ACMS 40390: Sample Test I

February 13, 2019

On a test to receive credit you must show your work.

Some Useful Results

The following results may be of some use. You can assume them in any argument you need to give.

Theorem 0.1 (Neville's Recursion Formula) *Let* $x_0, \ldots, x_n$ *be distinct real numbers in the interval,* $[a, b]$*. Let* f *be a real valued function on* $[a, b]$*, and let* $P(x)$ *denote the unique polynomial of degree* $\leq n$ *such that* $f(x_i) = P(x_i)$ *for* $i = 0, \ldots, n$*. For* $c = 0, \ldots, n$*, let* $P_{\widehat{c}}(x)$ *denote the unique polynomial of degree* $\leq n-1$ *such that* $P_{\widehat{c}}(x_i) = f(x_i)$ *for* $0 \leq i \leq n$ *with* $i \neq c$*. Then for two distinct points* x_j *and* x_k *in the set* $\{x_0, \ldots, x_n\}$*:*

$$P(x) = \frac{(x - x_j)P_{\widehat{j}}(x) - (x - x_k)P_{\widehat{k}}(x)}{x_k - x_j}$$

or equivalently:

$$P(x) = \frac{x - x_j}{x_k - x_j} P_{\widehat{j}}(x) + \frac{x - x_k}{x_j - x_k} P_{\widehat{k}}(x).$$

Problems

In the following you must show your work.

Problem 1 *Perform the following computations in a) 3 digit rounding arithmetic; and compute b) the absolute error, and c) the relative error in each case:*

1. $(81.0 + 0.8) + 99.8$;
2. $81.0 + (0.8 + 99.8)$.

Put your answers in the table below.

expression to be evaluated	3 digit rounding answer *(2 points each)*	absolute error *(1 point each)*	relative error *(1 point each)*
$(81.0 + 0.8) + 99.8$	182	$\lvert 182 - 181.6 \rvert$	$\left\lvert \frac{182-181.6}{181.6} \right\rvert$
$81.0 + (0.8 + 99.8)$	182	$\lvert 182 - 181.6 \rvert$	$\left\lvert \frac{182-181.6}{181.6} \right\rvert$

Problem 2 (16 points total) *A person wishes to find a zero of $f(x) = e^x - 3x$ for $0 \leq x \leq 1$. That person decides to use the bisection method to accomplish this.* Assume for simplicity that you are using exact arithmetic, i.e., that the errors of computer arithmetic play no role here.

1. *State a theorem and show that it applies to guarantee that $f(x)$ has a zero on $[0, 1]$.*

   The intermediate value theorem states that on the interval ($[0, 1]$ in this case), all values between $f(0)$ and $f(1)$ are taken on. So in this case $f(0) > 0$ and $f(1) < 0$ implies that there is at least one x in $[0, 1]$ with $f(x) = 0$.

2. *The first approximation to a solution by the bisection method is 0.5 with $f(0.5) = e^{0.5} - 3 \cdot 0.5 \approx 0.149$. What is the second approximation to a solution by the bisection method?*

   Since $f(0.5) > 0$ and $f(1) < 0$, the second approximation is 0.75.

3. *You would like to find an approximation to a zero of $f(x)$ on $[0, 1]$ with an absolute error of no more than 0.001. Using the bisection method as outlined in the previous part of this problem, and the error estimate for the bisection method, which is the smallest integer n for which you know that on the n-th approximation you will be within 0.001 of the correct answer. An answer without justification by the error estimate for the bisection method will receive no credit.*

We need the smallest n such that $\frac{1}{2^n} < 0.001$, i.e., $n = 10$.

Problem 3 *A person wishes to find a zero of* $f(x) = e^x - 3x$ *for* $0 \leq x \leq 1$*. That person decides to use Newton's method (also known as the Newton-Raphson method) for finding a solution of the equation* $f(x) = 0$ *on the interval,* $[0, 1]$. Assume for simplicity that you are using exact arithmetic, i.e., that the errors of computer arithmetic play no role here.

