```
> restart;
```

Homework 11

```
> with(LinearAlgebra):
```

```
> Newton2D := proc(f,g,v)
  local J11, J12, J21, J22, A,B, w;
  with(LinearAlgebra);
  J11:=unapply(diff(f(x,y),x),x,y);
  J12:=unapply(diff(f(x,y),y),x,y);
  J21:=unapply(diff(g(x,y),x),x,y);
  J22:=unapply(diff(g(x,y),y),x,y);
  A := (x,y) -> Matrix([[J11(x,y),J12(x,y)],[J21(x,y),J22(x,y)]]);
  w:= Vector([f(v[1],v[2]),g(v[1],v[2])]);
  B:= MatrixInverse(A(v[1],v[2]));
  VectorAdd(v,-MatrixVectorMultiply(B,w));
  end proc:
```

```
> Newton3D := proc(f,g,h,v)
  local J11, J12, J13, J21, J22, J23, J31, J32, J33, A,B, w;
  with(LinearAlgebra);
  J11:=unapply(diff(f(x,y,z),x),x,y,z);
  J12:=unapply(diff(f(x,y,z),y),x,y,z);
  J13:=unapply(diff(f(x,y,z),z),x,y,z);
  J21:=unapply(diff(g(x,y,z),x),x,y,z);
  J22:=unapply(diff(g(x,y,z),y),x,y,z);
  J23:=unapply(diff(g(x,y,z),z),x,y,z);
  J31:=unapply(diff(h(x,y,z),x),x,y,z);
  J32:=unapply(diff(h(x,y,z),y),x,y,z);
  J33:=unapply(diff(h(x,y,z),z),x,y,z);
  A := (x,y,z) -> Matrix([[J11(x,y,z),J12(x,y,z),J13(x,y,z)],[J21
  (x,y,z),J22(x,y,z),J23(x,y,z)],[J31(x,y,z),J32(x,y,z),J33(x,y,z)]
  ]);
  w:= Vector([f(v[1],v[2],v[3]),g(v[1],v[2],v[3]),h(v[1],v[2],v[3])
  ]);
  B:= MatrixInverse(A(v[1],v[2],v[3]));
  VectorAdd(v,-MatrixVectorMultiply(B,w));
  end proc:
```

Problem 3a on page 680

```
> with(LinearAlgebra):
```

```
> Continue := proc(H1,H2,t_Start,v,h)
  local J11, J12, J21, J22, J, H1_t, H2_t, H_t, A,B, w, u;
  with(LinearAlgebra);
  J11:=unapply(diff(H1(x,y,t),x),x,y,t);
  J12:=unapply(diff(H1(x,y,t),y),x,y,t);
  J21:=unapply(diff(H2(x,y,t),x),x,y,t);
  J22:=unapply(diff(H2(x,y,t),y),x,y,t);
  H1_t:=unapply(diff(H1(x,y,t),t),x,y,t);
  H2_t:=unapply(diff(H2(x,y,t),t),x,y,t);
  J := (x,y) -> Matrix([[J11(x,y,t_Start),J12(x,y,t_Start)],[J21(x,
  y,t_Start),J22(x,y,t_Start)]]);
  H_t := (x,y) -> Vector([H1_t(x,y,t_Start),H2_t(x,y,t_Start)]);
  B:= MatrixInverse(J(v[1],v[2]));
  #print(B);
```

```
  w:= h*H_t(v[1],v[2]);
  #print(w);
  u:=VectorAdd(v,-MatrixVectorMultiply(B,w));
  end proc:
```

```
> vv:=Vector([0,0]);
  f1 := (x,y) -> 4.0*x^2 - 20.0*x + 0.25*y^2 + 8.0;
  f2 := (x,y) -> 0.5*x*y^2 + 2.0*x -5.0*y + 8.0;
  H1 := (x,y,t) -> t*f1(x,y)+(1-t)*(f1(x,y)-f1(vv[1],vv[2]));
  H2 := (x,y,t) -> t*f2(x,y)+(1-t)*(f2(x,y)-f2(vv[1],vv[2]));
```

$$vv := \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$f1 := (x, y) \to 4.0\,x^2 - 20.0\,x + 0.25\,y^2 + 8.0$$

$$f2 := (x, y) \to 0.5\,x\,y^2 + 2.0\,x - 5.0\,y + 8.0$$

$$H1 := (x, y, t) \to t\,f1(x, y) + (1 - t)\,\left(f1(x, y) - f1\left(vv_1, vv_2\right)\right)$$

