Homework 9 (Due Thursday, April 9)
Pg. 421: 1a, 2, 3, 4.
Pg. 438: 1, 2, 3, 7.

pg 421 # 1a)
$$e^{\varepsilon} = 1 + \varepsilon + \frac{\varepsilon^2}{2!} + \cdots$$

ε^k is an asymptotic sequence

$$\frac{\varepsilon^{k+1}}{\varepsilon^k} \to 0 \quad \text{with } \varepsilon \to 0$$

1b)
$$e^{-\frac{1}{\varepsilon}} = 1 - \frac{1}{\varepsilon} + \frac{1}{\varepsilon^2 2!} - \frac{1}{\varepsilon^3 3!} \cdots$$

is not an asymptotic expansion

$$\frac{\frac{1}{\varepsilon^{k+1}}}{\frac{1}{\varepsilon^k}} \not\to 0 \quad \text{as } \varepsilon \to 0$$

if $\underset{\varepsilon \to 0}{\varepsilon < 0}$ $\quad e^{-\frac{1}{\varepsilon}} \to \infty$ so you have no asymp. expansion.

if $\underset{\varepsilon \to 0}{\varepsilon > 0}$ $\quad e^{-\frac{1}{\varepsilon}} \to 0$.

$$e^{-\frac{1}{\varepsilon}} - 0 \approx \alpha \varepsilon + O(\varepsilon^2) \Rightarrow \alpha = 0$$

and so on. So $e^{-\frac{1}{\varepsilon}} \sim 0$ in powers of $\varepsilon > 0$.

②

#2

$$\frac{1}{1+x} = \frac{1}{x}\,\frac{1}{1+\frac{1}{x}} = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} \cdots$$

$$\frac{\left(\frac{1}{x}\right)^{k+1}}{\left(\frac{1}{x}\right)^{k}} \to 0 \text{ as } x \to \infty \text{ so this}$$

is an asymptotic expansion

$$\text{if } \left(1+\frac{1}{e^x}\right)\left(\frac{1}{1+x}\right) = \sum_{j=0}^{N} \frac{a_j}{x^j} + O\left(\frac{1}{x^{N+1}}\right)$$

all N

~~Note~~

then the claim is $a_j = (-1)^{j+1}$, $j \geq 1$

Note $a_0 = 0$ since LHS $\to 0$

$$\left(1+\frac{1}{e^x}\right)\left(\frac{1}{1+x}\right) - \frac{1}{x}\left(\frac{1-\frac{1}{x}^{N}}{1+\frac{1}{x}}\right) = O\left(\frac{1}{x^{N+1}}\right) + \sum^{N}\left(a_j - (-1)^{j+1}\right)\frac{1}{x^j}$$

$$\| $$

$$\frac{1}{x^{N}(1+x)} + e^{-x}\left(\frac{1}{1+x}\right) = O\left(\frac{1}{x^{N+1}}\right)$$

$$\Rightarrow a_j = (-1)^{j+1}$$

3 (a) $\displaystyle\int_0^1 \frac{\sin(kt)}{t}\,dt =$

$$\int_0^1 \left(k - \frac{k^3t^2}{3!} + \frac{k^5t^4}{5!} \cdots\right) dt + O(k^{N+2}t^{N+1})\,dt$$

$$= k - \frac{k^3}{3\cdot 3!} + \frac{k^5}{5\cdot 5!} \quad \cdots \quad O(k^{N+1})$$

and N

(b) $\displaystyle\int_0^k t^{-\frac{1}{4}} e^{-t}\,dt \qquad k \to 0^+$

$$\parallel$$

$t = ku$

$$k^{3/4}\int_0^1 u^{-\frac{1}{4}} e^{-ku}\,du$$

$$= k^{3/4}\int_0^1 u^{-\frac{1}{4}}\left(1 - ku \cdots + O(k^{N+1}u^{N+1})\right)du$$

$$= k^{3/4}\left(\frac{4}{3} - \frac{4}{7}k + \frac{4}{11}\frac{k^2}{2!} \cdots\right.$$

(c) $\int_k^\infty e^{-t^4}\,dt = \int_0^\infty e^{-t^4}\,dt - \int_0^k e^{-t^4}\,dt$ (4)

$$\int_0^\infty e^{-t^4}\,dt \underset{\substack{\uparrow \\ \text{let } t^4 = u}}{=} \int_0^\infty e^{-u} \frac{du}{4u^{3/4}} = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)$$

