```
> restart;
```

Problem 3.1 Note I made a bad choice using cos (x) (error unreasonably small) due to the symmetry.

```
> with(student):
> a:=0;
  b:=Pi;
```

$$a := 0$$

$$b := \pi \qquad (1)$$

```
> #Digits:=40;
> f := x -> cos(x);
```

$$f := x \to \cos(x) \qquad (2)$$

```
> for j from 1 to 10 do
  print(j,evalf(( trapezoid(f(x),x=a..b,j)-trapezoid(f(x),x=a..b,2*j))/3),
  evalf(int(f(x),x=a..b)-trapezoid(f(x),x=a..b,2*j)));
  od;
```

$$1,\ 0.,\ -0.$$
$$2,\ 0.,\ -0.$$
$$3,\ 0.,\ -0.$$
$$4,\ 1.308996939\ 10^{-12},\ 3.926990818\ 10^{-12}$$
$$5,\ 0.,\ -1.256637062\ 10^{-11}$$
$$6,\ -5.235987756\ 10^{-13},\ -1.570796327\ 10^{-12}$$
$$7,\ 7.105983386\ 10^{-12},\ 3.365992129\ 10^{-12}$$
$$8,\ -1.701696021\ 10^{-12},\ -1.178097245\ 10^{-12}$$
$$9,\ -1.279908118\ 10^{-12},\ -3.490658505\ 10^{-13}$$
$$10,\ 4.188790206\ 10^{-12},\ -0. \qquad (3)$$

```
> f := x -> arctan(x);#a better choice
```

$$f := x \to \arctan(x) \qquad (4)$$

```
> for j from 1 to 10 do
  print(j,evalf(( trapezoid(f(x),x=a..b,j)-trapezoid(f(x),x=a..b,2*j))/3),
  evalf(int(f(x),x=a..b)-trapezoid(f(x),x=a..b,2*j)));
  od;
```

$$1,\ -0.1950778211,\ 0.205111638$$
$$2,\ -0.0523593864,\ 0.048033479$$
$$3,\ -0.0221759559,\ 0.020979739$$
$$4,\ -0.0120979433,\ 0.011739649$$
$$5,\ -0.0076299708,\ 0.007496492$$
$$6,\ -0.0052600052,\ 0.005199723$$
$$7,\ -0.0038488215,\ 0.003817512$$
$$8,\ -0.0029393976,\ 0.002921457$$
$$9,\ -0.0023186390,\ 0.002307596$$
$$10,\ -0.0018759167,\ 0.001868740 \qquad (5)$$

Problem 3.2 -- note simpson in the student package counts n for (f(a)+4 f((a+b)/2) +f(b))*(b=a)/6 as 2 instead of 1, so we must adjust the loop.

```
> f := x -> cos(x);
```

$$f := x \to \cos(x) \qquad (6)$$

```
> for j from 1 to 10 do
  print(j,evalf(( simpson(f(x),x=a..b,2*j)-simpson(f(x),x=a..b,4*j))/15),
```

```
evalf(int(f(x),x=a..b)-simpson(f(x),x=a..b,4*j)));
od;
```

$$1, 0., -0.$$
$$2, -1.047197551\ 10^{-12}, -1.570796327\ 10^{-11}$$
$$3, -2.327105671\ 10^{-13}, -3.490658506\ 10^{-12}$$
$$4, 1.134464014\ 10^{-12}, 1.308996939\ 10^{-12}$$
$$5, 1.396263401\ 10^{-13}, -6.283185310\ 10^{-12}$$
$$6, 2.559816238\ 10^{-13}, 3.490658504\ 10^{-13}$$
$$7, -4.138923655\ 10^{-13}, 2.767593527\ 10^{-12}$$
$$8, -7.853981636\ 10^{-14}, 1.308996940\ 10^{-13}$$
$$9, -3.102807559\ 10^{-13}, -1.163552835\ 10^{-12}$$
$$10, 4.258603375\ 10^{-13}, 1.047197552\ 10^{-13} \quad (7)$$

a better choice of function

```
> f := x -> arctan(x);
```

$$f := x \rightarrow \arctan(x) \quad (8)$$

```
> for j from 1 to 10 do
print(j,evalf(( simpson(f(x),x=a..b,2*j)-simpson(f(x),x=a..b,4*j))/15),
evalf(int(f(x),x=a..b)-simpson(f(x),x=a..b,4*j)));
od;
```

$$1, -0.0009573151, -0.004325908$$
$$2, 0.0002645076, -0.000358294$$
$$3, 0.0000757289, -0.000060282$$
$$4, 0.0000226901, -0.000017942$$
$$5, 0.0000084201, -0.000007176$$
$$6, 0.0000037909, -0.000003420$$
$$7, 0.0000019651, -0.000001833$$
$$8, 0.0000011248, -0.000001071$$
$$9, 6.917\ 10^{-7}, -6.65\ 10^{-7}$$
$$10, 4.494\ 10^{-7}, -4.34\ 10^{-7} \quad (9)$$

```
> restart;
```