$$H2 := (x, y, t) \to t\,f2(x, y) + (1 - t)\,\left(f2(x, y) - f2\left(vv_1, vv_2\right)\right) \qquad \textbf{(1)}$$

```
> N:=20;
  h:= 1.0/N;
  v:= Vector([0.0,0.0]);
  t_Start:=0;
  for j from 1 to N do
  v:=Continue(H1,H2,t_Start,v,h);
  t_Start:=t_Start+h;
  print(t_Start,v);
  od:
```

$$N := 20$$

$$h := 0.05000000000$$

$$v := \begin{bmatrix} 0. \\ 0. \end{bmatrix}$$

$$t\_Start := 0$$

$$0.05000000000, \begin{bmatrix} 0.0200000000000000 \\ 0.0880000000000000 \end{bmatrix}$$

$$0.1000000000, \begin{bmatrix} 0.0403568719989383 \\ 0.176189555884923 \end{bmatrix}$$

$$0.1500000000, \begin{bmatrix} 0.0610811495364354 \\ 0.264669427282326 \end{bmatrix}$$

$$0.2000000000, \begin{bmatrix} 0.0821848379774599 \\ 0.353546095401992 \end{bmatrix}$$

$$0.2500000000, \begin{bmatrix} 0.103681579695617 \\ 0.442932937386397 \end{bmatrix}$$

$$0.3000000000, \begin{bmatrix} 0.125586854984821 \\ 0.532951609452131 \end{bmatrix}$$

$$0.3500000000, \begin{bmatrix} 0.147918227141615 \\ 0.623733692979072 \end{bmatrix}$$

$$0.4000000000, \begin{bmatrix} 0.170695642792901 \\ 0.715422674387426 \end{bmatrix}$$

$$0.4500000000, \begin{bmatrix} 0.193941802150972 \\ 0.808176352559828 \end{bmatrix}$$

$$0.5000000000, \begin{bmatrix} 0.217682618891996 \\ 0.902169799494657 \end{bmatrix}$$

$$0.5500000000, \begin{bmatrix} 0.241947796433211 \\ 0.997599045049107 \end{bmatrix}$$

$$0.6000000000, \begin{bmatrix} 0.266771557549599 \\ 1.09468572162072 \end{bmatrix}$$

$$0.6500000000, \begin{bmatrix} 0.292193579142381 \\ 1.19368299986142 \end{bmatrix}$$

$$0.7000000000, \begin{bmatrix} 0.318260206180295 \\ 1.29488328898720 \end{bmatrix}$$

$$0.7500000000, \begin{bmatrix} 0.345026052767854 \\ 1.39862839326659 \end{bmatrix}$$

$$0.8000000000, \begin{bmatrix} 0.372556151488727 \\ 1.50532315854507 \end{bmatrix}$$

$$0.8500000000, \begin{bmatrix} 0.400928898008611 \\ 1.61545419575422 \end{bmatrix}$$

$$0.9000000000, \begin{bmatrix} 0.430240181072309 \\ 1.72961619201849 \end{bmatrix}$$

$$0.9500000000, \begin{bmatrix} 0.460609335985373 \\ 1.84854992201642 \end{bmatrix}$$

(2)

$$1.000000000,\ \begin{bmatrix} 0.492188008516533 \\ 1.97319897602792 \end{bmatrix} \tag{2}$$

N = 20 and Runge Kutta of order 4

```
> ContinueRK := proc(H1,H2,t_Start,v,h)
  local J11, J12, J21, J22, J, H1_t, H2_t, H_t, A,B, w, u, K1,K2,
  K3,K4, F;
  with(LinearAlgebra);
  J11:=unapply(diff(H1(x,y,t),x),x,y,t);
  J12:=unapply(diff(H1(x,y,t),y),x,y,t);
  J21:=unapply(diff(H2(x,y,t),x),x,y,t);
  J22:=unapply(diff(H2(x,y,t),y),x,y,t);
  H1_t:=unapply(diff(H1(x,y,t),t),x,y,t);
  H2_t:=unapply(diff(H2(x,y,t),t),x,y,t);
  J := (x,y,t) -> Matrix([[J11(x,y,t),J12(x,y,t)],[J21(x,y,t),J22
  (x,y,t)]]);
  H_t := (x,y,t) -> Vector([H1_t(x,y,t),H2_t(x,y,t)]);
  F:= unapply(-MatrixVectorMultiply(MatrixInverse(J(x,y,t)),H_t(x,
  y,t)),x,y,t);
  K1:=h*F(v[1],v[2],t_start);
  K2:=h*F(v[1]+K1[1]/2,v[2]+K1[2]/2,t_Start+h/2);
  K3:=h*F(v[1]+K2[1]/2,v[2]+K2[2]/2,t_Start+h/2);
  K4:=h*F(v[1]+K3[1],v[2]+K2[2],t_start+h);
  u:= v+ 1/6.0*(K1+2*K2+2*K3+K4);
  end proc:
```

```
> f1 := (x,y) -> 4.0*x^2 - 20.0*x + 0.25*y^2 + 8.0;
  f2 := (x,y) -> 0.5*x*y^2 + 2.0*x -5.0*y + 8.0;
  H1 := (x,y,t) -> t*f1(x,y)+(1-t)*(f1(x,y)-f1(vv[1],vv[2]));
  H2 := (x,y,t) -> t*f2(x,y)+(1-t)*(f2(x,y)-f2(vv[1],vv[2]));
```

