Homework set # 5

Due on 2/20

0. The following problems from Artin "Algebra" edition 2: 15.8.1; 16.1.1 parts a,b,c

1. (1) Let $\phi : F \to F'$ be an isomorphism of fields. Let $f(x) \in F[x]$ be a polynomial and let $f'(x) = \phi(f(x))$ (here we are just applying ϕ to the coefficients of $f(x)$). Let E be a splitting field for $f(x)$ over F and let E' be a splitting field for $f'(x)$ over F'. Prove that the isomorphism ϕ extends to an isomorphism $\sigma : E \to E'$ (so in other words that σ restricted to F is just ϕ). (Hint: first consider what happens when you adjoin one root of $f(x)$ to F and one root of $f'(x)$ to F', it might also be helpful to think of adjoining one root as a quotient of the polynomial ring).

(2) Using the first part, prove that any two splitting fields of a polynomial $f(x) \in F[x]$ over a field F are isomorphic.

2. (1) For every non constant monic polynomial $f \in F[x]$ where F is a field, let x_f denote a new variable in the polynomial ring $R_f = F[\ldots, x_f, \ldots]$ (i.e. there will be infinitely many variables in this new polynomial ring). Now let I be the ideal in R_f generated by the polynomials $f(x_f)$. Prove that I is a proper ideal (i.e. that $I \neq R_f$). (Hint: If it were proper then $1 \in I$ meaning that there would be a relation $g_1 f_1(x_{f_1}) + \cdots g_n f_n(x_{f_n}) = 1$ among finitely many of the $f(x_f)$'s. Now what would happen to this relation if you set each x_{f_i} equal to a root of f_i in some extension field of F and set the remaining variables showing up in the g_i's to 0?).

(2) Observe that if I is not equal to R_f then I is contained in some maximal ideal M of R_f. Prove that the field R_f/M contains a root of every non constant monic polynomial $f \in F[x]$.

(3) Using the above work, prove that for any field F there exists an algebraically closed field K containing F. (Hint: It might be useful to use the fact that a union of fields is a field (even a countable union)).