1. *Write down iteration formula that Newton's method gives for solving* $f(x) = 0$*, and*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{e^{x_n} - 3x_n}{e^{x_n} - 3}.$$

2. *using* 1.0 *as a starting guess find the first approximation to a solution of* $f(x) = 0$ *given by this formula.*

$$x_1 = 1 - \frac{e-1}{e-1} = 0.$$

Problem 4 *Write down the Lagrange form* (NOT THE NEWTON FORM) *of the interpolating polynomial of degree* ≤ 2 *with the value* 0 *at* $x_0 = 0$*, the value* 2 *at* $x_1 = 1$*, and the value* 3 *at* $x_2 = 3$*.*

$$2\frac{(x-0)(x-3)}{(1-0)(1-3)} + 3\frac{(x-0)(x-1)}{(3-0)(3-1)} = -x(x-3) + \frac{3}{2}x(x-1)$$

Problem 5 *Given the function* $f(x) = x^3$*:*

1. *Compute the divided differences,* $f[0], f[0,1], f[0,1,2]$*;*

   Using the divided difference table

$$\begin{array}{cccc} 2 & 8 & & \\ & & 7 & \\ 1 & 1 & & 3 \\ & & 1 & \\ 0 & 0 & & \end{array}$$

   we have $f[0] = 0, f[0,1] = 1, f[0,1,2] = 3.$

2. *Write down the interpolating polynomial,* $p_2(x)$*, of degree* ≤ 2 *for* $f(x)$ *with the node points* $x_0 = 0$*,* $x_1 = 1$*,* $x_2 = 2$ *using Newton's interpolatory divided-difference formula, i.e., the Newton form of the interpolation polynomial built out of divided differences* (NOT THE LAGRANGE FORM).

$$0 + 1x + 3x(x-1) = x + 3x(x-1)$$

Problem 6 *You wish to estimate*

$$\int_0^2 \exp(x)\ \mathrm{d}x$$

using the composite trapezoid method.

Denote the number computed by the composite trapezoid estimate by $T_1(h)$*, where* n *is the number of intervals in the partition and* $h = \frac{2-0}{n}$*. Assume that The trapezoid method satisfies a formula*

$$T_1(h) = \int_0^2 \exp(x)\ \mathrm{d}x + ch^2 + O(h^4), \tag{1}$$

where c *does not depend on* h*.*

By the composite trapezoid rule for $n = 2$ *and* $n = 6$ *(with* $h = 1$ $(= \frac{2-0}{2})$*) you compute the estimates:*

$$T_1(1) = 6.91$$
$$T_1\left(\frac{1}{3}\right) = 6.45.$$

Use these estimates with equation (1) to do Richardson extrapolation to compute an $O(h^4)$ *estimate to the integral.*

We have

$$T_1(h) = \int_0^2 \exp(x)\ \mathrm{d}x + ch^2 + O(h^4)$$

and

$$T_1\left(\frac{h}{3}\right) = \int_0^2 \exp(x)\ \mathrm{d}x + c\left(\frac{h}{3}\right)^2 + O(h^4) = \int_0^2 \exp(x)\ \mathrm{d}x + c\left(\frac{h^2}{9}\right) + O(h^4).$$

Therefore

$$\frac{9T_1\left(\frac{1}{3}\right) - T_1(h)}{8} = \int_0^2 \exp(x)\ \mathrm{d}x + O(h^4).$$

Thus

$$\int_0^2 \exp(x)\ \mathrm{d}x = \frac{9 \cdot 6.45 - 6.91}{8} + O(h^4).$$

Problem 7 *Neville's method is used to approximate $f(1)$ giving the following table:*

$$\begin{array}{llll} x_0 = 0 & P_0 = 0 & & \\ x_1 = 2 & P_1 = 16 & P_{01} = 8 & \\ x_2 = 3 & P_2 & P_{12} & P_{123} = 8 \end{array}$$

Determine $P_2 = f(3)$.

$$8 = P_{123} = \frac{(3-1)P_{01} + (1-0)P_{12}}{3-0} = \frac{16 + P_{12}}{3}.$$

So $P_{12} = 8$.

So

$$8 = P_{12} = \frac{(3-1)P_1 + (1-2)P_2}{3-2} = 32 - P_2.$$

Therefore $P_2 = 24$.

Problem 8 *Let* $f(x) = \sin(x)$. *Let* $H_3(x)$ *denote the unique polynomial of degree* ≤ 3 *such that* $f(0) = H_3(0)$, $f'(0) = H_3'(0)$, $f(\pi) = H_3(\pi)$, $f'(\pi) = H_3'(\pi)$.

1. *Fill in the four missing entries in the divided difference table for the divided differences*

$$\begin{array}{llllll}
\pi & 0 & & & & \\
 & & & -1 & & \\
\pi & 0 & & & f[0,\pi,\pi] = -\frac{1}{\pi} & \\
 & & f[0,\pi] = \frac{0-0}{\pi} = 0 & & & f[0,0,\pi,\pi] = 0 \\
0 & 0 & & & f[0,0,\pi] = -\frac{1}{\pi} & \\
 & & & 1 & & \\
0 & 0 & & & &
\end{array}$$

2. *Write down* H_3 *in the Newton divided difference form*

$$0 + 1x - \frac{1}{\pi}x^2 + 0x^2(x-\pi) = x - \frac{1}{\pi}x^2$$