$$f1 := (x, y) \to 4.0\,x^2 - 20.0\,x + 0.25\,y^2 + 8.0$$

$$f2 := (x, y) \to 0.5\,x\,y^2 + 2.0\,x - 5.0\,y + 8.0$$

$$H1 := (x, y, t) \to t\,f1(x, y) + (1 - t)\,\left(f1(x, y) - f1\left(vv_1, vv_2\right)\right)$$

$$H2 := (x, y, t) \to t\,f2(x, y) + (1 - t)\,\left(f2(x, y) - f2\left(vv_1, vv_2\right)\right) \tag{3}$$

```
> N:=20;
  h:= 1.0/N;
  v:= Vector([0.0,0.0]);
  t_Start:=0;
  for j from 1 to N do
  v:=ContinueRK(H1,H2,t_Start,v,h);
  t_Start:=t_Start+h;
  print(t_Start,v);
  od:
```

$$N := 20$$

$$h := 0.05000000000$$

$$v := \begin{bmatrix} 0. \\ 0. \end{bmatrix}$$

$$t\_Start := 0$$

$$0.05000000000, \begin{bmatrix} 0.0201784251866935 \\ 0.0880870272243411 \end{bmatrix}$$

$$0.1000000000, \begin{bmatrix} 0.0407206622440089 \\ 0.176414996797582 \end{bmatrix}$$

$$0.1500000000, \begin{bmatrix} 0.0616382526841100 \\ 0.265088444538119 \end{bmatrix}$$

$$0.2000000000, \begin{bmatrix} 0.0829443351824495 \\ 0.354218441899772 \end{bmatrix}$$

$$0.2500000000, \begin{bmatrix} 0.104653839095779 \\ 0.443923931241862 \end{bmatrix}$$

$$0.3000000000, \begin{bmatrix} 0.126783720817223 \\ 0.534333313903453 \end{bmatrix}$$

$$0.3500000000, \begin{bmatrix} 0.149353253727376 \\ 0.625586359606540 \end{bmatrix}$$

$$0.4000000000, \begin{bmatrix} 0.172384386029428 \\ 0.717836528021788 \end{bmatrix}$$

$$0.4500000000, \begin{bmatrix} 0.195902185673966 \\ 0.811253824492332 \end{bmatrix}$$

$$0.5000000000, \begin{bmatrix} 0.219935398542257 \\ 0.906028356132256 \end{bmatrix}$$

$$0.5500000000, \begin{bmatrix} 0.244517156090348 \\ 1.00237481836471 \end{bmatrix}$$

$$0.6000000000, \begin{bmatrix} 0.269685883395293 \\ 1.10053823593034 \end{bmatrix}$$

$$0.6500000000, \begin{bmatrix} 0.295486480663279 \\ 1.20080142365361 \end{bmatrix}$$

$$0.7000000000, \begin{bmatrix} 0.321971885253457 \\ 1.30349484970798 \end{bmatrix}$$

$$0.7500000000, \begin{bmatrix} 0.349205174938074 \\ 1.40900992795449 \end{bmatrix}$$

$$0.8000000000,\ \begin{bmatrix} 0.377262460464936 \\ 1.51781732637627 \end{bmatrix}$$

$$0.8500000000,\ \begin{bmatrix} 0.406236962697153 \\ 1.63049282457475 \end{bmatrix}$$

$$0.9000000000,\ \begin{bmatrix} 0.436244927946173 \\ 1.74775491572958 \end{bmatrix}$$

$$0.9500000000,\ \begin{bmatrix} 0.467434510524975 \\ 1.87052141201811 \end{bmatrix}$$

$$1.000000000,\ \begin{bmatrix} 0.499999677363961 \\ 1.99999828681953 \end{bmatrix} \qquad \textbf{(4)}$$