Problem 3.3

```
> with(CurveFitting);
```

[*ArrayInterpolation*, *BSpline*, *BSplineCurve*, *Interactive*, *LeastSquares*, *PolynomialInterpolation*, *RationalInterpolation*, *Spline*, *ThieleInterpolation*] **(10)**

```
> t:=Vector(4):
y:=Vector(4):
g0:= h-> f0+h*df0;
g1:= h-> f1+h*df1;
```

$$g0 := h \rightarrow f0 + h\, df0$$
$$g1 := h \rightarrow f1 + h\, df1 \quad (11)$$

```
> t[1]:= a+h1;
t[2]:= a+h2;
t[3]:= b+h3;
t[4]:= b+h4;
```

$$t_1 := a + h1$$
$$t_2 := a + h2$$
$$t_3 := b + h3$$

$$t_4 := b + h4 \quad (12)$$

```
> for j from 1 to 2 do y[j]:= g0(h||j); od;
  for j from 3 to 4 do y[j]:= g1(h||j); od;
```

$$y_1 := f0 + h1\, df0$$

$$y_2 := f0 + h2\, df0$$

$$y_3 := f1 + h3\, df1$$

$$y_4 := f1 + h4\, df1 \quad (13)$$

```
> limit(limit(limit(limit(PolynomialInterpolation(t,y,x),h4=0),h3=0),h2=0),
  h1=0);
```

$$-\frac{1}{(-b+a)^3}\left(a^2 b^2 df0 - a^2 b^2 df1 + 3 a^2 b f1 - 3 a b^2 f0 - 2 a^2 b\, df0\, x - a^2 b\, df1\, x - 2 b^2 df0\, x^2 + b\, df0\, x^3 + b\, df1\, x^3 - 3 b f0 x^2 + 3 b f1 x^2 - b^2 df1\, x^2 + b^3 df0\, x + a^3 b\, df1 - a^3 df1\, x + 2 a^2 df1\, x^2 + a^2 df0\, x^2 - 3 a f0 x^2 + 3 a f1 x^2 - a\, df0\, x^3 - a\, df1\, x^3 - a\, df0\, b^3 + b^3 f0 + 2 f0 x^3 - 2 f1 x^3 - a^3 f1 - 6 a b f1 x + 6 a b f0 x - a b\, df1\, x^2 + 2 a b^2 df1\, x + a b^2 df0\, x + a b\, df0\, x^2\right) \quad (14)$$

```
> P:= unapply(%,f0,df0,f1,df1,x);
```

$$P := (f0, df0, f1, df1, x) \rightarrow -\frac{1}{(-b+a)^3}\left(a^2 b^2 df0 + 3 a f1 x^2 + 2 a^2 df1\, x^2 - a\, df0\, b^3 - a\, df0\, x^3 - a^2 b^2 df1 - a^3 df1\, x - 3 b f0 x^2 + 3 b f1 x^2 - b^2 df1\, x^2 - 2 b^2 df0\, x^2 + b^3 df0\, x + b\, df1\, x^3 - 3 a b^2 f0 + 3 a^2 b f1 - a\, df1\, x^3 + b\, df0\, x^3 - 3 a f0 x^2 + a^3 b\, df1 + a^2 df0\, x^2 + b^3 f0 + 2 f0 x^3 - 2 f1 x^3 - a^3 f1 - 2 a^2 b\, df0\, x - a^2 b\, df1\, x - 6 a b f1 x + 6 a b f0 x - a b\, df1\, x^2 + 2 a b^2 df1\, x + a b^2 df0\, x + a b\, df0\, x^2\right) \quad (15)$$

```
> int(P(f0,df0,f1,df1,x),x=a..b);
```

$$-\frac{1}{4}\,\frac{(-2 f1 + df0\, b + b\, df1 + 2 f0 - df0\, a - a\, df1)(b^4 - a^4)}{(-b+a)^3} - \frac{1}{3}\,\frac{1}{(-b+a)^3}\left((-a b\, df1 + 3 f1 b - 2\, df0\, b^2 - b^2 df1 - 3 f0 b + df0\, b a + 2 a^2 df1 + df0\, a^2 - 3 f0 a + 3 f1 a)(b^3 - a^3)\right) - \frac{1}{2}\,\frac{(-6 f1 b a + 6 f0 b a - 2\, df0\, b a^2 - a^2 b\, df1 + df0\, b^2 a - a^3 df1 + 2 a b^2 df1 + df0\, b^3)(b^2 - a^2)}{(-b+a)^3} - \frac{(a^2 b^2 df0 - a^2 b^2 df1 + 3 a^2 b f1 - 3 a b^2 f0 + a^3 b\, df1 - a\, df0\, b^3 + b^3 f0 - a^3 f1)(b - a)}{(-b+a)^3} \quad (16)$$

```
> simplify(%);
```

$$\frac{1}{12} df0\, a^2 - \frac{1}{12} a^2 df1 - \frac{1}{6} df0\, b a + \frac{1}{6} a b\, df1 - \frac{1}{2} f1 a - \frac{1}{2} f0 a + \frac{1}{2} f1 b + \frac{1}{12} df0\, b^2 + \frac{1}{2} f0 b - \frac{1}{12} b^2 df1 \quad (17)$$

```
> factor(%);
```

$$\frac{1}{12}(-b+a)(df0\, a - a\, df1 - df0\, b + b\, df1 - 6 f1 - 6 f0) \quad (18)$$

Let us see what this looks like setting b-a=h

```
> subs(b=a+h,%);
```

$$-\frac{1}{12} h\,(df0\, a - a\, df1 - df0\,(a+h) + (a+h)\, df1 - 6 f1 - 6 f0) \quad (19)$$

```
> simplify(%);
```

$$\frac{1}{12} h\,(df0\, h - df1\, h + 6 f1 + 6 f0) \quad (20)$$

The error is bounded by the max of the 4th derivative of f on [a,b} divided by 4! times

```
> int((x-a)^2*(x-b)^2,x=a..b);
```

$$\frac{1}{5}b^5-\frac{1}{5}a^5+\frac{1}{4}(-2a-2b)(b^4-a^4)+\frac{1}{3}(a^2+4ba+b^2)(b^3-a^3)+\frac{1}{2}(-2ba^2-2b^2a)(b^2-a^2)+b^2a^2(b-a) \quad \mathbf{(21)}$$

```
> factor(simplify(%));
```

$$-\frac{1}{30}(-b+a)^5 \quad \mathbf{(22)}$$