ACMS 40390: Sample Test II

February 27, 2019

By signing you confirm that you are following the honor code for this test.

Name: ....

To receive credit you must show your work.

Problem Number	Maximum Points	Points attained
1	7	
2	12	
3	7	
4	7	
5	7	
6	7	
7	7	
8	7	
9	7	
10	10	
11	10	
12	12	
TOTAL	100	

Some Useful Results

The following result may be of use. You may assume it in any argument you need to give.
Runge-Kutta Method of Order Four For the ordinary differential equation $y' = f(t, y)$ on $[a, b]$ with initial condition $y(a) = \alpha$ and stepsize h we have

$$\begin{aligned} w_0 &= \alpha \\ k_1 &= hf(t_i, w_i) \\ k_2 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{k_1}{2}\right) \\ k_3 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{k_2}{2}\right) \\ k_4 &= hf(t_i + h, w_i + k_3) \\ w_{i+1} &= w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{aligned}$$

Letting $f(x) \in C^5[a, b]$ with $a < b$, we use the book's notation $S(f, a, b)$ for the Simpson rule approximation to $\int_a^b f(x)\mathrm{dx}$. We use

$$\mathrm{Err(f, a, b)} := \frac{\left|\mathrm{S(f, a, b)} - \mathrm{S}\left(\mathrm{f, a, \frac{a+b}{2}}\right) - \mathrm{S}\left(\mathrm{f, \frac{a+b}{2}, b}\right)\right|}{15}$$

as an estimate for the error

$$\left|\int_a^b f(x)\mathrm{dx} - \left(\mathrm{S}\left(\mathrm{f, a, \frac{a+b}{2}}\right) + \mathrm{S}\left(\mathrm{f, \frac{a+b}{2}, b}\right)\right)\right|.$$

Problems

In the following you must show your work.

Problem 1 (7 points total) *We want to approximate*

$$\int_0^1 (1 + 1.06\pi x)\sqrt{\sin(\pi x)}\ \mathrm{d}x$$

with at most an error of 0.005. *We use the Adaptive Simpson's rule and find*

$$\frac{|S(f,0,1) - S(f,0,0.5) - S(f,0.5,1)|}{15} = 0.00508.$$

Should we accept the result or should we subdivide further and if so how? Explain your answer.

Since $0.00508 > 0.005$. we need to subdivide further. Use the subintervals $[0, 0.25], [0.25, 0.5], [0.5, 0.75], [0.75, 1.0]$.

Problem 2 (12 points total) *Let $y' = t^2y^2$ on $[0,1]$ with initial condition $y(0) = 24$.*

1. *What is an explicit local solution near* 0*?*
2. *Is it unique and if so why?*
3. *Is there a continuously differentiable solution on all of* $[0,1]$*? (Either show there is by explicitly constructing the solution or show there is no solution on all of* $[0,1]$*.)*

$$y = \frac{1}{\frac{1}{24} - \frac{t^3}{3}}$$

The solution is unique since t^2y^2 is continuously differentiable with respect to t and y.

The solution does not exist at $t = 0.5$.

Problem 3 (7 points total) *What integration method does the Runge-Kutta Method of Order Four applied to solving $y' = f(t))$ on $[1, 2]$ with initial value $y(1) = 0$ and $h = 1.0$ reduce to. (To receive credit you must show this explicitly.)*

$k_1 = hf(1)$
$k_2 = hf(1.5)$
$k_3 = hf(1.5)$
$k_4 = hf(2)$
So we get

$$(k_1 + 2k_2 + 2k_3 + k_4)/6 = (f(1) + 4f(1.5) + f(2))/6,$$

i.e., Simpson's method.

Problem 4 (7 points total) *Use the midpoint method to approximate the solution of $y' = t\sin(y)$ on $[1, 1.5]$ with initial value $y(1) = 2$ and $h = 0.5$.*

$f(t, y) = tsin(y)$.

$$2.0+hf(1+h/2, 2.0+f(1.0, 2)h/2) = 2.0+0.5(1.25\sin(2.0+\sin(2.0)0.25)).$$

Problem 5 (7 points total) *Use Taylors method of order three to approximate the solution of* $y' = ty$ *on* $[0, 0.5]$ *with initial value* $y(0) = -1$ *and* $h = 0.5$.

Note
$y' = ty$
$y'' = y + t^2 y$
$y''' = 2ty + ty + t^3 y = 3ty + t^3 y.$
So we get

$$-1 + 0.5 \cdot 0 + 0.25/2 \cdot (-1 + 0 \cdot (-1)) + 0.125/6 \cdot (0 + 0) = -1 - 0.125 = -1.125.$$

Problem 6 (7 points) *Use the Euler method with* $h = 0.5$ *to solve* $y'' - 3y' + 2y$ *on* $[1, 1.5]$ *and the initial conditions* $y(0) = 2, y'(0) = 5$.

Letting $y_1 = y$ and $y_2 = y'$ we have

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} y_2 \\ 3y_2 + 2y_1 \end{bmatrix} \quad \text{with initial condition} \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

Therefore we get

$$\begin{bmatrix} 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 5 \\ 19 \end{bmatrix} \cdot 0.5 = \begin{bmatrix} 4.5 \\ 14.5 \end{bmatrix}.$$

Problem 7 (7 points) *Let*

$$v = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

Compute $||v||_1$, $||v||_2$, *and* $||v||_\infty$.

$$4, \quad \sqrt{10}, \quad 3.$$

Problem 8 (7 points) *Let*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Compute $||A||_\infty$.

3

Problem 9 (7 points) *Let*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Compute $||A||_1$.

2

Problem 10 (10 points) *Let*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Compute $||A||_2$.

We need to take the largest of the square roots of absolute values of the eigenvalues of $A^t \cdot A$.

This gives us

$$3 + \sqrt{5}$$

Problem 11 (10 points) *Let*

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Use the Gram-Schmidt process on v_1, v_2 *to find a set of orthogonal vectors.*

First let

$$e_1 = \frac{v_1}{||v_1||_2} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}.$$

The we have

$$e_2 = v_2 - (v_2^t e_1)e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ -\frac{2}{5} \end{bmatrix}.$$

Problem 12 (12 points) *Let*

$$A = \begin{bmatrix} \frac{5}{2}\pi & \frac{1}{2}\pi \\ \frac{1}{2}\pi & \frac{5}{2}\pi \end{bmatrix}$$

Compute $\cos(A)$ *explicitly.*

Note the eigenvalues of A are 3π and 2π with eigenvectors $[1,1]^t$ and $[-1,1]^t$ respectively. Therefore letting

$$T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

we have

$$T^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \quad \text{and} \quad A = T \begin{bmatrix} 3\pi & 0 \\ 0 & 2\pi \end{bmatrix} \cdot T^{-1}.$$

Therefor

$$\cos(A) = T \cdot \cos\left(\begin{bmatrix} 3\pi & 0 \\ 0 & 2\pi \end{bmatrix} \right) \cdot T^{-1} = T \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot T^{-1} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$