Problem 4a of page 692

```
> rkVect2Step := proc(f1,f2,a,b,w1,w2)
  local j, K11,K21,K31,K41,K12,K22,K32,K42,h,W1,W2;
  h := b-a;

  K11 := h*f1(a,w1,w2);
  K12 := h*f2(a,w1,w2);

  K21 := h*f1(a+h/2,w1+K11/2,w2+K12/2);
  K22 := h*f2(a+h/2,w1+K11/2,w2+K12/2);

  K31 := h*f1(a+h/2,w1+K21/2,w2+K22/2);
  K32 := h*f2(a+h/2,w1+K21/2,w2+K22/2);

  K41 := h*f1(b,w1+K31,w2+K32);
  K42 := h*f2(b,w1+K31,w2+K32);

  W1 := w1+(K11+2*K21+2*K31+K41)/6;
  W2 := w2+(K12+2*K22+2*K32+K42)/6;

  return( [W1,W2])
  end proc;
```

$$rkVect2Step := \textbf{proc}(f1, f2, a, b, w1, w2) \qquad \textbf{(5)}$$

local $j, K11, K21, K31, K41, K12, K22, K32, K42, h, W1, W2$;

$h := b - a$;

$K11 := h * f1(a, w1, w2)$;

$K12 := h * f2(a, w1, w2)$;

$K21 := h * f1(a + 1/2 * h, w1 + 1/2 * K11, w2 + 1/2 * K12)$;

$K22 := h * f2(a + 1/2 * h, w1 + 1/2 * K11, w2 + 1/2 * K12)$;

$K31 := h * f1(a + 1/2 * h, w1 + 1/2 * K21, w2 + 1/2 * K22)$;

$K32 := h * f2(a + 1/2 * h, w1 + 1/2 * K21, w2 + 1/2 * K22)$;

$K41 := h * f1(b, w1 + K31, w2 + K32)$;

$K42 := h * f2(b, w1 + K31, w2 + K32)$;

$W1 := w1 + 1/6 * K11 + 1/3 * K21 + 1/3 * K31 + 1/6 * K41$;

```
    W2 := w2 + 1/6*K12 + 1/3*K22 + 1/3*K32 + 1/6*K42;
    return [W1, W2]
end proc
```

```
> f1:= (t,y1,y2) -> y2;
  f2:= (t,y1,y2) -> -y1;
```

$$f1 := (t, y1, y2) \rightarrow y2$$
$$f2 := (t, y1, y2) \rightarrow -y1 \quad \textbf{(6)}$$

```
> h:= evalf(Pi/20);
```

$$h := 0.1570796327 \quad \textbf{(7)}$$

```
> with(LinearAlgebra):
```

```
> Sol1w1:= Vector(6):
  Sol1w2:= Vector(6):
  w1:=1; w2:=0;
  Sol1w1[1]:= 1;Sol1w2[1]:=0;
  for j from 1 to 5 do
  a:=(j-1)*h;
  b:=j*h;
  A:=rkVect2Step(f1,f2,a,b,w1,w2):
  w1:=A[1];
  w2:=A[2];
  Sol1w1[j+1]:= w1;Sol1w2[j+1]:=w2;
  print(j*h,w1,w2);
  od:
```

$$w1 := 1$$
$$w2 := 0$$
$$Sol1w1_1 := 1$$
$$Sol1w2_1 := 0$$

$$0.1570796327, 0.9876883614, -0.1564336685$$
$$0.3141592654, 0.9510568067, -0.3090154275$$
$$0.4712388981, 0.8910073220, -0.4539882466$$
$$0.6283185308, 0.8090185150, -0.5877824516$$
$$0.7853981635, 0.7071092062, -0.7071036208 \quad \textbf{(8)}$$

```
> w1:=0; w2:=1;
  Sol2w1:= Vector(6):
  Sol2w2:= Vector(6):
  w1:=0; w2:=1;
  Sol2w1[1]:= 0;Sol2w2[1]:=1;
  for j from 1 to 5 do
  a:=(j-1)*h;
  b:=j*h;
  A:=rkVect2Step(f1,f2,a,b,w1,w2):
  w1:=A[1];
  w2:=A[2];
  Sol2w1[j+1]:= w1;Sol2w2[j+1]:=w2;
  print(j*h,w1,w2);
  od:
```