So Not $\int_0^k e^{-t^4}\,dt \underset{\substack{\uparrow \\ t = uk}}{=} k\int_0^1 e^{-t^4 u^4}\,du$

$$= k\left(1 - \frac{k^4}{5} + \frac{k^8}{2!\,9} - \frac{k^{12}}{3!\,13} + \cdots O(k^{N+1})\right.$$

So asympt. expn is

$$\frac{1}{4}\Gamma\left(\frac{1}{4}\right) - k + \frac{k^5}{5} - \frac{k^9}{2!\,9} + \frac{k^{13}}{3!\,13}$$

4

$$\left|\int_k^\infty \frac{e^{-t^2}}{2t^2}\,dt\right| = \left|\int_k^\infty \frac{(2t\,e^{-t^2})}{4t^3}\,dt\right|$$

$$\le \frac{1}{4k^3}\int_k^\infty 2t\,e^{-t^2}\,dt = \frac{e^{-k^2}}{4k^3}$$

since $t > k$, $\frac{1}{t^3} < \frac{1}{k^3}$

pg 438

1) $$\int_1^4 e^{-kt}\sin(t)\,dt =$$

$$\frac{\sin(1)}{k}e^{-k} + \frac{e^{-k}}{k^2}\cos(1) + O\left(\frac{1}{k^3}\right)$$

$$\int_1^4 e^{-kt}\,\frac{e^{it}-e^{-it}}{2i}\,dt = \int_1^4 \frac{e^{-(k-i)t}}{2i}\,dt$$

$$-\int \frac{e^{-(k+i)t}}{2i}\,dt = \frac{e^{-(k-i)4}-e^{-(k-i)}}{2i\,(i-k)}$$

$$+\left(\frac{e^{-(k+i)4}-e^{-(k+i)1}}{2i\,(k+i)}\right) =$$

Exact $\nearrow$

$$\frac{-1}{k^2+1}\left(e^{-4k}\cos(4)-e^{-k}\cos(1)+ke^{-4k}\sin(4) - k\,e^{-k}\sin(1)\right)$$

