Homework for Mathematics 860, Spring Semester 2006
April 26, 2006

You may use Maple, Mathematica, or Matlab.

1 Homework 1: Due February 1, 2006

Problem 1.1 *Let B_t be a Brownian motion on $[0, \infty)$. We want numerical answers for the following probabilities.*

1. *Compute $P(B_3 \leq 0)$.*
2. *Compute $P(-1 \leq B_{1.5} < 1)$.*
3. *Compute $P(-1 \leq B_{1.5} < 1, 1 < B_2 < 4)$.*

Problem 1.2 *Using the Gaussian Process characterization of Brownian motion, show that for any positive real number a, $B_{t+a} - B_a$ is a Brownian motion on $[0, \infty)$.*

Problem 1.3 *Let B_t be a Brownian motion on $[0, \infty)$. For ω in the space on which B_t is defined, let $T_2(\omega)$ be the first t with $B_t(\omega) = 2$. Compute $P(T_2 < 2)$ (we want a numerical answer for this probability).*

2 Homework 2: Due February 8, 2006

Problem 2.1 *Let B_t be a Brownian motion on $[0, \infty)$. Let $\mathcal{F}_t$ for $t \in [0, \infty)$ be the usual filtration with respect to which B_t is a martingale. Use the fact that*

$$e^{\theta B_t - \frac{\theta^2 t}{2}}$$

is a martingale with respect to $\mathcal{F}_t$ for any $\theta \in \mathbb{R}$ to compute, for $n = 3, 4$, a polynomial $p_n(x, t)$ of degree n in x, such that $p_n(B_t, t)$ is a martingale with respect to $\mathcal{F}_t$.

Problem 2.2 *Let N_t for $t \geq 0$ be the standard Poison Process. Let $\mathcal{G}_t$ denote the filtration associated this process. Letting $X_t := N_t - t$, show that X_t and $X_t^2 - t$ are martingales relative to $\mathcal{G}_t$.*

3 Homework 3: Due February 15, 2006

Problem 3.1 *Let* $X_t := \int_0^t xt \ \mathrm{dB_x}$ *for* $t \in [0, \infty)$. *Say a few words about why* X_t *is (or isn't) a Gaussian process. Compute the mean* $m(t)$ *and the covariance* $\rho(s, t)$ *for the process* X_t.

Problem 3.2 *Verify that the Brownian bridge*

$$X_t = (1 - t) \int_0^t \frac{\mathrm{dB_s}}{1 - s}$$

satisfies

$$dX_t = \ \mathrm{dB_t} - \frac{\mathrm{X_t}}{1 - \mathrm{t}} \ \mathrm{dt \ for \ t} \in [0, 1).$$

Problem 3.3 *The Langevin (or Ornstein-Uhlenbeck) equation is given by* $\mathrm{dX_t} = \mu \mathrm{X_t} \ \mathrm{dt} + \sigma \ \mathrm{dB_t}$ *for* $t \in [0, \infty)$ *and* μ, σ *real constants. Assuming that* $X_0 = 0$, *find a function* $f(s, t)$ *such that the solution* X_t *of this equation is*

$$\int_0^t f(s, t) \ \mathrm{dB_s}.$$

Compute the mean $m(t)$ *and the covariance* $\rho(s, t)$ *for the process* X_t.

4 Homework 4: Due March 1, 2006

Problem 4.1 *Show that*

$$\int_0^t \sin(t - s) \mathrm{dB_s} = \int_0^\mathrm{t} \cos(\mathrm{t - s}) \mathrm{B_s ds}$$

and

$$\int_0^t \cos(t - s) \mathrm{dB_s} = \mathrm{B_t} - \int_0^\mathrm{t} \sin(\mathrm{t - s}) \mathrm{B_s ds}.$$

As a consequence show that

$$\mathrm{d} \int_0^\mathrm{t} \sin(\mathrm{t - s}) \mathrm{dB_s} = \left(\int_0^\mathrm{t} \cos(\mathrm{t - s}) \mathrm{dB_s} \right) \mathrm{dt}.$$

Problem 4.2 *Verify that* $\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = \begin{bmatrix} \cos(B_t) \\ \sin(B_t) \end{bmatrix}$ *is a solution of*

$$\mathrm{d} \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} \mathrm{dt} - \begin{bmatrix} Y(t) \\ -X(t) \end{bmatrix} \mathrm{dB_t}.$$

5 Homework 5: Due April 26, 2006

Problem 5.1 *Given the Black-Scholes Differential Equation*

$$\frac{\partial V}{\partial t} + \frac{S^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

show that under the changes of variables

$$\begin{aligned} X &= e^{r(T-t)}S \\ Y &= e^{r(T-t)}V \\ \tau &= T - t \end{aligned}$$

it transforms into

$$\frac{\partial Y}{\partial \tau} - \frac{X^2}{2}\frac{\partial^2 Y}{\partial X^2} = 0.$$

6 Homework 6: Due May 3, 2006

Problem 6.1 *You are given the heat equation*

$$\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = 0$$

on $0 \le t \le 1; 0 \le x \le 1$ *with initial conditions* $u(t,0) = 0$, $u(t,1) = 0$, *and* $u(0,x) = \sin(\pi x)$. *Using the Crank-Nicolson with a gridsize of* 200×200, *i.e., with* $h = 1/200$ *and* $k = 1/200$:

1. *compute the approximate value of* $u(0.2, 0.25)$*;*
2. *give a three-dimensional plot of the approximate solution; and*
3. *give a three-dimensional plot of the difference of the exact solution* $\sin(\pi x)e^{-\pi^2 t}$ *and the approximate solution.*