$$w1 := 0$$

$$w2 := 1$$
$$w1 := 0$$
$$w2 := 1$$
$$Sol2w1_1 := 0$$
$$Sol2w2_1 := 1$$

0.1570796327, 0.1564336685, 0.9876883614
0.3141592654, 0.3090154275, 0.9510568067
0.4712388981, 0.4539882466, 0.8910073220
0.6283185308, 0.5877824516, 0.8090185150
0.7853981635, 0.7071036208, 0.7071092062 **(9)**

```
> Sol := Vector(6):
> Sol := Sol1w1+ (1.0-Sol1w1[6])/Sol2w1[6]*Sol2w1;
  trueSol:= x -> cos(x)+(sqrt(2.0)-1.0)*sin(x);
```

$$Sol := \begin{bmatrix} 1. \\ 1.05248506162507 \\ 1.07905470007319 \\ 1.07905469442766 \\ 1.05248505055518 \\ 1.00000000000657 \end{bmatrix}$$

$$trueSol := x \rightarrow \cos(x) + \left(\sqrt{2.0} - 1.0\right) \sin(x) \quad \textbf{(10)}$$

```
> for j from 1 to 6 do
  err:= abs(Sol[j]-trueSol((j-1)*h));
  od;
```

$$err := 0.$$
$$err := 5.56374929905346\ 10^{-7}$$
$$err := 8.45926813708431\ 10^{-7}$$
$$err := 8.51572335092143\ 10^{-7}$$
$$err := 5.66444823713042\ 10^{-7}$$
$$err := 3.06572323083287\ 10^{-10} \quad \textbf{(11)}$$

Problem 3d on page 699
y1' = y2;
y2' = 0.5*(1.0 - y2^2 - y1*sin(x));
z1' = z2;
z2' = -0.5*sin(x)*z1 - y2*z2;
y1(0) = 2
y2(0) = t
z1(0) = 0
z2(0) = 1

```
> with(LinearAlgebra):

> f1 := (x,y1,y2,z1,z2) -> y2;
  f2 := (x,y1,y2,z1,z2) ->  0.5*(1.0 - y2^2 - y1*sin(x));
```

```
f3 := (x,y1,y2,z1,z2) -> z2;
f4 := (x,y1,y2,z1,z2) ->  -0.5*sin(x)*z1 - y2*z2;
```

$$f1 := (x, y1, y2, z1, z2) \rightarrow y2$$

$$f2 := (x, y1, y2, z1, z2) \rightarrow 0.5\,\left(1.0 - y2^2 - y1\,\sin(x)\right)$$

$$f3 := (x, y1, y2, z1, z2) \rightarrow z2$$

$$f4 := (x, y1, y2, z1, z2) \rightarrow -0.5\,\sin(x)\,z1 - y2\,z2 \qquad (12)$$

```
> F:= proc(x,w::Vector(4))
  local j,v;
  v:=Vector(4);
  v(1):= f1(x,w(1),w(2),w(3),w(4));
  v(2):= f2(x,w(1),w(2),w(3),w(4));
  v(3):= f3(x,w(1),w(2),w(3),w(4));
  v(4):= f4(x,w(1),w(2),w(3),w(4));
  return(v);
  end proc;
```

$F := \mathbf{proc}(x, w::(Vector(4)))$ (13)

  local j, v;
  $v := Vector(4)$;
  $v(1) := f1(x, w(1), w(2), w(3), w(4))$;
  $v(2) := f2(x, w(1), w(2), w(3), w(4))$;
  $v(3) := f3(x, w(1), w(2), w(3), w(4))$;
  $v(4) := f4(x, w(1), w(2), w(3), w(4))$;
  return v

end proc

```
> with(LinearAlgebra):

  rkVect4Step := proc(F,a,h,w::Vector(4))
  local j, K1,K2,K3,K4,W;
  K1:=Vector(4);
  K2:=Vector(4);
  K3:=Vector(4);
  K4:=Vector(4);
  W:=Vector(4);

  K1 := h*F(a,w);

  K2 := h*F(a+h/2,w+K1/2);

  K3 := h*F(a+h/2,w+K2/2);

  K4 := h*F(a+h,w + K3);

  W := w+(K1+2*K2+2*K3+K4)/6;

  return(W)
  end proc;
```

