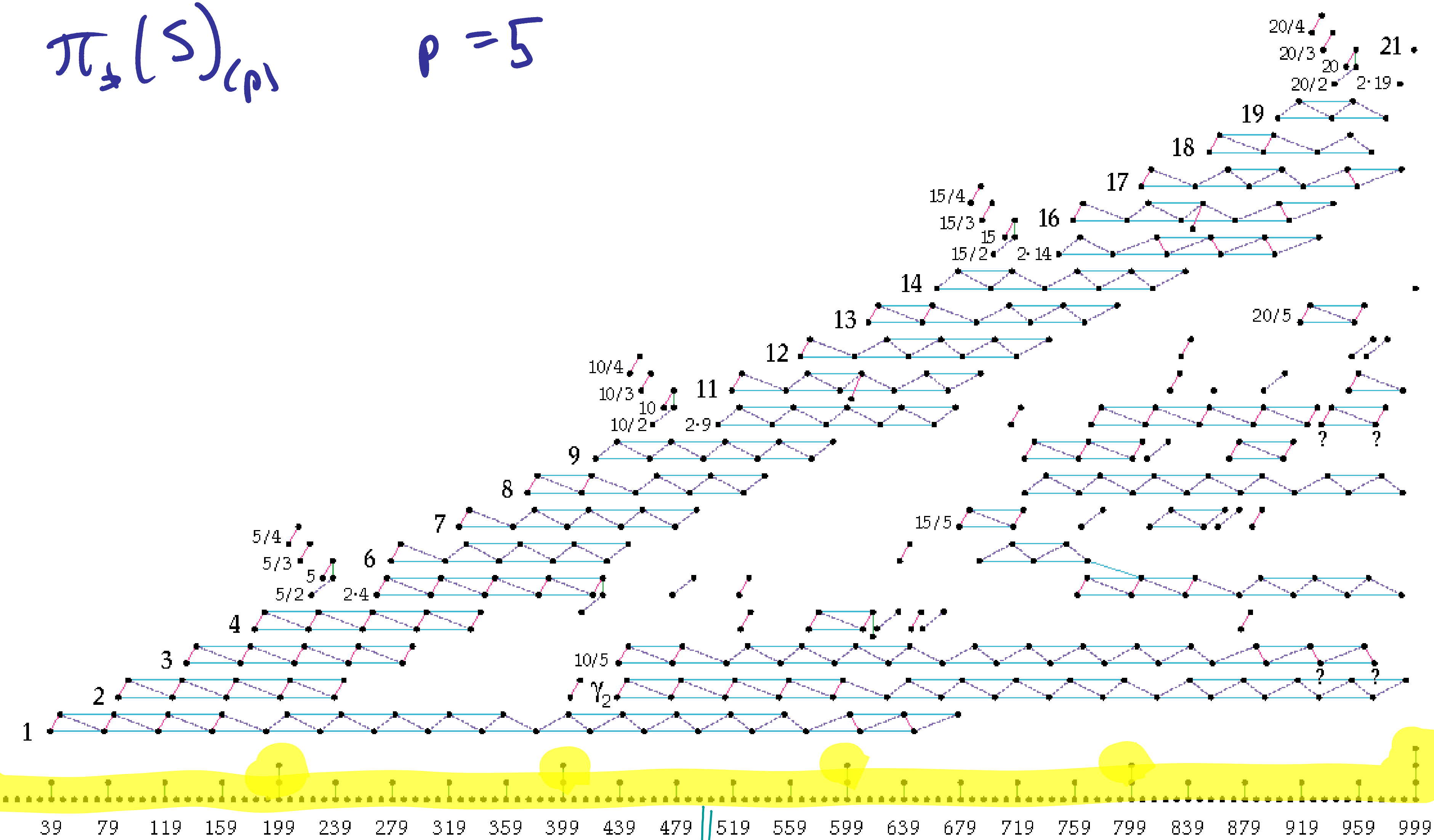
stable homotopy elts live in periodic families

$\pi_*(S)_{(p)}$ $p=5$



picture: Hatcher
computation: Ravenel

$\pi_*(S)_{(p)}$ $\quad p=5$



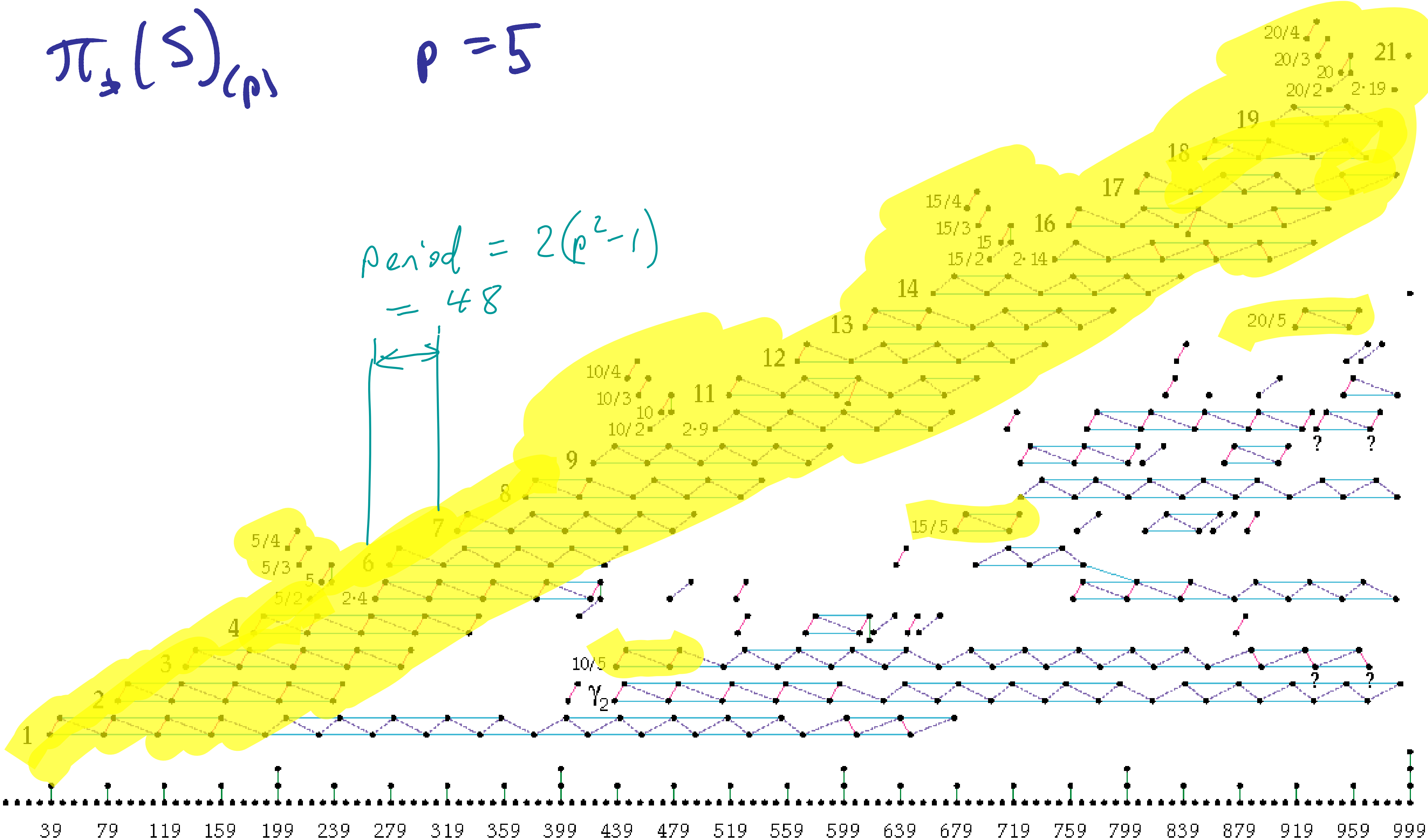
period $= 2(p-1) = 8$

v_1-periodic

picture: Hatcher

computation: Ravenel

$\pi_*(S)_{(p)}$ $p=5$

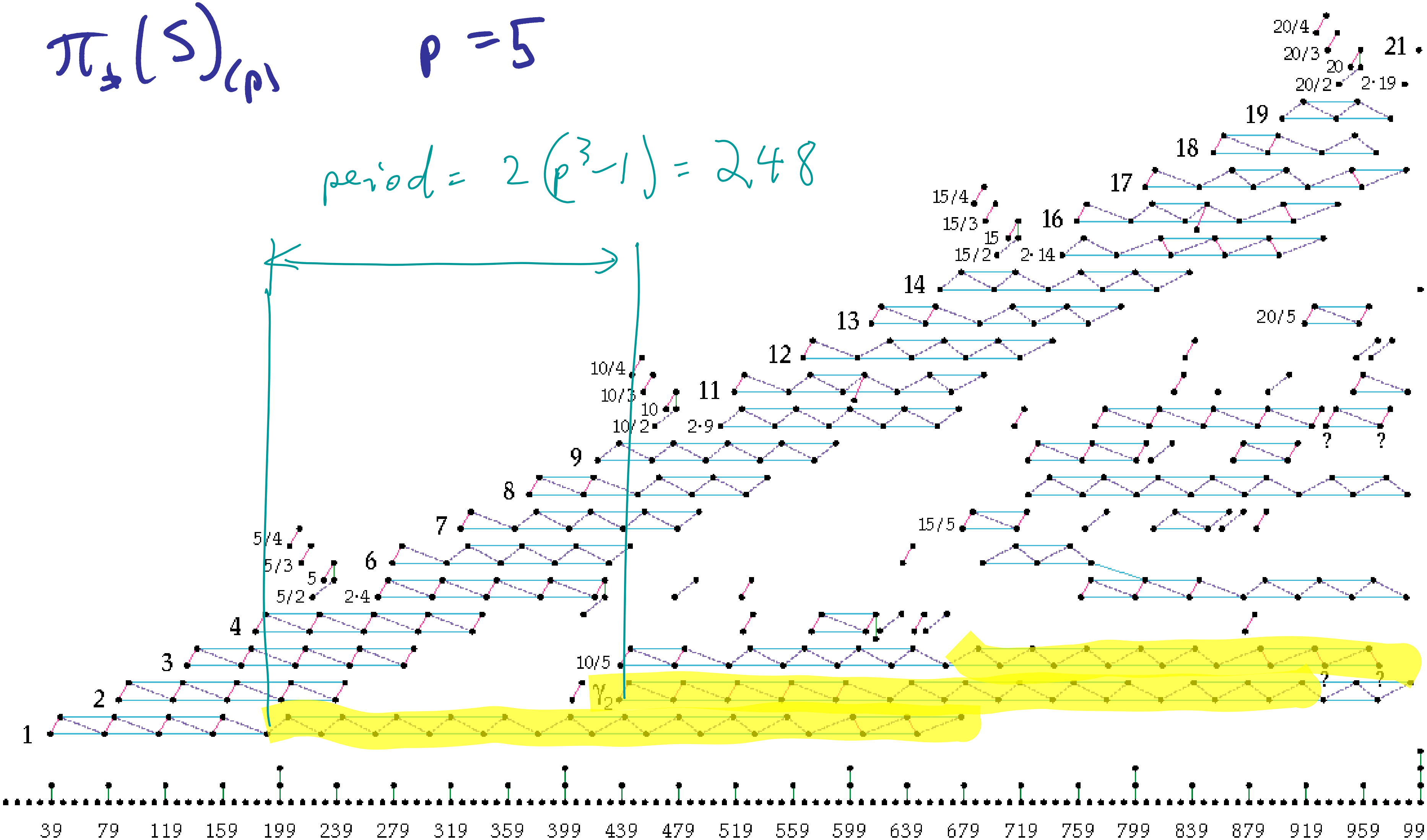


picture: Hatcher
computation: Ravenel

v_2-periodic

$\pi_*(S)_{(p)}$ $p = 5$

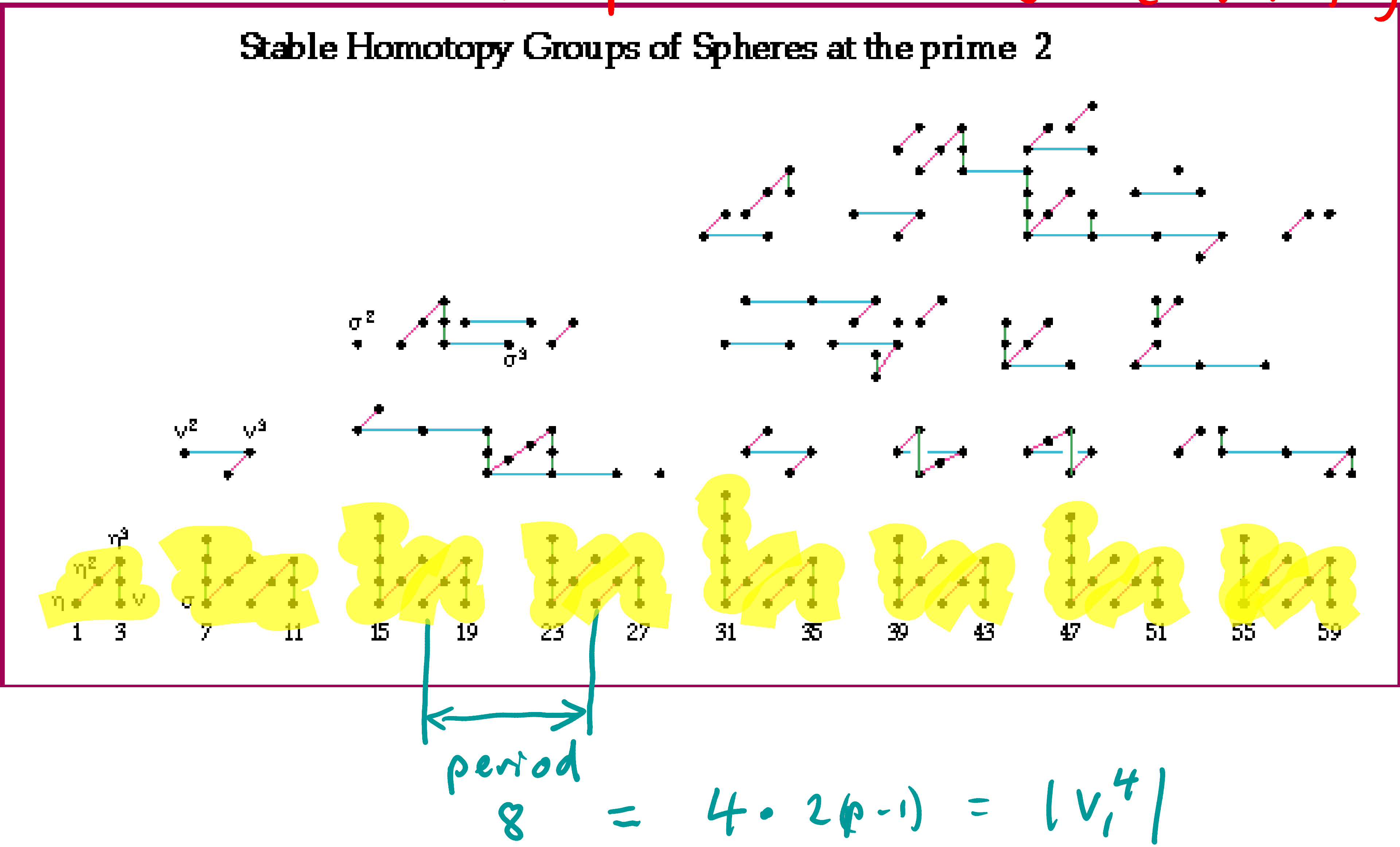
period $= 2(p^3-1) = 248$



picture: Hatcher
computation: Ravenel

v_3-periodic (?)