which agrees with the answer on the graphs

7

```
> #assume(k>0);
  f:= k-> -(exp(-4*k)*cos(4)-exp(-k)*cos(1)+k*exp(-4*k)*sin(4)-k*exp(-k)*sin
  (1))/(k^2+1);
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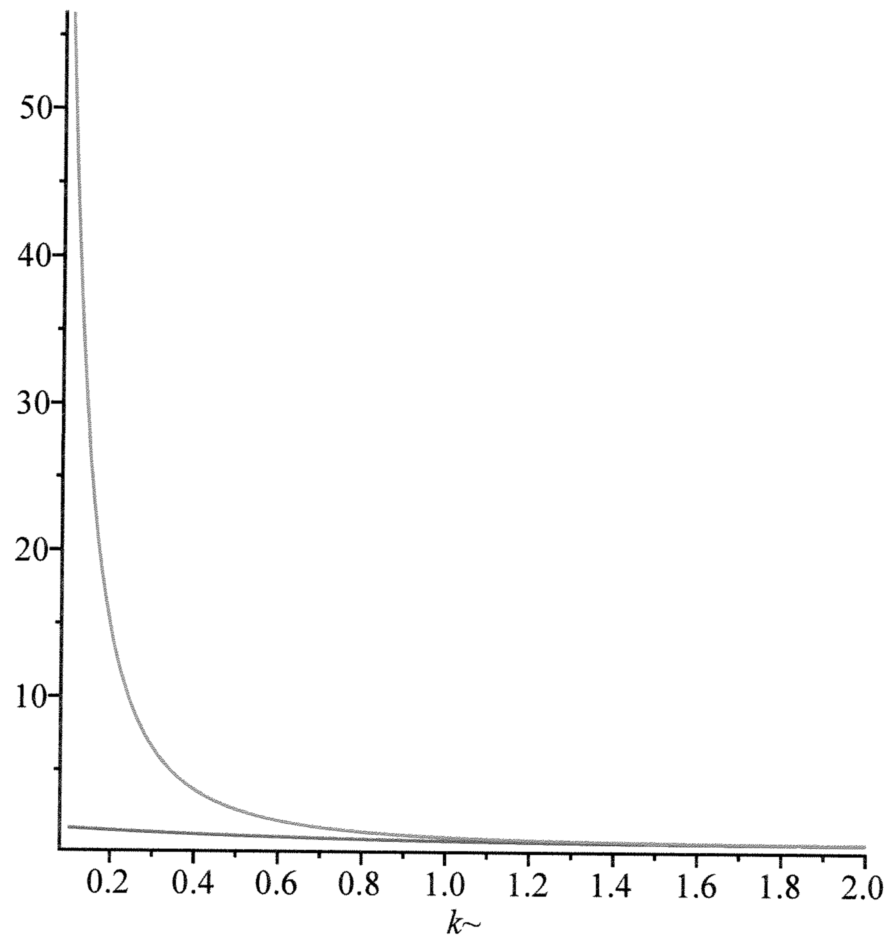
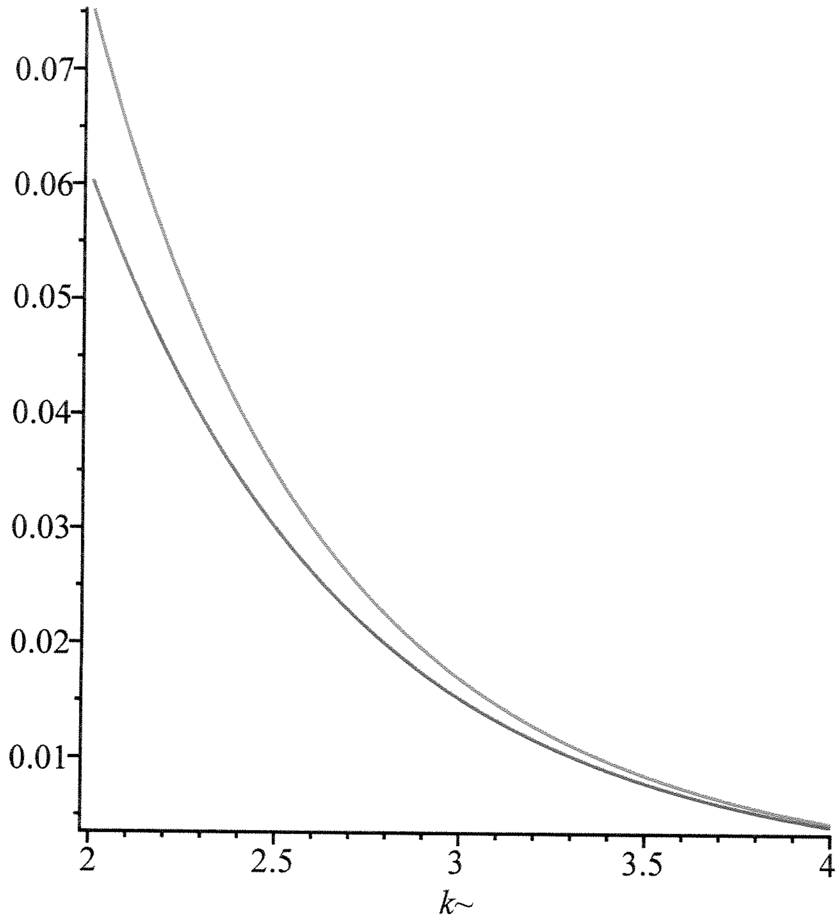
$$f := k \to -\frac{e^{-4k}\cos(4) - e^{-k}\cos(1) + k\,e^{-4k}\sin(4) - k\,e^{-k}\sin(1)}{k^2+1} \tag{1}$$

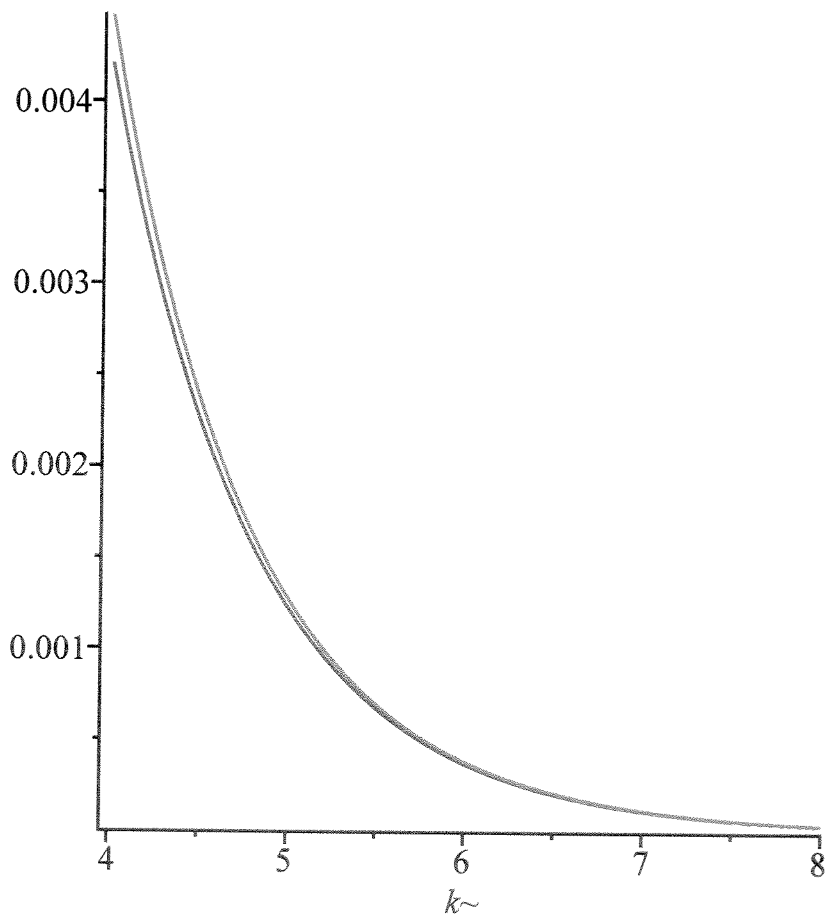
```
> plot({f(k),(sin(1)*exp(-k)/k+cos(1)*exp(-k)/k^2)},k=0.1..2);
```