$rkVect4Step := \mathbf{proc}(F, a, h, w::(Vector(4)))$ (14)

  local $j, K1, K2, K3, K4, W$;

```
    K1 := Vector(4);
    K2 := Vector(4);
    K3 := Vector(4);
    K4 := Vector(4);
    W := Vector(4);
    K1 := h*F(a, w);
    K2 := h*F(a + 1/2*h, w + 1/2*K1);
    K3 := h*F(a + 1/2*h, w + 1/2*K2);
    K4 := h*F(a + h, w + K3);
    W := w + 1/6*K1 + 1/3*K2 + 1/3*K3 + 1/6*K4;
    return W
end proc
```

```
> ft0:=4.0; #a guess to start Newton
  w0:=Vector([2,t0,0,1]);
  h:= evalf(Pi/20);
```

$$ft0 := 4.0$$

$$w0 := \begin{bmatrix} 2 \\ 1.0 \\ 0 \\ 1 \end{bmatrix}$$

$$h := 0.1570796327$$

(15)

```
> a:=0;
  w:=Vector(4):
  w:=w0;
  t:=t0;
  for k from 1 to 3 do
  for j from 1 to 20 do
  w:=rkVect4Step(F,a,h,w);
  a:=a+h:
  od:
  t:= t - (w(1)-2.0)/w(3);
  od;
```

$$a := 0$$

$$w := \begin{bmatrix} 2 \\ 1.0 \\ 0 \\ 1 \end{bmatrix}$$

$$t := 1.0$$

$$t := 0.999992738217714$$

$$t := 0.999986198363812$$

$$t := 0.999979923491883$$

(16)

```
> w0:=Vector([2,t,0,1]);
  for j from 1 to 20 do
  w:=rkVect4Step(F,a,h,w);
```

```
a:=a+h:
err:= abs(2.0+evalf(sin(a))-w(1));
print(err);
od:
```

$$w0 := \begin{bmatrix} 2 \\ 0.999979923491883 \\ 0 \\ 1 \end{bmatrix}$$

0.000478594134026178
0.000493759151042550
0.000504928899386137
0.000511877373449821
0.000514418120435067
0.000512388443001388
0.000505641269989976
0.000494022529607729
0.000477356516756622
0.000455430799310097
0.000427973353883448
0.000394629858542217
0.000354936823590002
0.000308283670217513
0.000253874041066027
0.000190669763810103
0.000117332636281553
0.0000321537410474804
0.0000670147480041905
0.000182819635737630 **(17)**

Problem 3a on page 704

```
> with(LinearAlgebra):
> p:= -3.0;
  q:= 2.0;
```

```
r:= x -> 2*x+3.0;
h:=0.1;
alpha:=2.0;
beta:=1.0;
```

$$p := -3.0$$
$$q := 2.0$$
$$r := x \mapsto 2\,x + 3.0$$
$$h := 0.1$$
$$\alpha := 2.0$$
$$\beta := 1.0 \quad (18)$$

```
> A:=Matrix(9);
b:=Vector(9);
w:=Vector(9);
```

$$A := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$b := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(19)

$$w := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (19)$$

```
> for j from 1 to 9
  do
  A(j,j):= 2+h^2*q;
  od:

  for j from 1 to 8 do
  A(j,j+1):= -1.0+h/2*p;
  od:

  for j from 1 to 8 do
  A(j+1,j):= -1.0+h/2*p;
  od:
```

```
> A;
```

$$\begin{bmatrix} [2.020, -1.150000000, 0, 0, 0, 0, 0, 0, 0], \\ [-1.150000000, 2.020, -1.150000000, 0, 0, 0, 0, 0, 0], \\ [0, -1.150000000, 2.020, -1.150000000, 0, 0, 0, 0, 0], \\ [0, 0, -1.150000000, 2.020, -1.150000000, 0, 0, 0, 0], \\ [0, 0, 0, -1.150000000, 2.020, -1.150000000, 0, 0, 0], \\ [0, 0, 0, 0, -1.150000000, 2.020, -1.150000000, 0, 0], \\ [0, 0, 0, 0, 0, -1.150000000, 2.020, -1.150000000, 0], \\ [0, 0, 0, 0, 0, 0, -1.150000000, 2.020, -1.150000000], \\ [0, 0, 0, 0, 0, 0, 0, -1.150000000, 2.020] \end{bmatrix} \quad (20)$$

```
> Determinant(A);
```

$$-7.08344548668919 \quad (21)$$

```
> ConditionNumber(A,infinity);
```

$$34.78968855 \quad (22)$$

```
> b(1):= -h^2*r(h) + (1+h/2*p)* alpha:
  for j from 2 to 8 do

  b(j):= -h^2*r(j*h):
  od:

  b(9):= -h^2*r(9*h) + (1-h/2*p)* beta:
> b;
```

$$\begin{bmatrix} 1.668000000 \\ -0.034 \\ -0.036 \\ -0.038 \\ -0.040 \\ -0.042 \\ -0.044 \\ -0.046 \\ 1.102000000 \end{bmatrix} \quad (23)$$

```
> w:=LinearSolve(A,b);
```

$$w := \begin{bmatrix} 1.09442433147831 \\ 0.471945347466248 \\ -0.235876851581073 \\ -0.854963817199960 \\ -1.23284220123973 \\ -1.27576770149938 \\ -0.971549761393976 \\ -0.392519705470904 \\ 0.322080365697258 \end{bmatrix} \quad (24)$$