Bad primes: fundamental period of v_1-periodicity at $p=2$ is 8 (instead of 2)



v_n-self maps

$K(i)_* = \mathbb{F}_p[v_i^{\pm 1}]$

X = type n complex

$$K(i)_* X = 0 \quad i < n$$
$$K(n)_* X \neq 0$$

A <u>v_n-self map</u> is a map

$$\Sigma^{|v_n^k|} X \xrightarrow{v_n^k} X$$

inducing $\cdot v_n^k$ on $K(n)_*$

Generalized Moore spectra

$$S \xrightarrow{p^{i_0}} S \longrightarrow M(i_0)$$

type 1

$$\Sigma^{|v_1^{i_1}|} M(i_0) \xrightarrow{v_1^{i_1}} M(i_0) \longrightarrow M(i_0, i_1)$$

type 2

$$\Sigma^{|v_2^{i_2}|} M(i_0, i_1) \xrightarrow{v_2^{i_2}} M(i_0, i_1) \longrightarrow M(i_0, i_1, i_2)$$

type 3

$\vdots$ etc.

Minimal sequences

$(i_0, \ldots, i_n)$

If, inductively, i_k is minimal s.t. $M(i_0, \ldots, i_{k-1})$ has a $v_k^{i_k}$-self map $(0 \lesssim k \leq n)$

then $(i_0, \ldots, i_n)$ is a minimal sequence.

i.e. $p \geq 5$, $(1,1,1)$ is a minimal sequence [Smith Toda]

but for $p = 2$ $(1,4)$ is minimal (Adams)

(Since $\Sigma^8 M(1) \xrightarrow{v_1^4} M(1)$ is minimal)

$p=3$: $(1,1,9)$ is minimal [B-Pemmaraju]

This talk will discuss the proof of the following theorem:

Thm [BHHM] $(1,4,32)$ is minimal at $p=2$

i.e., There is a self map (minimal)

$$\Sigma^{192} M(1,4) \xrightarrow{v_2^{32}} M(1,4)$$

v_n-self maps $\rightsquigarrow$ v_n-periodic families

Given a factorization $\quad x \in \pi_m S$

$$\begin{array}{ccc} S^m & \xrightarrow{\;x\;} & S \\ & {\scriptstyle x'}\searrow \quad \nearrow{\scriptstyle x''} & \\ & X & \end{array}$$

X: type n complex

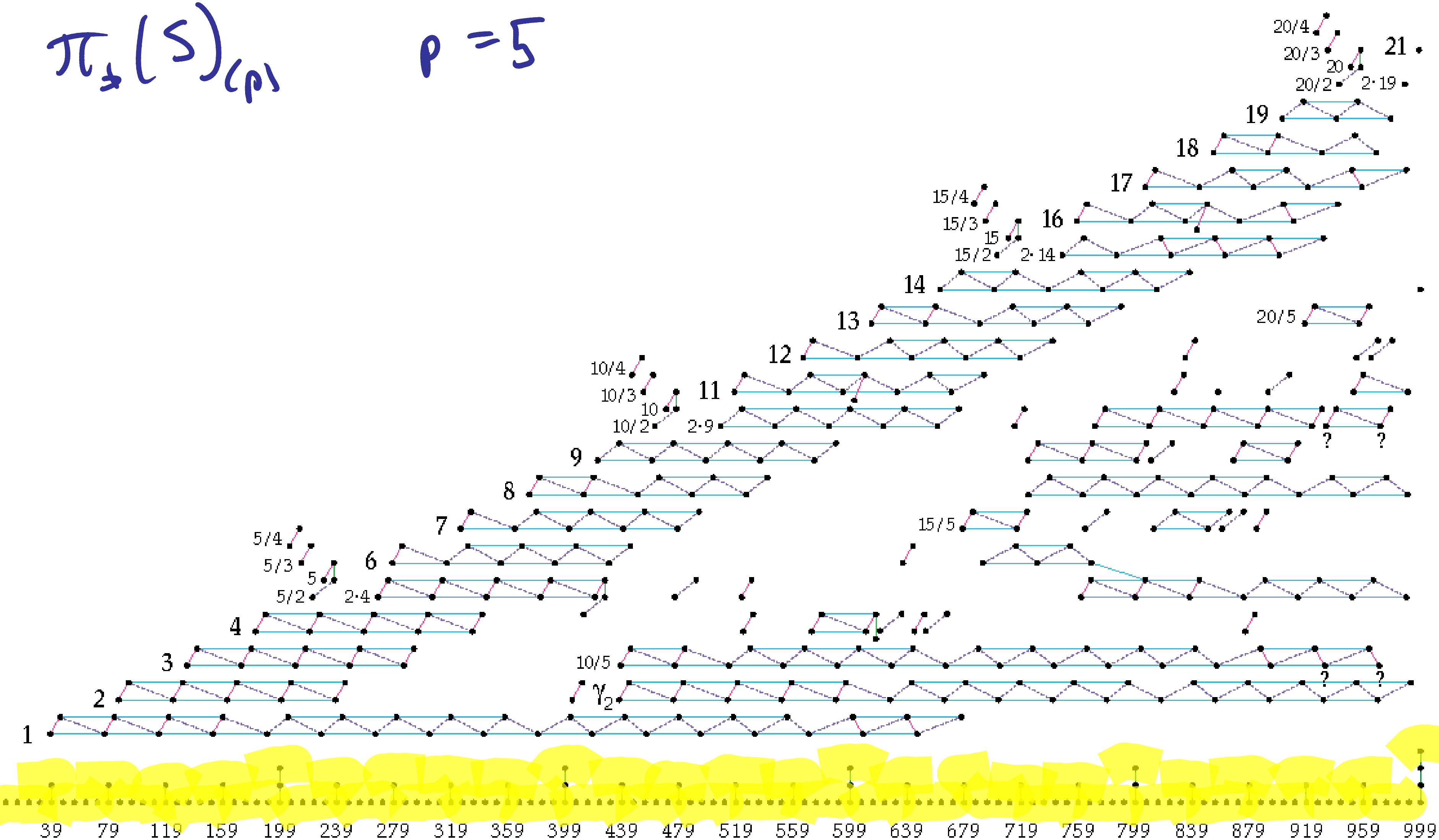
$$\Sigma^{|v_n^k|} X \xrightarrow{\;v_n^k\;} X$$

v_n-self map

We get a family:

$$x_j : S^{m+j|v_n^k|} \xrightarrow{\;x'\;} \Sigma^{j|v_n^k|} X \xrightarrow{\;(v_n^k)^j\;} X \xrightarrow{\;x''\;} S$$

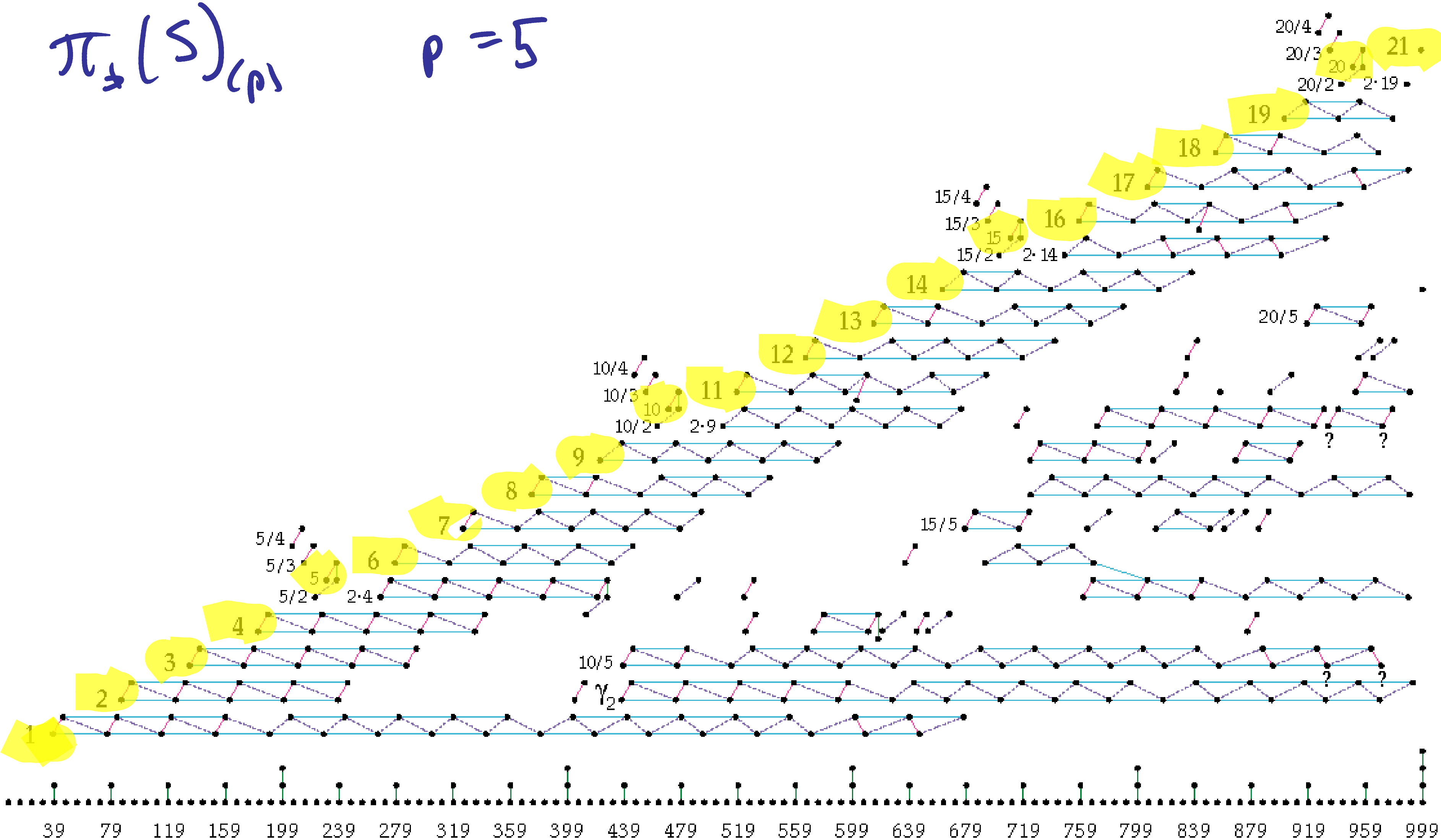
$\pi_*(S)_{(p)}$ $p=5$



picture: Hatcher
computation: Ravenel

$\alpha_j : S^{8j-1} \xrightarrow{\text{bottom cell}} \Sigma^{8j-1} M(1) \xrightarrow{v_1^j} \Sigma^{-1} M(1) \xrightarrow{\text{top cell}} S$

$\pi_*(S)_{(p)}$ $\quad p = 5$



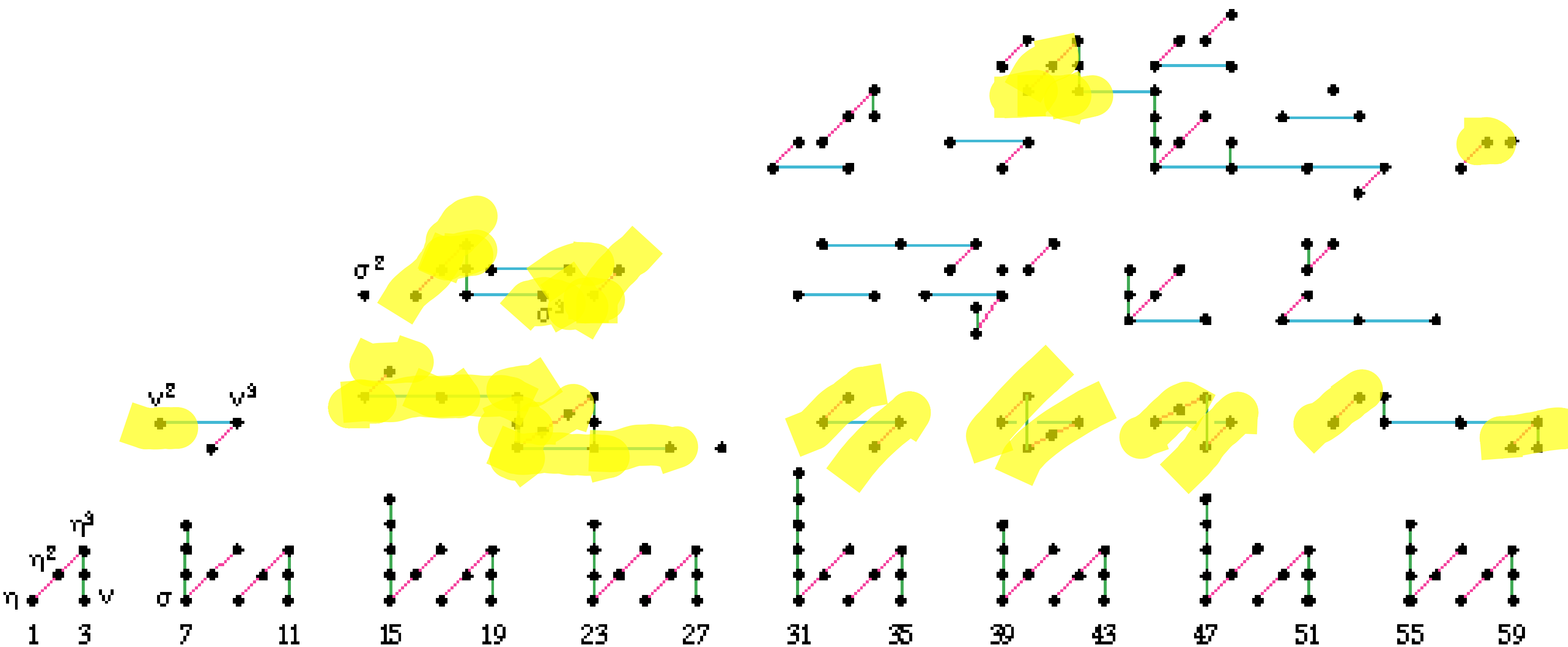
picture: Hatcher
computation: Ravenel

$$\beta_j : S^{48j-10} \xrightarrow[\text{bottom cell}]{} \Sigma^{48j-10} M(1,1) \xrightarrow[v_2^j]{} \Sigma^{-10} M(1,1) \xrightarrow[\text{top cell}]{} S$$