```
> plot({f(k),(sin(1)*exp(-k)/k+cos(1)*exp(-k)/k^2)},k=2..4);
```




```
> plot({f(k),(sin(1)*exp(-k)/k+cos(1)*exp(-k)/k^2)},k=4..8);
```



1b) $$\int_5^9 \frac{e^{-kt}}{t}\,dt \sim$$

$$\frac{e^{-5k}}{5k} - \frac{e^{-5k}}{k^2\,25}$$

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2) $u = t^2+1$ $\qquad dt = \dfrac{du}{2\sqrt{u-1}}$

$$\int_2^\infty \frac{e^{-ku}}{\sqrt{u-1}}\,du \sim$$

$$\frac{e^{-2k}}{1k} + \frac{1}{2}\,\frac{e^{-2k}}{k^2}$$

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3) $$t^{-\frac{1}{3}}\cos(t) = t^{-1/3} - \frac{t^{5/3}}{2!} + \frac{t^{11/3}}{4!}$$

so $\displaystyle\int_0^\pi \sim$

$$\sum_{m=0}^{\infty} \frac{\Gamma((6m+2)/3)}{k^{(6m+2)/3}(2m)!}(-1)^m$$

7 This is trickier than expected.

Maybe someone sees an easier (and sharper) approach?

Show

$$\left| \int_0^\infty \frac{e^{-kx}}{1+k^{\alpha x}x}\,dx - \frac{1}{k} \right| = O\left(\frac{1}{k^{1+\varepsilon}}\right)$$

$$\int_0^\infty \frac{e^{-kx}}{1+k^{\alpha}x}\,dx - \frac{1}{k} =$$

$$\int_0^\infty \frac{e^{-kx}}{1+k^{\alpha x}x}\,dx - \int_0^\infty e^{-kx}\,dx$$

$$= \int_0^\infty \frac{e^{-kx}\,x\,k^{\alpha x}}{1+x\,k^{\alpha x}}\,dx = \int_0^\infty \frac{e^{-kx}\,x}{k^{-\alpha x}+x}\,dx$$

$$\leq \int_0^\infty \frac{e^{-kx}\,x}{k^{-\alpha x}}\,dx = \int_0^\infty e^{-(k-(\ln k)\alpha)x}\,x\,dx$$

$$= \frac{1}{(k-\ln k)^2} \leq \frac{1}{k^{2-\varepsilon}}$$ for any $0<\varepsilon<1$ and k large enough