Problems 4a and 5a on page 711
The below is sloppy programming. I have a number of constants, e.g., h, that should be incorporated into the variable lists of F, JF.

```
> f:= (y1,y2) -> y1^3-y1*y2;
  h:=0.1;
  a:=1.0;
  b:=2.0;
  alpha:= 0.5;
  beta := 1.0/3.0;

  dy1_f := (y1,y2) -> 3*y1^2-y2;
  dy2_f := (y1,y2) -> -y1;
```

$$f := (y1, y2) \rightarrow y1^3 - y1\, y2$$
$$h := 0.1$$
$$a := 1.0$$
$$b := 2.0$$
$$\alpha := 0.5$$
$$\beta := 0.3333333333$$
$$dy1\_f := (y1, y2) \rightarrow 3\, y1^2 - y2$$
$$dy2\_f := (y1, y2) \rightarrow -y1 \quad (25)$$

```
> F:= proc(w)
  local j,v,N;
  N:=op(1,w);
  v:=Vector(N);
  for j from 2 to N-1 do
  v(j) :=-w(j-1)+2*w(j)-w(j+1)+h^2*f(w(j),(w(j+1)-w(j-1))/(2*h));
  od ;
  v(1):=-alpha+2*w(1)-w(2)+h^2*f(w(1),(w(2)-alpha)/(2*h));
  v(N):=-w(N-1)+2*w(N)-beta+h^2*f(w(N),(beta-w(N-1))/(2*h));
  return(v);
  end proc;
```

$$F := \mathbf{proc}(w) \tag{26}$$

```
    local j, v, N;
    N := op(1, w);
    v := Vector(N);
    for j from 2 to N - 1 do
        v(j) := -w(j - 1) + 2 * w(j) - w(j + 1) + h^2 * f(w(j), 1 / 2 * (w(j
         + 1) - w(j - 1)) / h)
    end do;
    v(1) := -alpha + 2 * w(1) - w(2) + h^2 * f(w(1), 1 / 2 * (w(2) - alpha) / h);
    v(N) := -w(N - 1) + 2 * w(N) - beta + h^2 * f(w(N), 1 / 2 * (beta - w(N - 1))
    / h);
    return v
end proc
```

```
> JF:= proc(w)
  local A,j,v,N;
  N:=op(1,w);
  A:=Matrix(N);

  A(1,1) := 2.0+h^2*dy1_f(w(1),(w(2)-alpha)/(2*h));
  for j from 2 to N-1 do
  A(j,j) := 2.0+h^2*dy1_f(w(j),(w(j+1)-w(j-1))/(2*h));
  od;
  A(N,N) := 2.0+h^2*dy1_f(w(N),(beta-w(N-1))/(2*h));

  A(1,2) :=-1.0 + h/2*dy2_f(w(1),(w(2)-alpha)/(2*h));
  for j from 2 to N-2 do
  A(j,j+1) :=-1.0 + h/2*dy2_f(w(j),(w(j+1)-w(j-1))/(2*h));
  A(j+1,j) :=-1.0 - h/2*dy2_f(w(j+1),(w(j+2)-w(j))/(2*h));
  od ;
  A(2,1) :=-1.0 - h/2*dy2_f(w(2),(w(3)-w(1))/(2*h));
  A(N,N-1) :=-1.0 - h/2*dy2_f(w(N),(beta-w(N-1))/(2*h));
  A(N-1,N) :=-1.0 + h/2*dy2_f(w(N-1),(w(N)-w(N-2))/(2*h));

  return(A);
  end proc;
```