Cor: [Assembled from Davis-Mahowald, Mahowald, Hopkins-Mahowald]

The following elements generate v_2^{32}-periodic families [Period 192]

Stable Homotopy Groups of Spheres at the prime 2



Proof of main thm breaks up into 2 similar steps:

We'll focus on this today

Step 1: $\exists\ v_2^{32} \in \pi_{192}(M(1,4))$

Step 2: $S^{192} \xrightarrow{v_2^{32}} M(1,4)$ extends

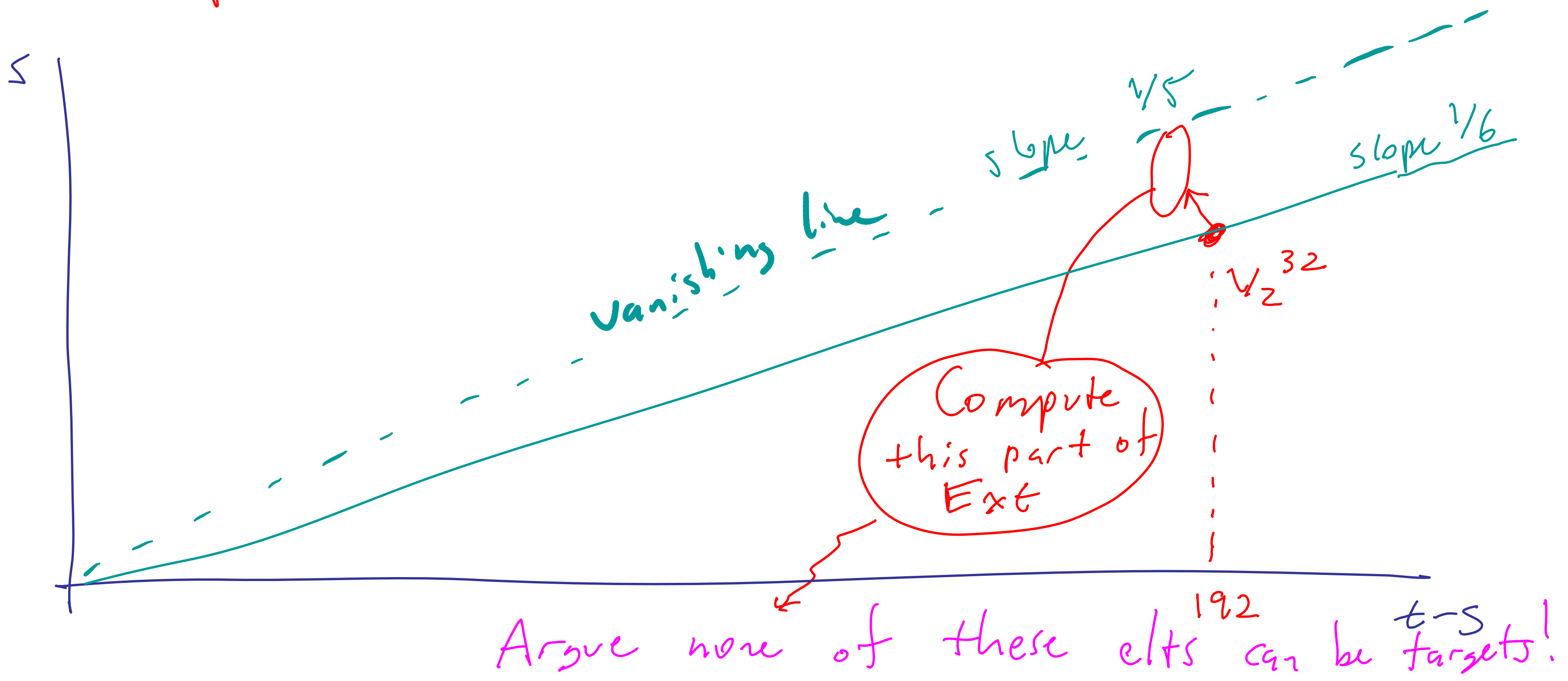
to $\Sigma^{192} M(1,4) \longrightarrow M(1,4)$

$\Longleftrightarrow$ $S^{192} \longrightarrow M(1,4) \wedge D(M(1,4))$

Proof strategy $v_2^{32} \in \pi_* M(1,4)$:

Use ASS:

$$\mathrm{Ext}_A^{s,t}(M(1,4)) \Longrightarrow \pi_*(M(1,4))$$



Side remark:

ASS = Modified Adams spectral sequence

Accounts for:

$$H^* M(1,4) \cong H^* M(1) \oplus H^* \Sigma^{9} M(1)$$

as an A-module

joined by a "quadrary" operation

i.e. v_1^4 has Adams filt 4

Our computational tool: tmf

- tmf is a connective version of TMF

TMF = "universal elliptic cohomology theory"

Our computational tool: tmf

- $H^* tmf = A /\!/ A(2) = A \otimes_{A(2)} \mathbb{F}_2$

  ($A(2)$ = subalg of A generated by Sq^1, Sq^2, Sq^4)

- "Can compute tmf_* anything"

  $$\mathrm{Ext}_{A(2)}(H^*(X)) \Longrightarrow tmf_* X$$

- $\pi_* TMF$ is 192-periodic

Refresher on Brown-Gitler spectra

$$\begin{array}{ccccccccc}
H\mathbb{Z}_0 & \longrightarrow & H\mathbb{Z}_1 & \longrightarrow & H\mathbb{Z}_2 & \longrightarrow & \cdots & \longrightarrow & H\mathbb{Z} \\
\| & & \downarrow & & \downarrow & & & & \\
S & & \Sigma^{2}(H\mathbb{Z}/2)_1 & & \Sigma^{4}(H\mathbb{Z}/2)_2 & & & &
\end{array}$$

$$\begin{array}{ccccccccc}
bo_0 & \longrightarrow & bo_1 & \longrightarrow & bo_2 & \longrightarrow & \cdots & \longrightarrow & bo \\
\| & & \downarrow & & \downarrow & & & & \\
S & & \Sigma^{4} H\mathbb{Z}_1 & & \Sigma^{8} H\mathbb{Z}_2 & & & &
\end{array}$$

$$\begin{array}{ccccccccc}
tmf_0 & \longrightarrow & tmf_1 & \longrightarrow & tmf_2 & \longrightarrow & \cdots & \longrightarrow & tmf \\
\| & & \downarrow & & \downarrow & & & & \\
S & & \Sigma^{8} bo_1 & & \Sigma^{16} bo_2 & & & &
\end{array}$$

Conjectural

bo resolutions!

$$
\begin{array}{ccccccc}
X & \leftarrow & X \wedge \overline{bo} & \leftarrow & X \wedge \overline{bo}^{\wedge 2} & \leftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
bo \wedge X & & bo \wedge \overline{bo} \wedge X & & bo \wedge \overline{bo}^{\wedge 2} \wedge X & &
\end{array}
$$

Key

$$bo \wedge \overline{bo} \simeq \bigvee_{j \geq 1} \Sigma^{4j}\, bo \wedge H\mathbb{Z}_j$$

tmf resolutions

$$
\begin{array}{ccccccc}
X & \leftarrow & \overline{tmf} \wedge X & \leftarrow & \overline{tmf}^{\wedge 2} \wedge X & \leftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
tmf \wedge X & & tmf \wedge \overline{tmf} \wedge X & & tmf \wedge \overline{tmf}^{\wedge 2} \wedge X & &
\end{array}
$$

Hope:

$$
tmf \wedge \overline{tmf} \simeq \bigvee_{i \geq 1} \Sigma^{8i} \, tmf \wedge bo_i
$$

FALSE!

(Mahowald, Rezk)

However, it $\underline{\underline{\text{is}}}$ true on the level of Ext

$$H^*(tmf \wedge \overline{tmf}) \cong \bigoplus_{j \geq 0} A/\!/A(2) \otimes H^* \Sigma^{8j} bo_j$$

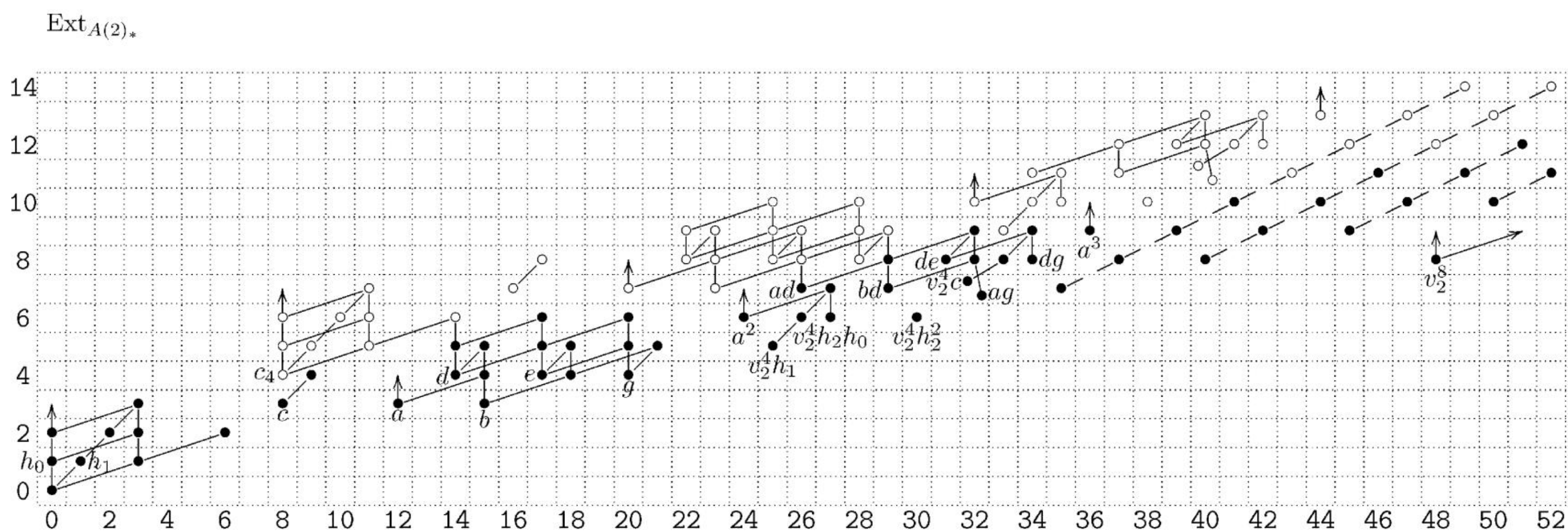
Algebraic tmf resolution: (our TOOL for computing Ext)

Spectral sequence:

$$\mathrm{Ext}_{A(2)}(bo_{i_1} \wedge \cdots \wedge bo_{i_s} \wedge X) \Rightarrow \mathrm{Ext}_A(X)$$

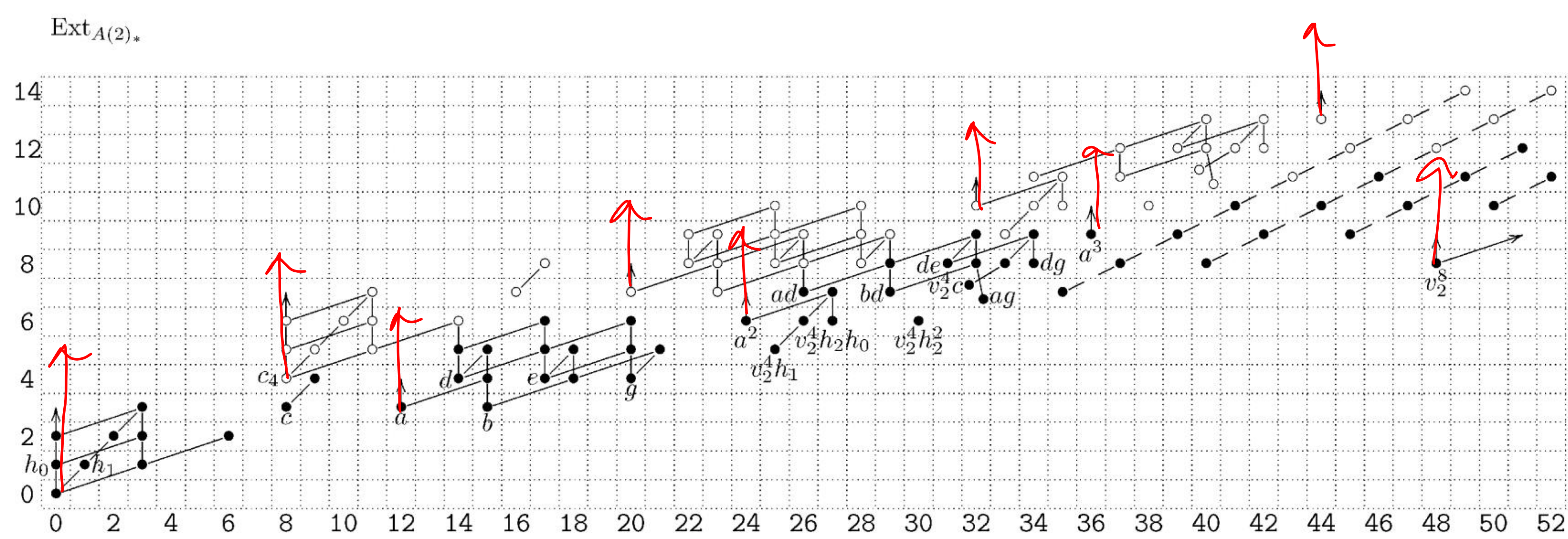
Remainder of talk: Proof of main thm.

