Homework set # 1

Due on 1/23

1. Prove that if R with size function σ is a Euclidean domain and $a \in R$ is a non unit with least σ-value among all non-units then the quotient ring $R/(a)$ is represented by 0 and units.

2. Let $\sigma : \mathbb{Z}[\sqrt{-n}] \to \mathbb{Z}_{\geq 0}$ (where $n \in \mathbb{N}$) be the function that takes $a + b\sqrt{-n}$ to the integer $a^2 + nb^2$.

(1) Show that $\sigma(\alpha\beta) = \sigma(\alpha)\sigma(\beta)$ for $\alpha, \beta \in \mathbb{Z}[\sqrt{-n}]$.

(2) Observe (using high school geometry and the number line) that for any rational number x that you can find an integer X such that the distance between x and X is less than or equal to $\frac{1}{2}$.

(3) Now fix $n = 2$. One can extend σ to be a map defined on $\mathbb{Q}[\sqrt{-2}] \to \mathbb{Q}_{\geq 0}$. Show that if $\frac{\alpha}{\beta} = a + b\sqrt{-2}$ for $a, b \in \mathbb{Q}$ and $\gamma \in \mathbb{Z}[\sqrt{-2}]$ then $\sigma(\frac{\alpha}{\beta} - \gamma) < 1$.

(4) Using the previous part, show that if $\alpha, \beta \in \mathbb{Z}[\sqrt{-2}]$ then there exists a $\gamma, \delta \in \mathbb{Z}[\sqrt{-2}]$ such that $\alpha = \gamma\beta + \delta$ where $\sigma(\delta) < \sigma(\beta)$ or $\delta = 0$. (In other words show that with this σ that $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain.)

(5) Prove that $\mathbb{Z}[\sqrt{-2}]$ is a UFD (note this step is trivial given earlier parts).

3. Does the proof in number 2. not work for $\mathbb{Z}[\sqrt{-3}]$? (Quick test, try to see if $\mathbb{Z}[\sqrt{-3}]$ contains an element with a non-unique factorization).

4. In a ring $\mathbb{Z}[\sqrt{-n}]$ for any $n \in \mathbb{N}$ let σ be the function $\sigma(a + b\sqrt{n}) = a^2 + nb^2$ (as in problem 2.). Prove that the units of $\mathbb{Z}[\sqrt{n}]$ are the elements α where $\sigma(\alpha) = 1$ (i.e. $\alpha \in \mathbb{Z}[\sqrt{-n}]$ is a unit if and only if $\sigma(\alpha) = 1$). Conclude that in $\mathbb{Z}[\sqrt{-2}]$ the only units are $1, -1$ and that in $\mathbb{Z}[\sqrt{-1}]$ the units are $1, -1, i, -i$.