$$JF := \mathbf{proc}(w) \tag{27}$$

```
    local A, j, v, N;
    N := op(1, w);
```

```
    A := Matrix(N);
    A(1, 1) := 2.0 + h^2 * dy1_f(w(1), 1/2 * (w(2) - alpha) / h);
    for j from 2 to N - 1 do
        A(j,j) := 2.0 + h^2 * dy1_f(w(j), 1/2 * (w(j+1) - w(j-1)) / h)
    end do;
    A(N, N) := 2.0 + h^2 * dy1_f(w(N), 1/2 * (beta - w(N-1)) / h);
    A(1, 2) := -1.0 + 1/2 * h * dy2_f(w(1), 1/2 * (w(2) - alpha) / h);
    for j from 2 to N - 2 do
        A(j,j+1) := -1.0 + 1/2 * h * dy2_f(w(j), 1/2 * (w(j+1) - w(j-1)) / h);
        A(j+1,j) := -1.0 - 1/2 * h * dy2_f(w(j+1), 1/2 * (w(2+j) - w(j)) / h)
    end do;
    A(2, 1) := -1.0 - 1/2 * h * dy2_f(w(2), 1/2 * (w(3) - w(1)) / h);
    A(N, N-1) := -1.0 - 1/2 * h * dy2_f(w(N), 1/2 * (beta - w(N-1)) / h);
    A(N-1, N) := -1.0 + 1/2 * h * dy2_f(w(N-1), 1/2 * (w(N) - w(N-2)) / h);
    return A
end proc
```

```
> N:=9;
  h:=0.1;
  w:=Vector(N):
  for j from 1 to N do
  w(j):= (beta-alpha)/(b-a)*(j*h)+alpha;#initial guess for Newton
  od:

  for k from 1 to 3 do
  w:= w-MatrixInverse(JF(w)) . F(w);
  od:
```

$$N := 9$$
$$h := 0.1 \qquad \textbf{(28)}$$

```
> for j from 1 to 9 do
  print(abs(1/(2+j*h)-w(j)));
  od:
```

$$0.00000676771966251488$$
$$0.0000108837022041453$$
$$0.0000130034547041191$$
$$0.0000136058021765195$$
$$0.0000130438473507910$$
$$0.0000115802627453920$$
$$0.00000941210924010338$$
$$0.00000668845343859692$$
$$0.00000352221719812640 \qquad \textbf{(29)}$$

```
> N:=2*9+1;
  w2:=Vector(N):
```

```
extrap1_w:=Vector(9):
h:=0.05:
for j from 1 to N do
w2(j):= (beta-alpha)/(b-a)*(j*h)+alpha;#initial guess for Newton
od:

for k from 1 to 3 do
w2:= w2-MatrixInverse(JF(w2)) . F(w2);
od:
```

$$N := 19 \tag{30}$$

```
> for j from 1 to 9 do
extrap1_w(j):= (4*w2(2*j)-w(j))/3;
od:
```

```
> for j from 1 to 9 do
print(1/(2+j*0.1),w(j),w2(2*j),extrap1_w(j));
od:
```

0.4761904762, 0.476197243919663, 0.476192171974287, 0.476190481325828
0.4545454545, 0.454556338202204, 0.454548181340725, 0.454545462386899
0.4347826087, 0.434795612154704, 0.434785866260482, 0.434782617629075
0.4166666667, 0.416680272502177, 0.416670074835605, 0.416666675613414
0.4000000000, 0.400013043847351, 0.400003267139733, 0.400000008237193
0.3846153846, 0.384626964862745, 0.384618284959740, 0.384615391658738
0.3703703704, 0.370379782509240, 0.370372727549872, 0.370370375896750
0.3571428571, 0.357149545553439, 0.357144532092237, 0.357142860938503
0.3448275862, 0.344831108417198, 0.344828468203636, 0.344827588132449 **(31)**

```
> for j from 1 to 9 do
err1:=abs(1/(2+j*0.1)-w(j)):
err2:=abs(1/(2+j*0.1)-w2(2*j)):
err3:=abs(1/(2+j*0.1)-extrap1_w(j)):
print(err1,err2,err3);
od:
```

0.0000067677196625148, 0.0000016957742866030, $5.12582787326465\ 10^{-9}$
0.0000108837022041453, 0.0000027268407252201, $7.88689891173533\ 10^{-9}$
0.0000130034547041191, 0.0000032575604822504, $8.92907492389128\ 10^{-9}$
0.0000136058021765195, 0.0000034081356047321, $8.91341417341707\ 10^{-9}$
0.0000130438473507910, 0.0000032671397326988, $8.23719326081829\ 10^{-9}$
0.0000115802627453920, 0.0000029003597396548, $7.05873770545651\ 10^{-9}$
0.0000094121092401033, 0.0000023571498723451, $5.49674972205594\ 10^{-9}$
0.0000066884534385969, 0.0000016749922369663, $3.83850307095202\ 10^{-9}$
0.0000035222171981264, $8.82003636182294\ 10^{-7}$, $1.93244886759203\ 10^{-9}$ **(32)**