$\mathrm{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$:



Non-Nilpotent phenomena

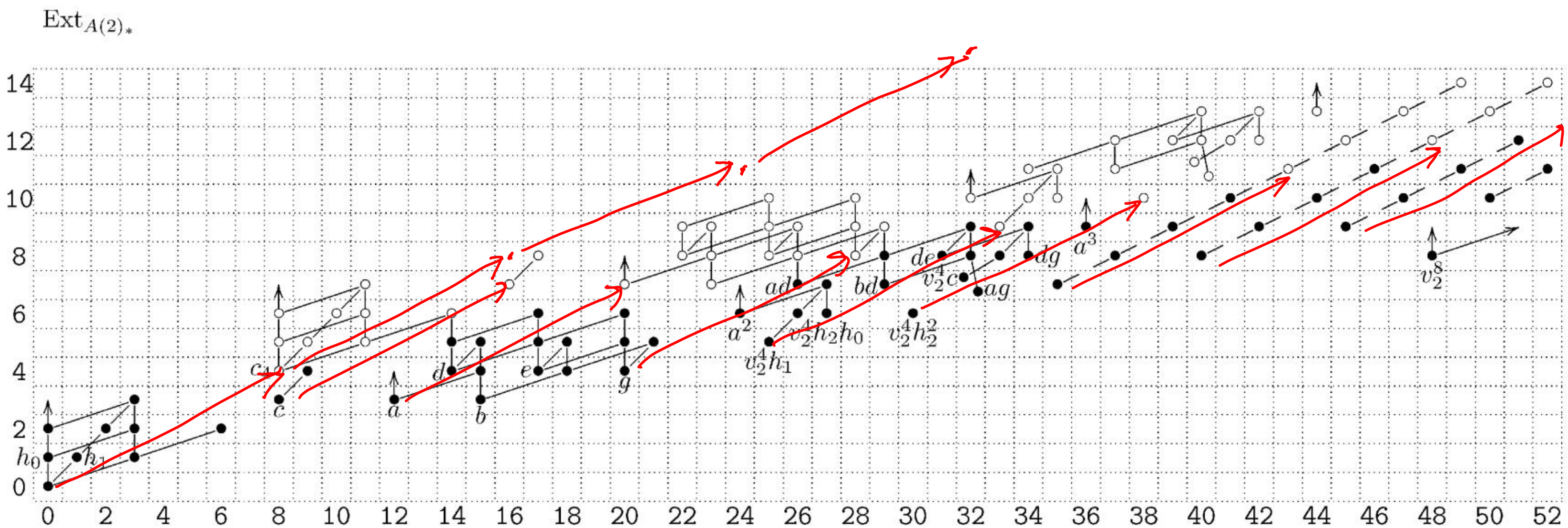
h_0 - multiplication (multiplication by 2)
(slope ∞)



Non-Nilpotent phenomena

v_1^4-multiplication (only first multipl. shown)

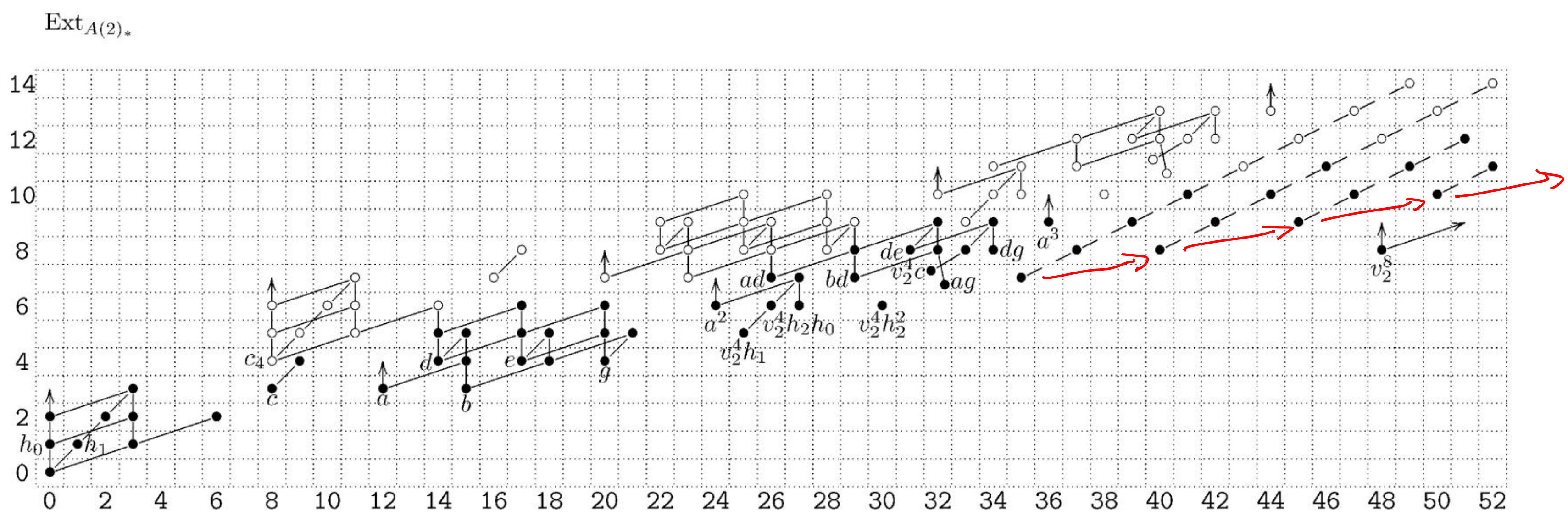
Slope 1/4



Non-Nilpotent phenomena

h_{21}-multiplication $(h_{21}^4 = \bar{\kappa})$

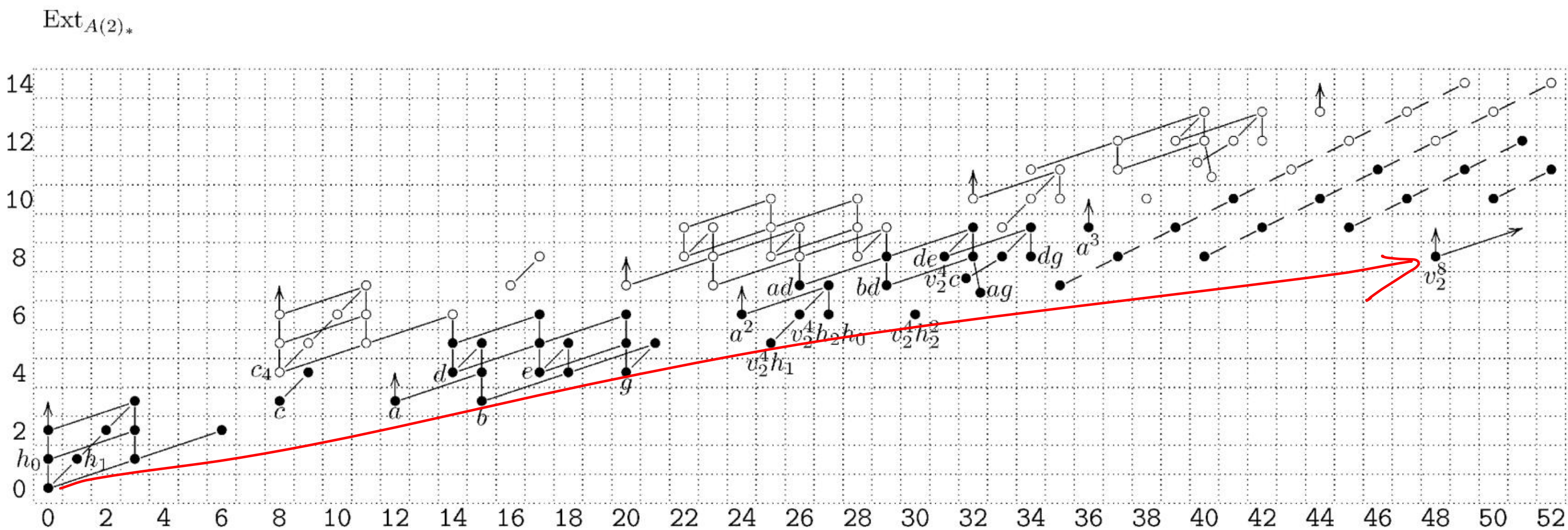
Slope $1/5$



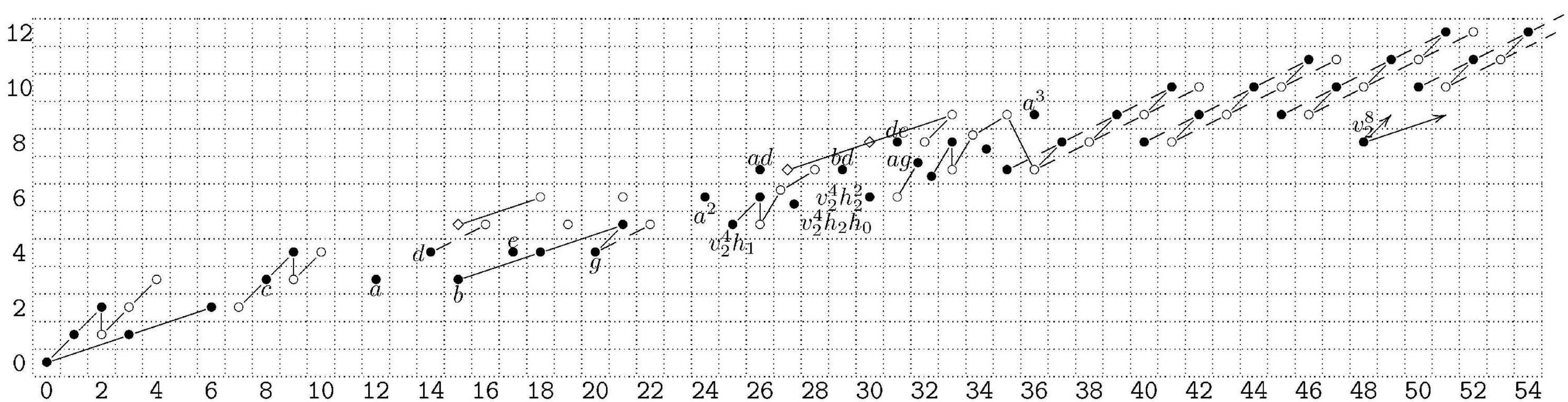
Non-Nilpotent phenomena

v_2^8-periodicity

(Slope 1/6)



$Ext_{A(2)}(M(1,4))$



12
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c
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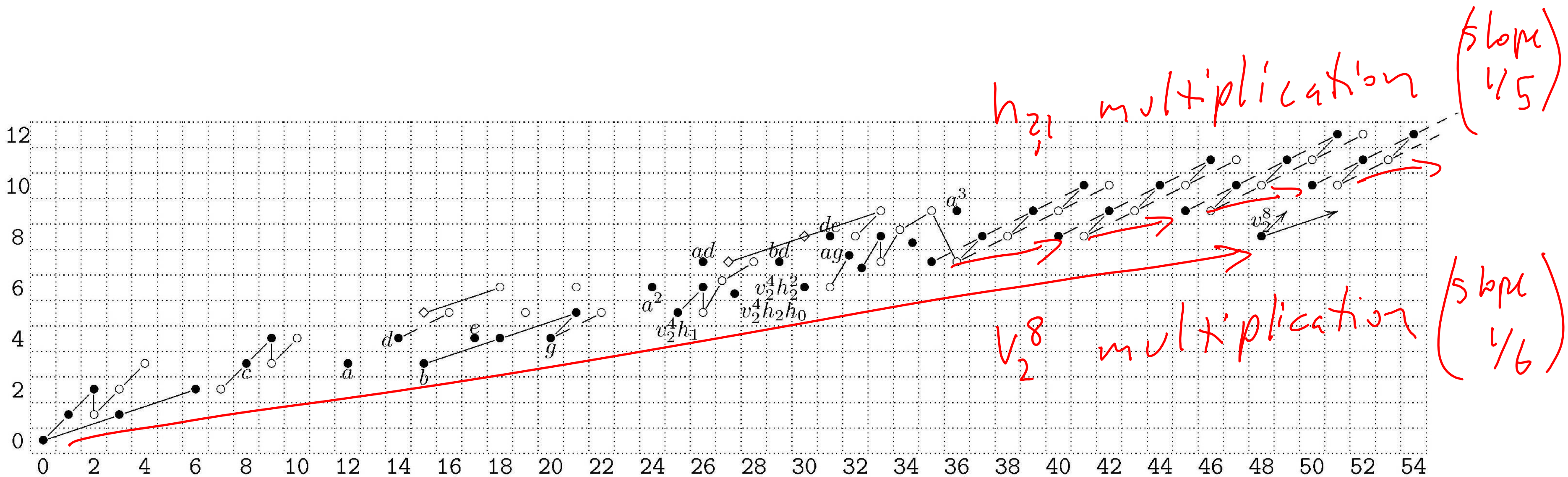
ad
bd
v₂⁴h₂²
v₂⁴h₂h₀
de
ag
a³
v₂⁸

$\mathrm{Ext}_{A(2)}(M(1,4))$

No h_0-multiplication

No v_1^4-multiplication



$\Longrightarrow$ vanishing line slope $1/5$

Prop $v_2^8 \in \mathrm{Ext}_{A(2)}(\mathbb{F}_2)$

lifts to an elt

$$v_2^8 \in \mathrm{Ext}_A(M(1,4) \wedge DM(1,4))$$

Cor: In MASS

$$\mathrm{Ext}_{A_*}\big(M(1,4)\wedge DM(1,4)\big) \Rightarrow [M(1,4), M(1,4)]_*$$

$$d_2(v_2^{16}) = 0$$

$$d_3(v_2^{32}) = 0$$

[**Note:** we shall see

$$d_2(v_2^8) \neq 0$$

$$d_3(v_2^{16}) \neq 0$$
]

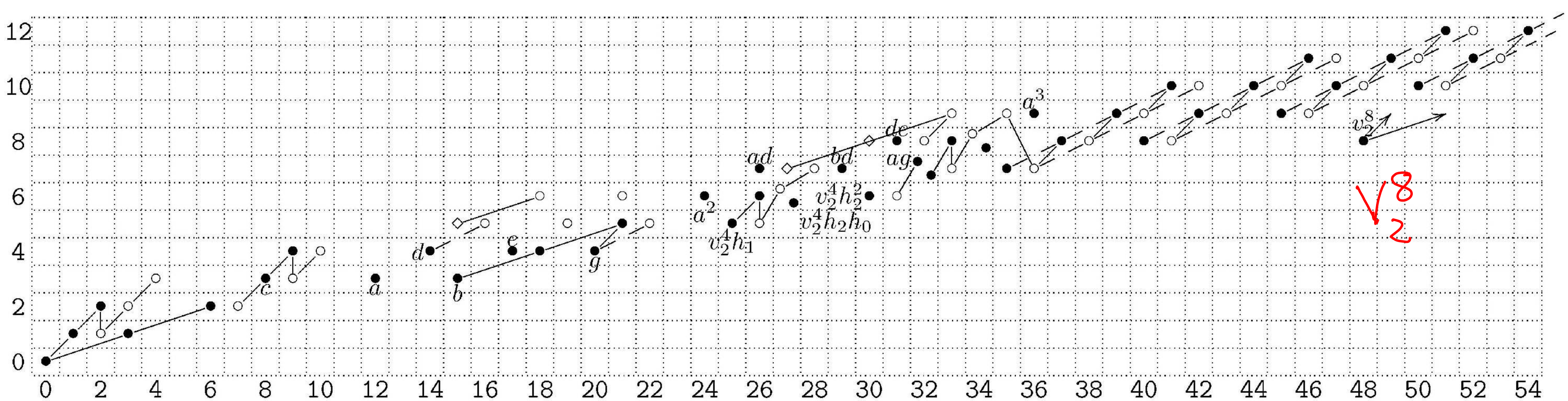
consider map of ASS's

$$\begin{array}{ccc}
\mathrm{Ext}_A(M(1,4)) & \longrightarrow & \mathrm{Ext}_{A(2)}(M(1,4)) \\
\rotatebox{90}{$\in$}\; v_2^8 & \longmapsto & v_2^8 \;\rotatebox{90}{$\in$} \\
\Downarrow & & \Downarrow \\
\pi_* M(1,4) & \xrightarrow{\text{hurewicz}} & tmf_* M(1,4)
\end{array}$$

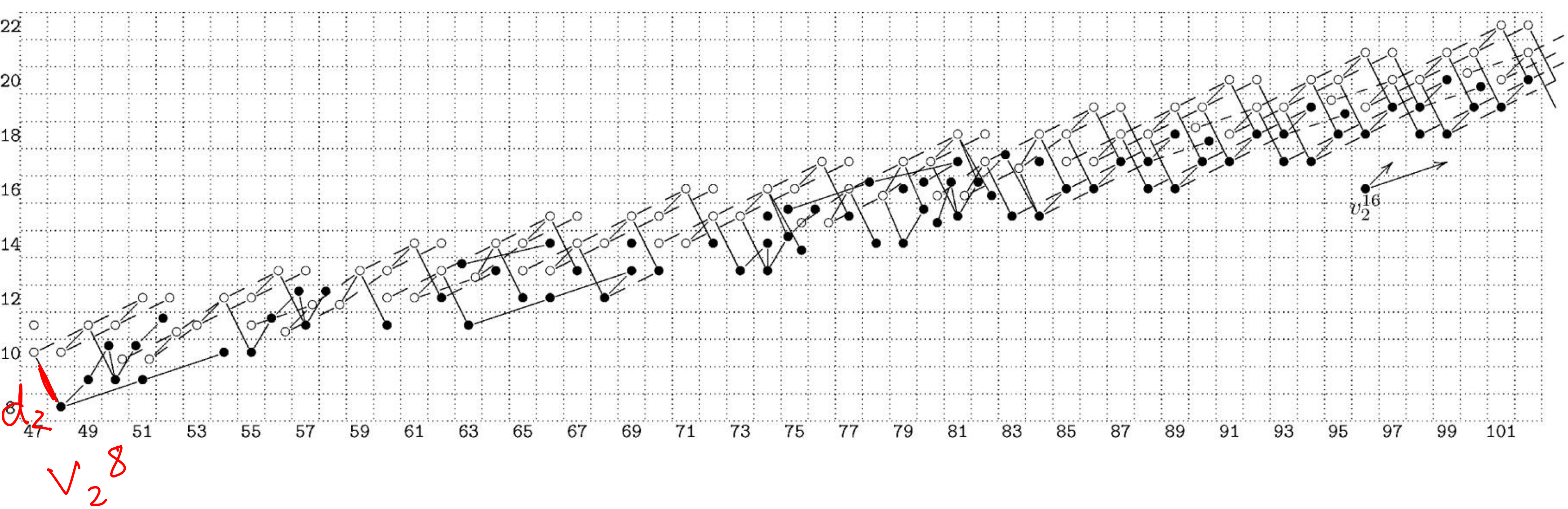
$d_2(v_2^8) \neq 0$, $d_3(v_2^{16}) \neq 0$

HERE!

$\mathrm{Ext}_{A(2)}(M(1,4)) \Rightarrow tmf_* M(1,4)$

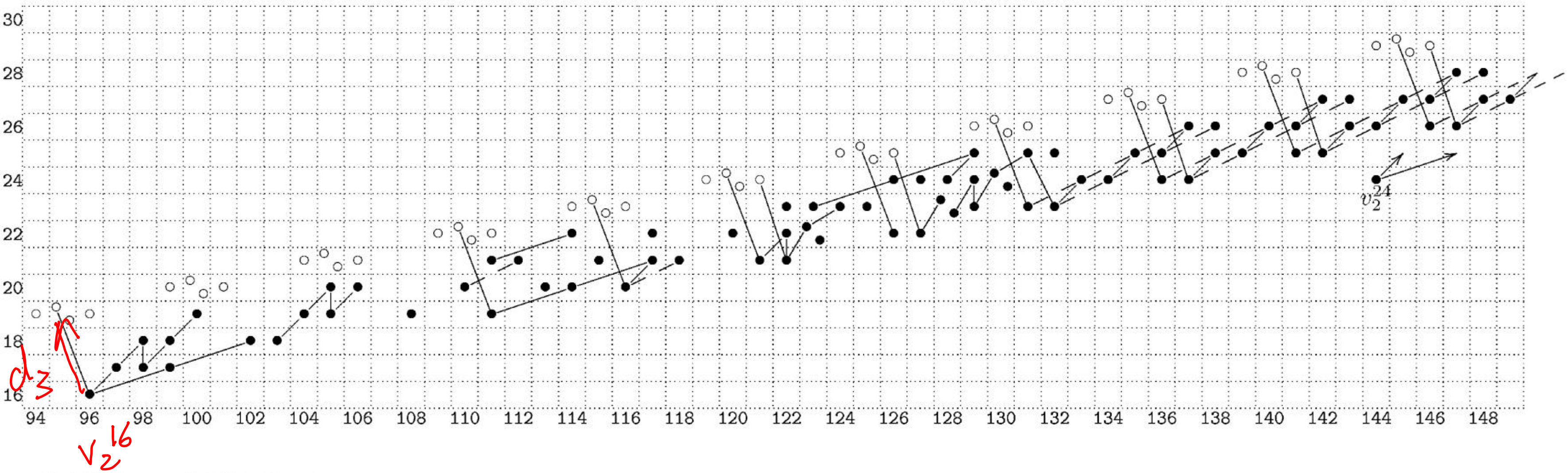


MASS for $tmf_* M(1,4)$, p2:

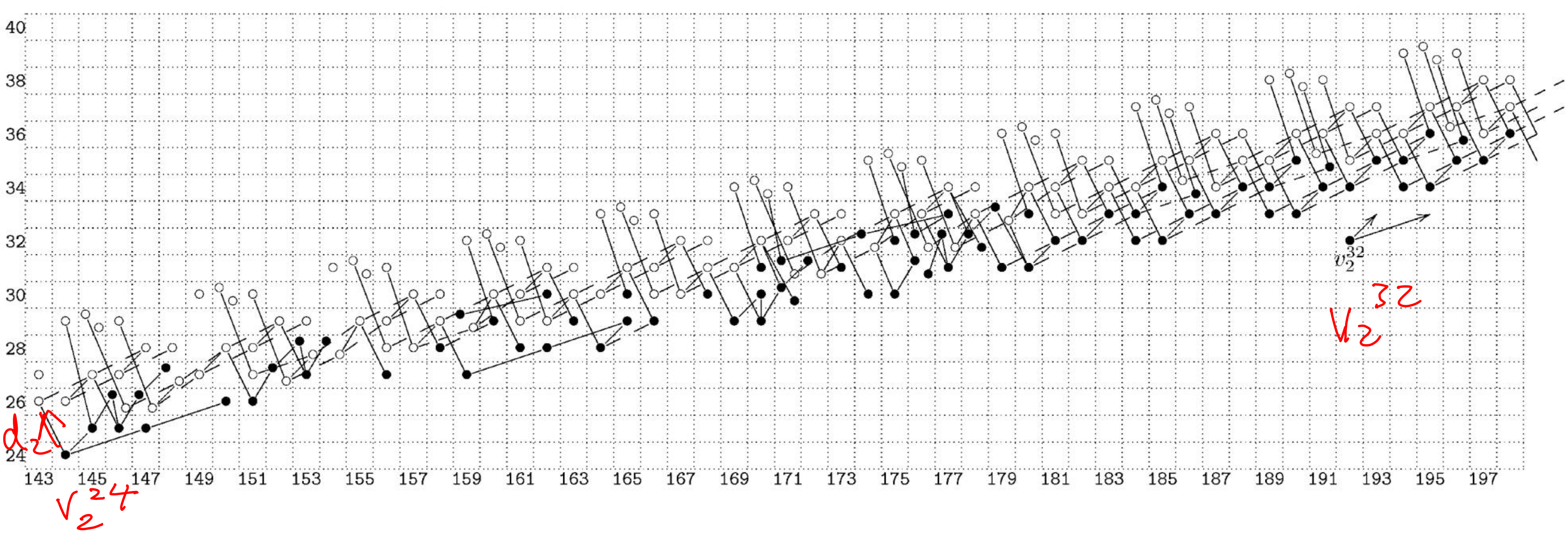


$\mathrm{Ext}_{A(2)}(M(1,4)) \Longrightarrow tmf_* M(1,4)$

MASS for $tmf_*M(1,4)$, p3:



MASS for $tmf_*M(1,4)$, p4:



$$\mathrm{Ext}_A(M(1,4)) \longrightarrow \mathrm{Ext}_{A(2)}(M(1,4))$$

$$v_2^{32} \longmapsto v_2^{32}$$

Permanent cycle

Idea: use Algebraic tmf-resolution to interpolate

$$\mathrm{Ext}_A(M(1,4)) \leftsquigarrow \mathrm{Ext}_{A(2)}(M(1,4))$$

$$\mathrm{Ext}_{A}(M(1,4)) \xrightarrow[\text{Homomorphism}]{\text{Edge}} \mathrm{Ext}_{A(2)}(M(1,4))$$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ v_2^{32} & \longmapsto & v_2^{32} \end{array}$$

Permanent cycle

Alg tmf-resolution:

$$\mathrm{Ext}_{A(2)}\left(bo_{i_1} \wedge \cdots \wedge bo_{i_s} \wedge M(1,4)\right) \Longrightarrow \mathrm{Ext}_{A}(M(1,4))$$

Problem

Compute

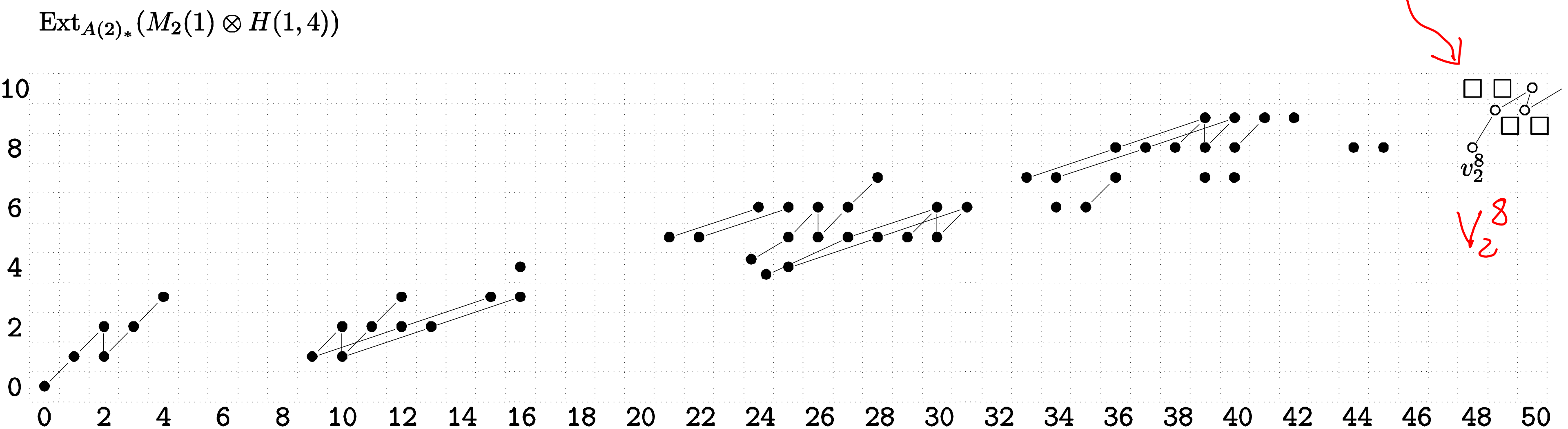
$$\mathrm{Ext}_{A(2)}\left(bo_{i_1} \wedge \cdots \wedge bo_{i_s} \wedge M(1,4)\right)$$

We computed

- $\mathrm{Ext}_{A(2)}\left(bo_1 \wedge M(1,4)\right)$
- $\mathrm{Ext}_{A(2)}\left(bo_1^{\wedge 2} \wedge M(1,4)\right)$
- $\mathrm{Ext}_{A(2)}\left(bo_1^{\wedge 3} \wedge M(1,4)\right)$

$\mathrm{Ext}_{A(2)}(bo_1 \wedge M(1,4))$

$\square = P[h_{2,1}]$

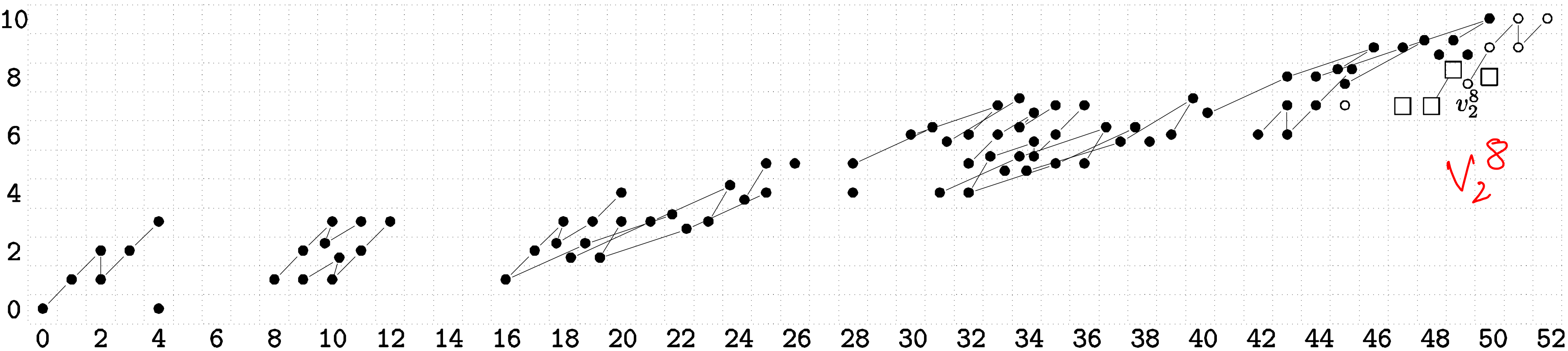


whole thing is v_2^8-periodic

$\mathrm{Ext}_{A(2)}(b_{0,1}^{\wedge 2} \wedge M(1,4))$

$\square = P[h_{2,1}]$

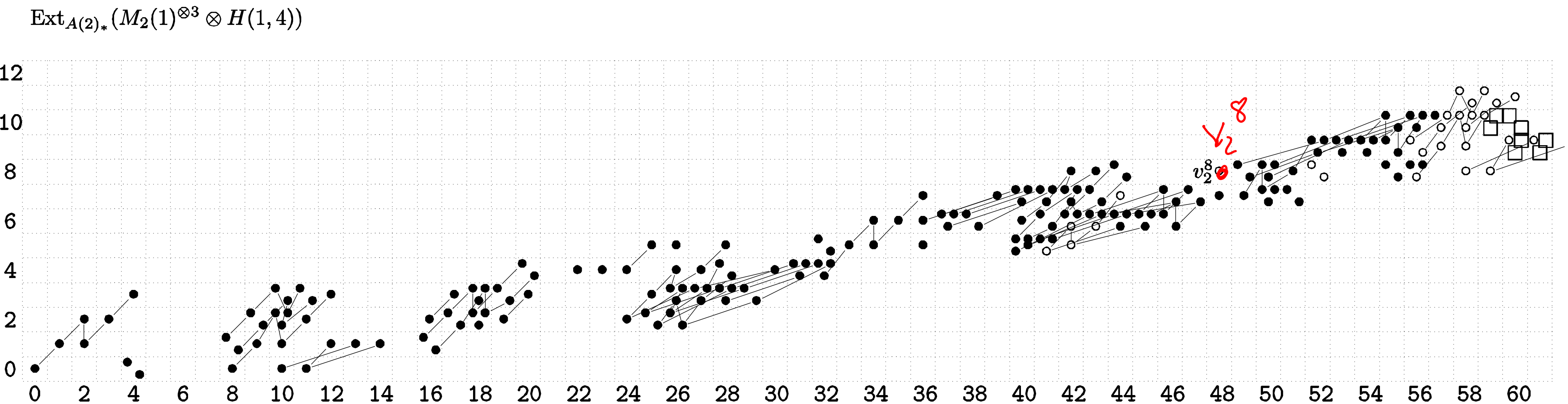
$\mathrm{Ext}_{A(2)_*}(M_2(1)^{\otimes 2} \otimes H(1,4))$



whole thing is v_2^8-periodic

$\mathrm{Ext}_{A(2)}(b_{0,1}^3 \wedge M(1,4))$

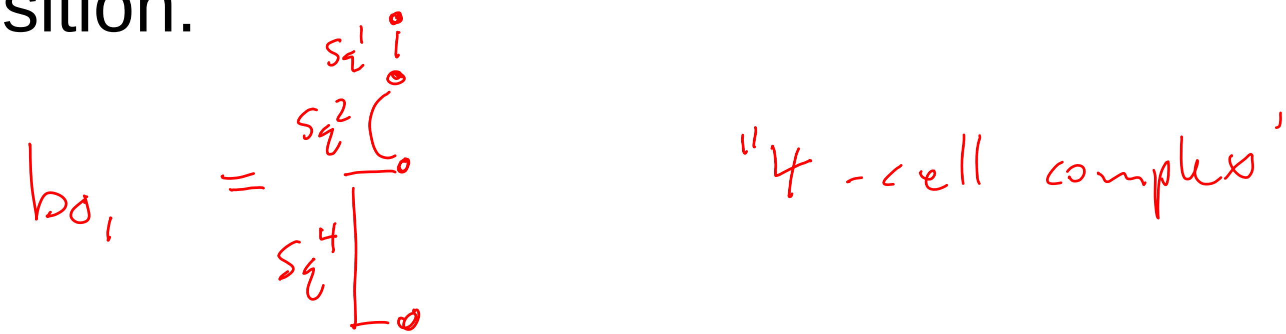
$\square = P[h_{2,1}]$



whole thing is v_2^8-periodic

How we made these computations:

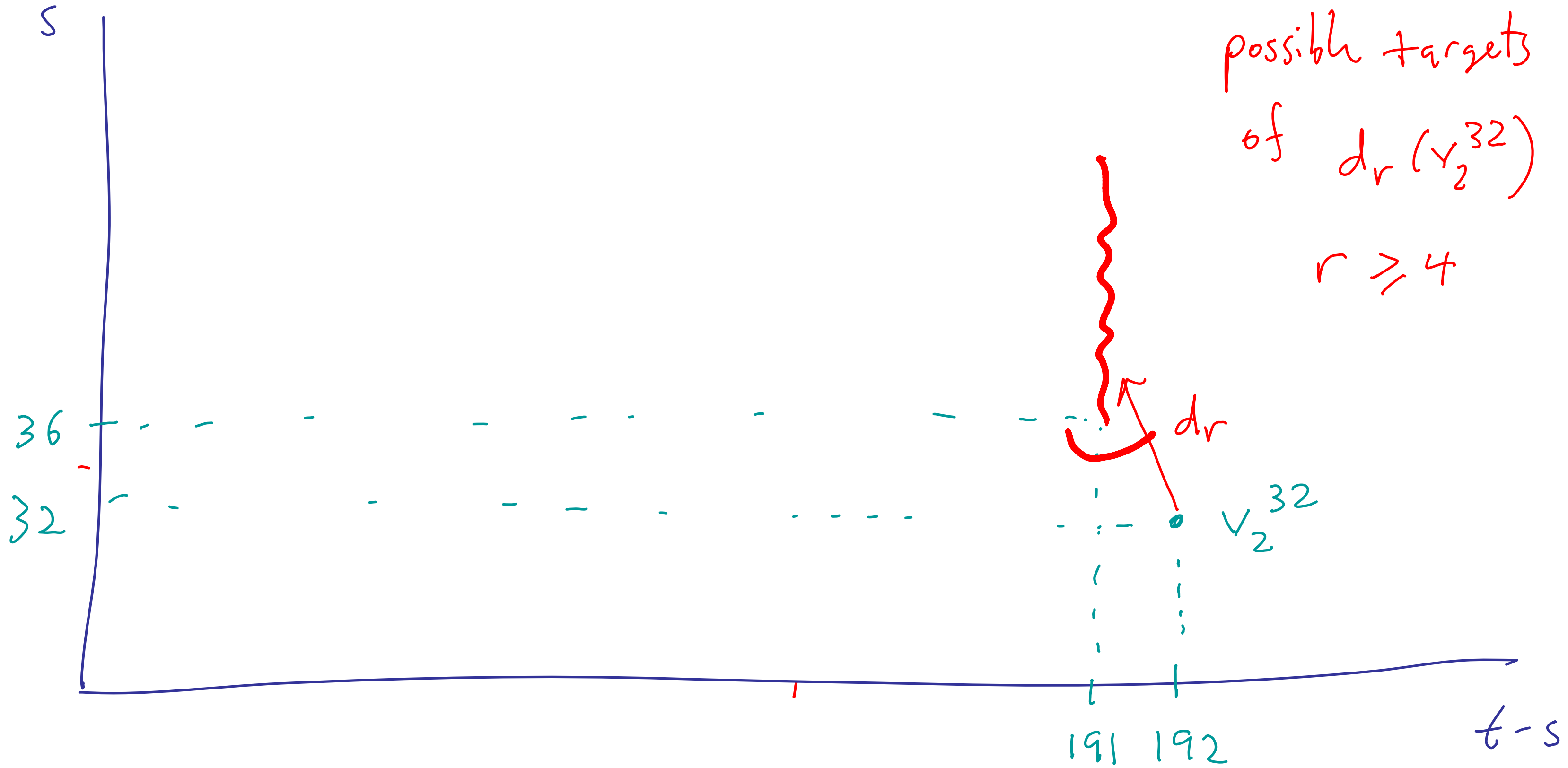
- Computation of $Ext_{A(2)}(bo_1)$ done by Davis-Mahowald
- Inductively get $Ext_{A(2)}(bo_1^s)$ by the cellular decomposition:



- Then use cellular decomposition of M(1,4) to get $Ext_{A(2)}(bo_1^s \wedge M(1,4))$
- Double check everything with Bob Bruner's ext software!

We want to understand:

$\mathrm{Ext}_A^{s,t}(M(1,4))$:



Vanishing Lemma

The only terms in the E_1-term of

$$\mathrm{Ext}_{A(2)}(bo_{i_1} \wedge \cdots \wedge bo_{i_s} \wedge M(1,4)) \Rightarrow \mathrm{Ext}_A(M(1,4))$$

which could yield potential targets of

$$d_r(v_2^{32}) \qquad r \geq 4$$

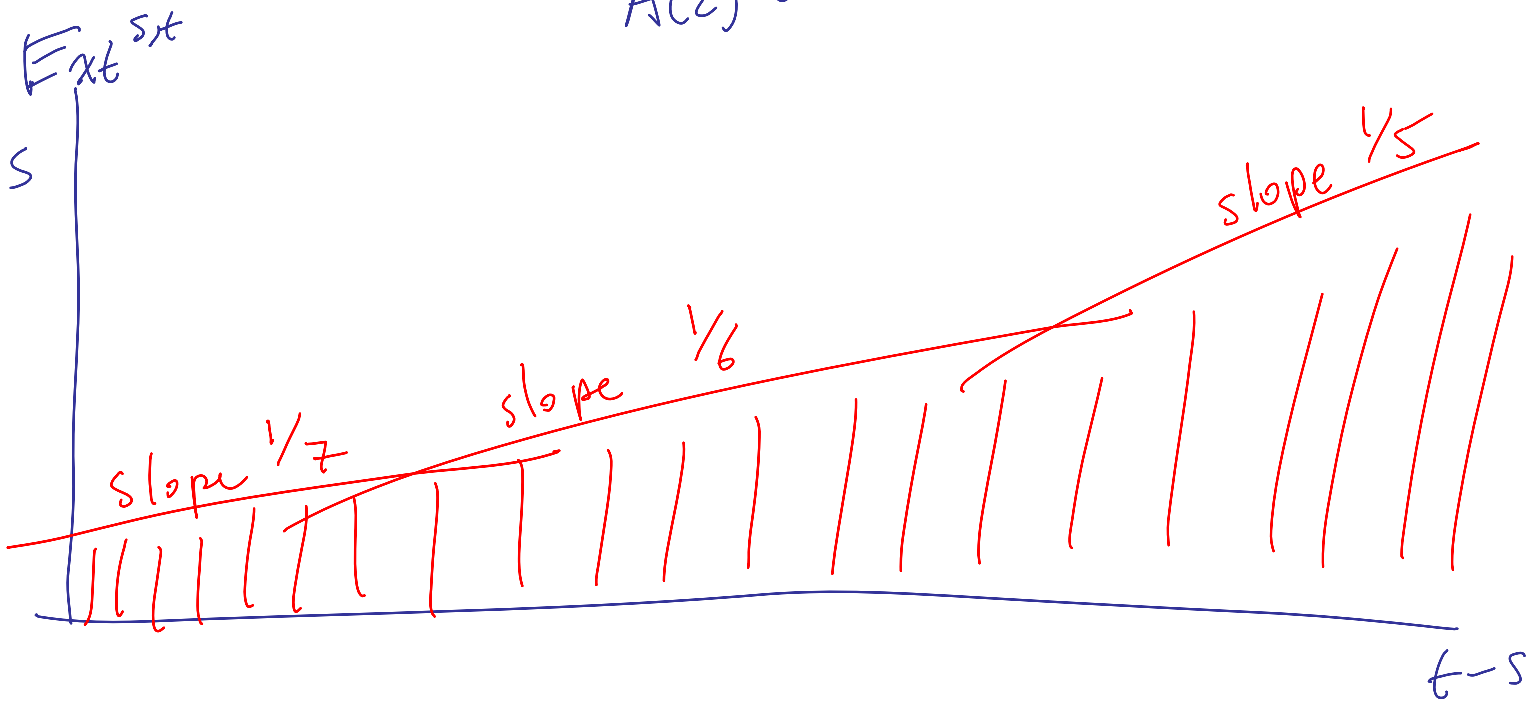
are

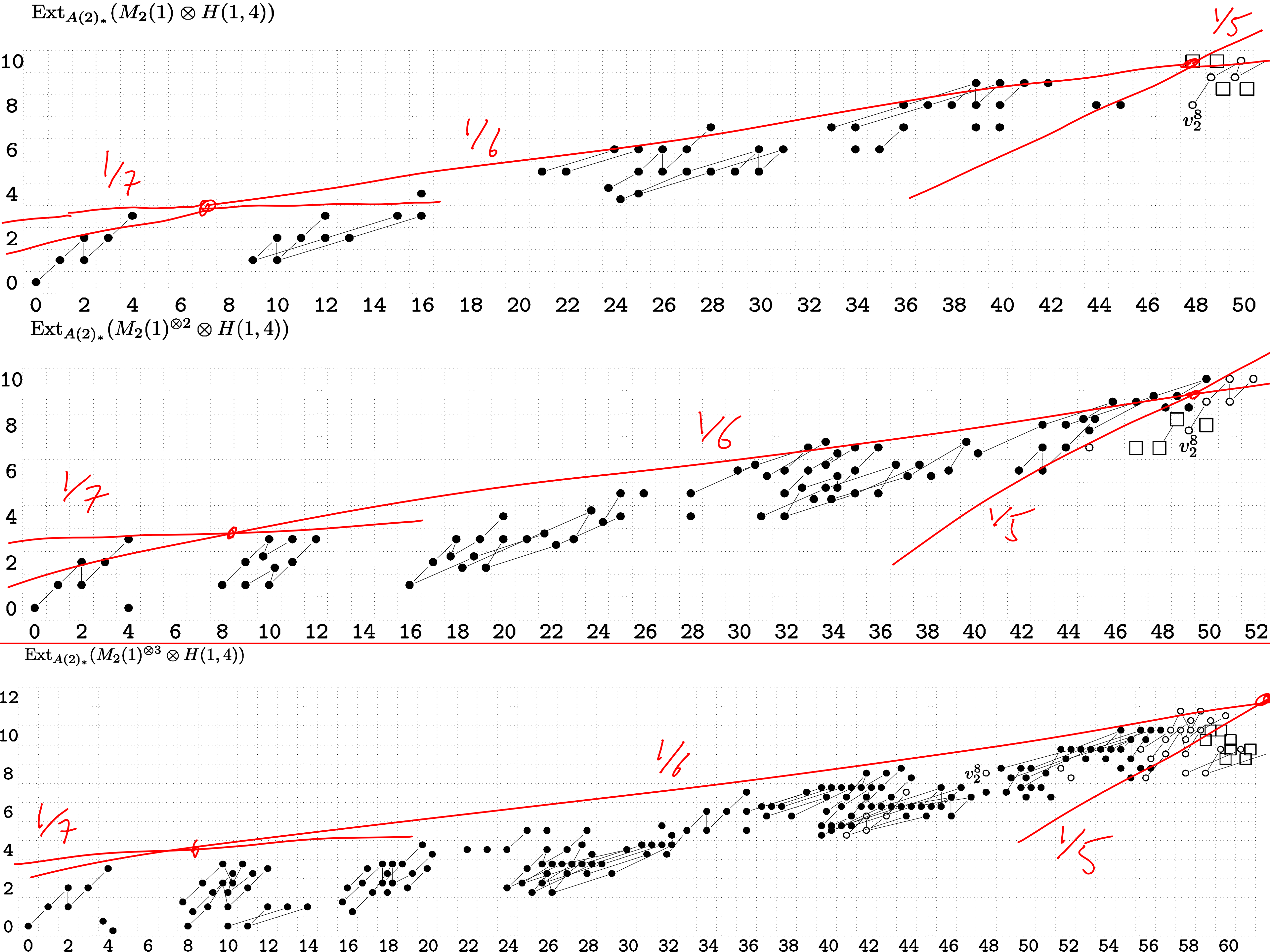
$$\mathrm{Ext}_{A(2)}(bo_1^{\wedge s} \wedge M(1,4)), \qquad s \leq 3$$

Sketch proof of "vanishing lemma":

Step 1: Establish vanishing polygons for

$$\mathrm{Ext}_{A(2)}(bo_1^{\wedge s} \wedge M(1,4)) \quad s \leq 3$$





$\mathrm{Ext}_{A(2)_*}(M_2(1) \otimes H(1,4))$
1/5
1/6
1/7
v_2^8
$\mathrm{Ext}_{A(2)_*}(M_2(1)^{\otimes 2} \otimes H(1,4))$
1/6
1/7
1/5
v_2^8
$\mathrm{Ext}_{A(2)_*}(M_2(1)^{\otimes 3} \otimes H(1,4))$
1/6
1/7
1/5
v_2^8

Sketch proof of "vanishing lemma", cont'd

Step 2: Inductively establish vanishing polygons for $\mathrm{Ext}_{A(2)}(bo_{i_1} \wedge \cdots \wedge bo_{i_s} \wedge M(1,4))$

Using exact sequences:

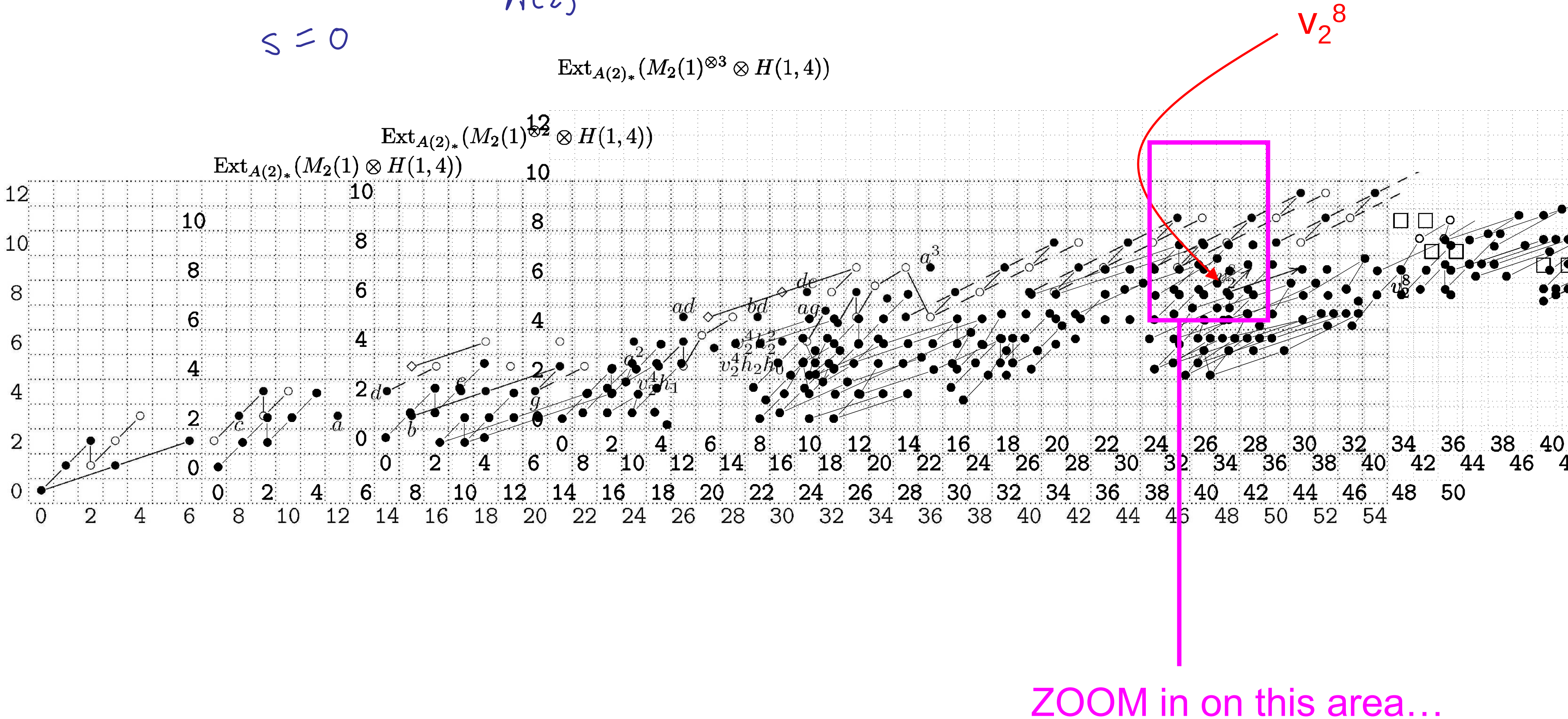
$$0 \to tmf_{j-1} \otimes \left(A(2)/\!/A(1)\right)_* \to bo_{2j+1} \to \Sigma^{8j} bo_j \otimes bo_1 \to 0$$

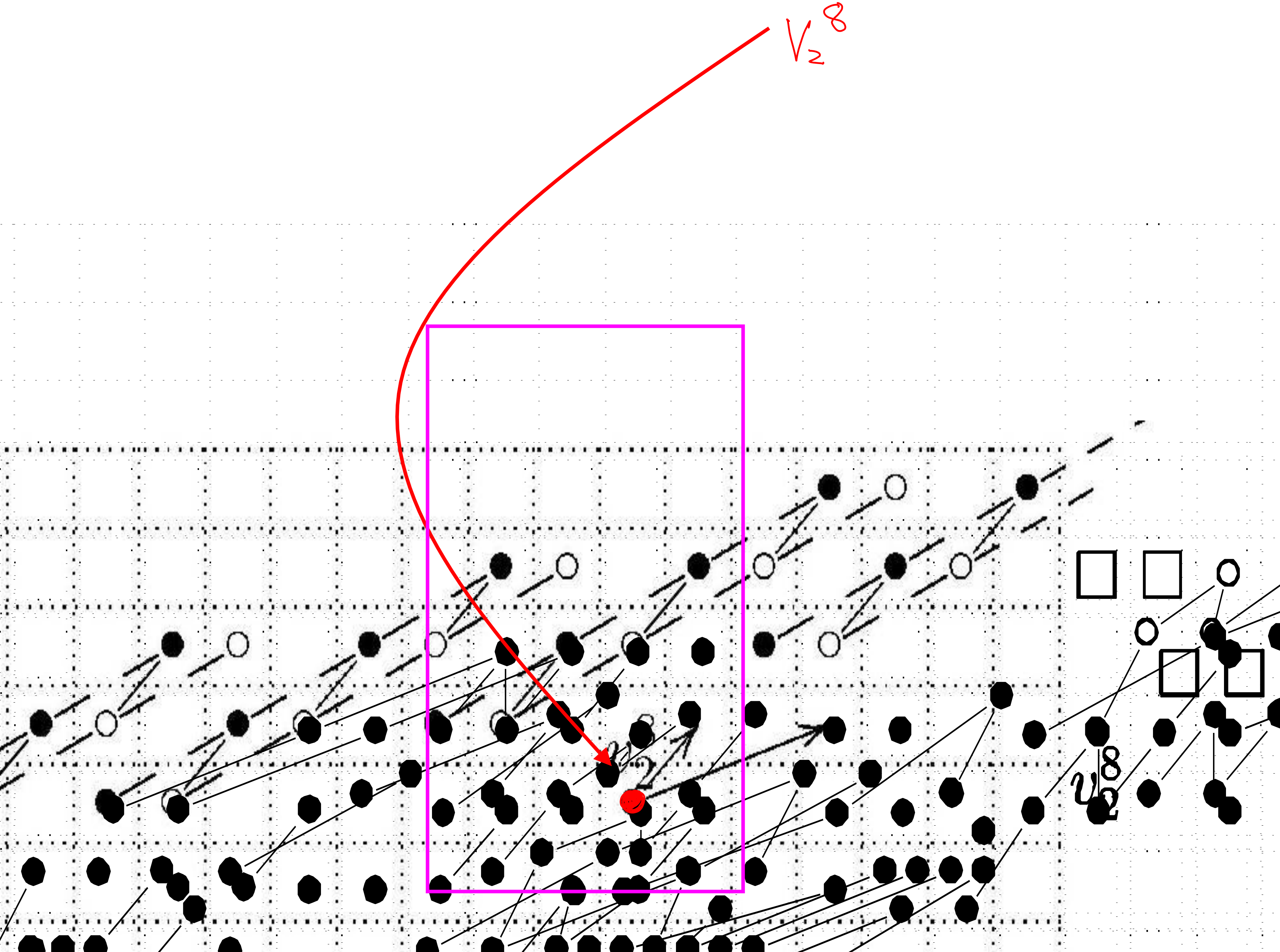
$$0 \to \Sigma^{8j} bo_j \to bo_{2j} \to tmf_{j-1} \otimes \left(A(2)/\!/A(1)\right)_* \to \Sigma^{8j+9} bo_{j-1} \to 0$$

The vanishing lemma follows from these vanishing polygons. $\square$

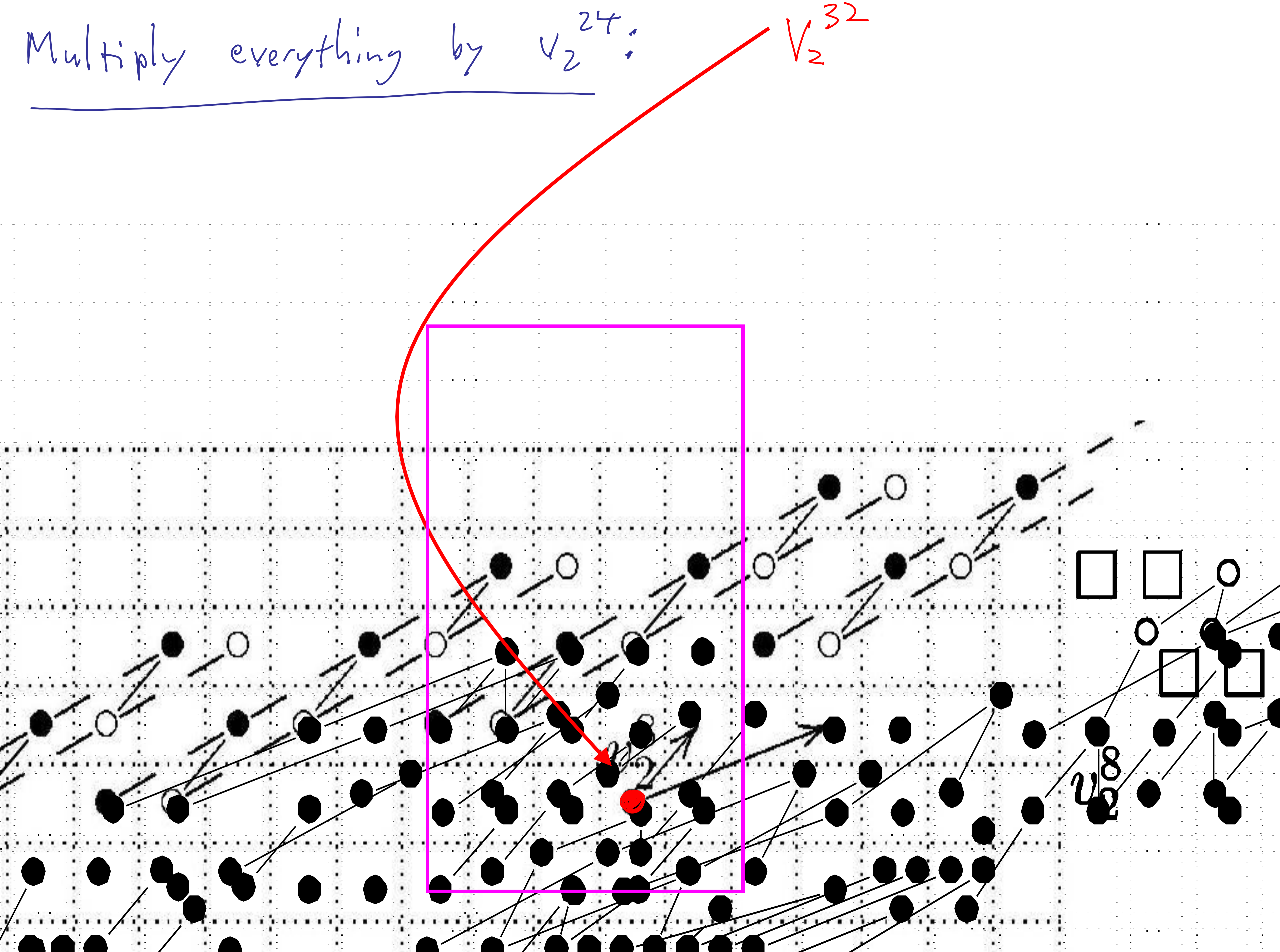
Part of the E_1-term of the Alg tmf Res'n:

$$\bigoplus_{s=0}^{3} \operatorname{Ext}_{A(2)}\left(\underline{bo}_1^{\wedge s} \wedge M(1,4)\right)$$




V_2^8
v_2^8

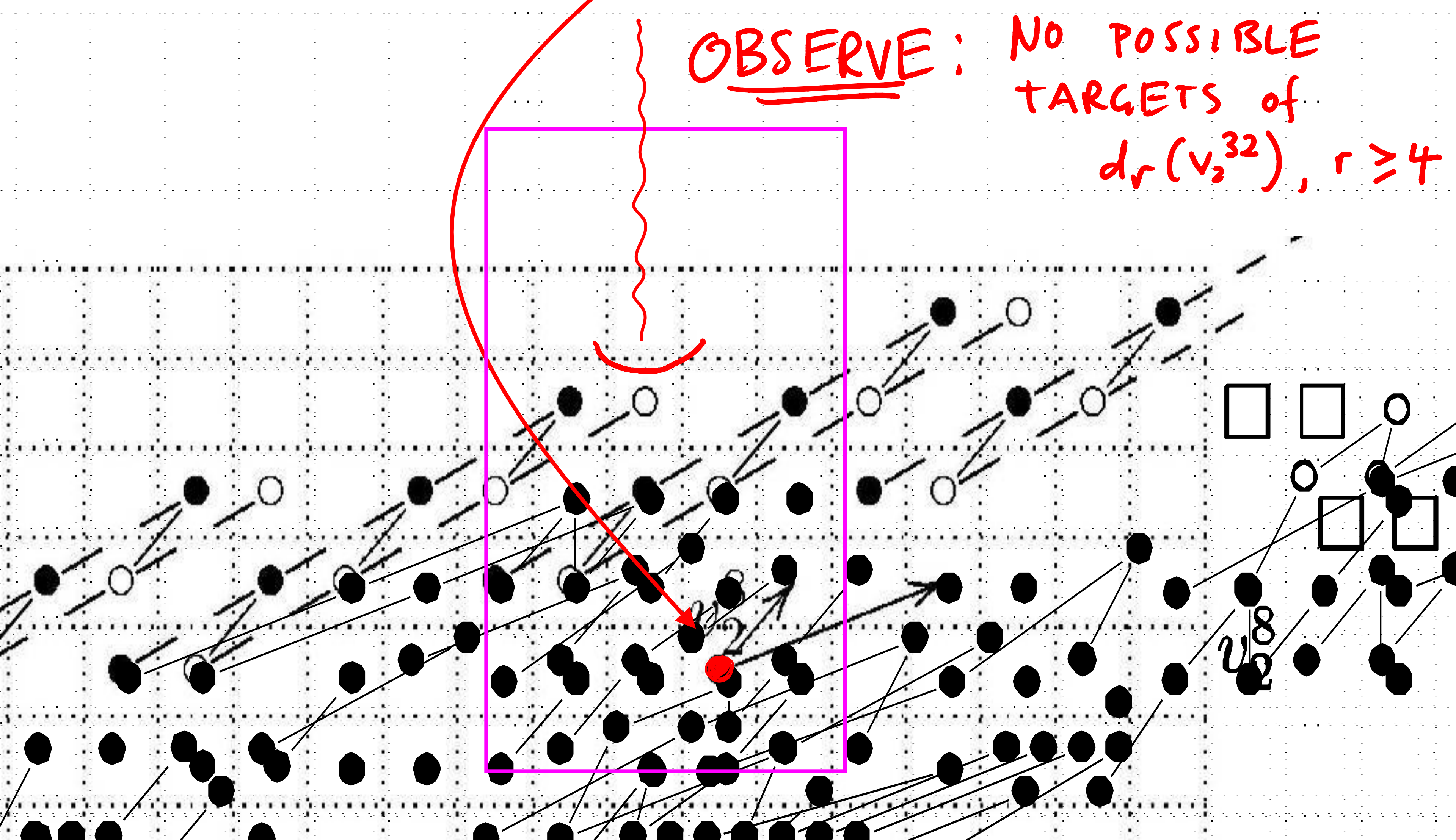
Multiply everything by v_2^{24}:



Multiply everything by v_2^{24}:

V_2^{32}

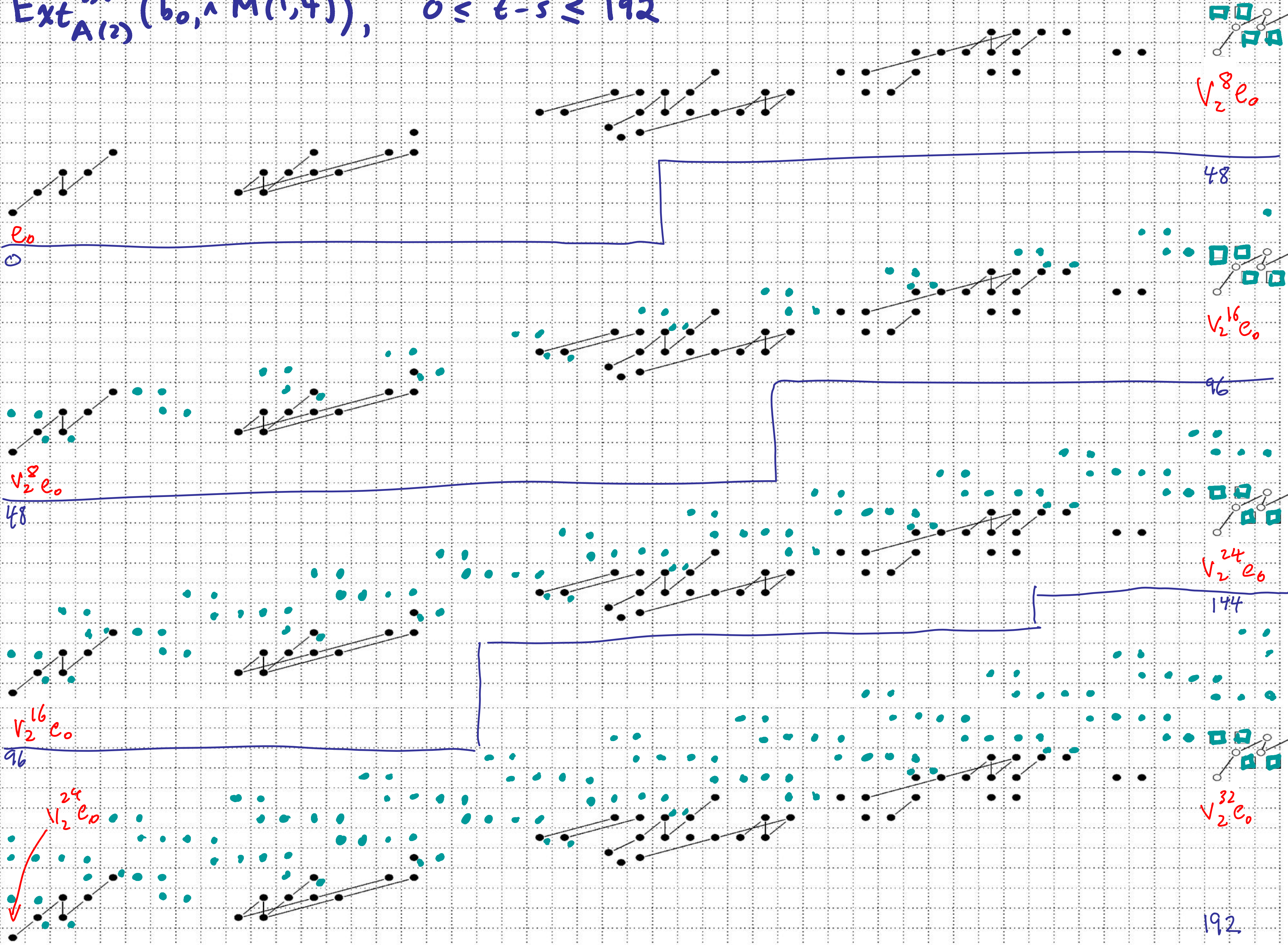
OBSERVE: NO POSSIBLE TARGETS of $d_r(v_2^{32})$, $r \geq 4$



So, we're done, right?!

NO!! We ignored the terms coming from $h_{2,1}$ multiplication!

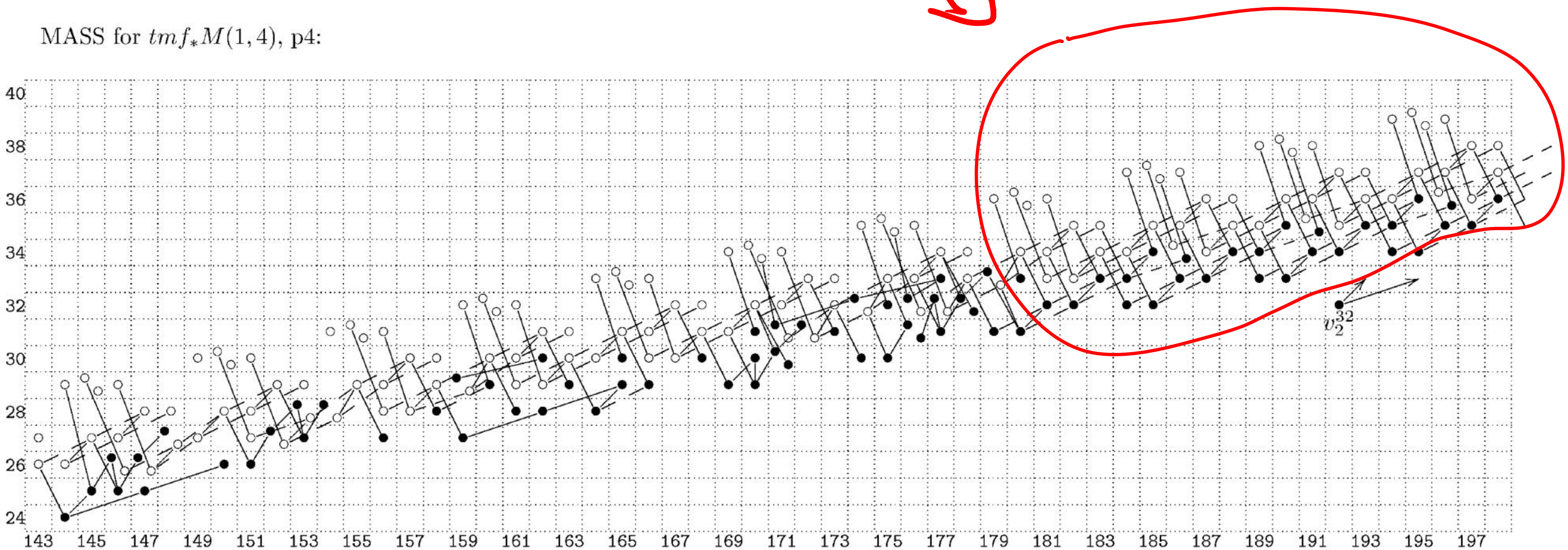
$\mathrm{Ext}_{A(2)}^{s,t}(bo \wedge M(1,4))$, $0 \leq t-s \leq 192$



Recall, in the ASS for $tmf \wedge M(1,4)$:

all h_{21}-towers annihilate each other near $t-s=192$

MASS for $tmf_* M(1,4)$, p4:



$\Rightarrow$ Same happens for $tmf \wedge bo_1^{\wedge s} \wedge M(1,4)$

CAVEAT:

$$tmf \wedge tmf \not\simeq \bigvee_{j \geq 0} \Sigma^{8j} tmf \wedge bo_j$$

$\Rightarrow$ We cannot directly deduce differentials amongst $h_{2,1}$-multiples in

$$\mathrm{Ext}_A(M(1,4)) \Rightarrow \pi_* M(1,4)$$

from diff'ls in

$$\mathrm{Ext}_{A(2)}(\widehat{bo}_1^{\,s} \wedge M(1,4)) \Rightarrow tmf_*(\widehat{bo}_1^{\,s} \wedge M(1,4))$$

Instead, to complete the proof that

v_2^{32} is a permanent cycle in

$$\mathrm{Ext}_A(M(1,4)) \Longrightarrow \pi_* M(1,4)$$

We explicitly deduce adams diff'ls amongst $h_{2,1}$-multiples using: $h_{21}^4 = \bar{\kappa}$

Prop: (1) $\bar{\kappa} \in \pi_{20}(S)$ lifts to $\tilde{\kappa} : \Sigma^{20} M(1,4) \to M(1,4)$

(2) In ASS for $M(1,4) \wedge DM(1,4)$

$\bar{\kappa}^6$ is killed by a d_3-diff'l.

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