A formal derivation of implicit differentiation

Suppose there exists a function \( y(t) \) such that \( f(t, y(t)) = 0 \). Let \( x(t) = t \). Use the Chain Rule to compute \( \frac{dy}{dx} \) at the point \((a, b)\).

\[
0 = \frac{df(t, y(t))}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y',
\]

\[
y'(a, b) = -\left. \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \right|_{(a, b)}
\]

If \( 0 = x^2 + xy + y^2 = f(x, y) \), find \( y' \).

\[
0 = 2x + y + xy' + 2yy'; \quad 2x + y = (-2y - x)y'; \quad y' = -\frac{2x + y}{2y + x}
\]

and \( \frac{\partial f}{\partial x} = 2x + y; \quad \frac{\partial f}{\partial y} = 2y + x \).
**MULTI-VARIABLE IMPLICIT DIFFERENTIATION**

An equation $F(x_1, \cdots, x_n) = 0$ is said to define one of the variables *implicitly* as a function of the other variables.

Once you have selected the variable to be the function, say $x_i$, then select one of the remaining variables, say $x_j$, and implicitly compute $\frac{\partial x_i}{\partial x_j}$.

$F(x, y) = 0$ defines $y$ implicitly as a function of $x$ and defines $x$ implicitly as a function of $y$.

$F(x, y, z) = 0$ defines
- $z$ implicitly as a function of $x$ and $y$;
  - Compute $\frac{\partial z}{\partial x}$
  - Compute $\frac{\partial z}{\partial y}$
- $x$ implicitly as a function of $y$ and $z$;
  - Compute $\frac{\partial x}{\partial y}$
  - Compute $\frac{\partial x}{\partial z}$
- $y$ implicitly as a function of $x$ and $z$.
  - Compute $\frac{\partial y}{\partial x}$
  - Compute $\frac{\partial y}{\partial z}$

Given that $ze^{x+2y} + z^2 - x - y = 0$. Find $z_x = \frac{\partial z}{\partial x}$ and $z_y(1, -1, 0)$. 

\[ ze^{x+2y} + z^2 - x - y = 0 \]

\[ z_x: \]
\[
0 = \left( \frac{\partial z}{\partial e} e^{x+2y} + z \frac{\partial e^{x+2y}}{\partial x} \right) + 2zz_x - 1 - 0 = z_x e^{x+2y} + z(1)e^{x+2y} + 2zz_x - 1 = \left( ze^{x+2y} - 1 \right) + \left( xe^{x+2y} + 2z \right) z_x. 
\]
Hence
\[
z_x = -\frac{ze^{x+2y} - 1}{e^{x+2y} + 2z}.
\]

\[ z_y(1, -1, 0): \]
\[
0 = \left( \frac{\partial z}{\partial e} e^{x+2y} + z \frac{\partial e^{x+2y}}{\partial y} \right) + 2zz_y - 0 - 1 = z_y e^{x+2y} + z2e^{x+2y} + 2zz_y - 1 = \left( 2ze^{x+2y} - 1 \right) + \left( e^{x+2y} + 2z \right) z_y. 
\]
Hence
\[
z_y = -\frac{2ze^{x+2y} - 1}{e^{x+2y} + 2z}
\]
and
\[
z_y(1, -1, 0) = -\frac{0 - 1}{e^{-1}} = e
\]

\[ y_x: \]
\[
0 = \left( z_x e^{x+2y} + z \frac{\partial e^{x+2y}}{\partial x} \right) + 2zz_x - x - y_x = 0 \cdot e^{x+2y} + z(x_x + 2yz)e^{x+2y} + 2z \cdot 0 - 1 - y_x = \left( ze^{x+2y} - 1 \right) + \left( 2ze^{x+2y} - 1 \right) y_x. 
\]
\[
y_x = -\frac{ze^{x+2y} - 1}{2ze^{x+2y} - 1}
\]

\[ x_z: \]
\[
0 = \left( z_x e^{x+2y} + z \frac{\partial e^{x+2y}}{\partial z} \right) + 2zz_x - x - y_z = 1 \cdot e^{x+2y} + z(x_x + 2yz)e^{x+2y} + 2z - 0 - 0 = \left( e^{x+2y} + 2z \right) + \left( 2ze^{x+2y} \right) x_z. 
\]
\[
y_x = -\frac{e^{x+2y} + 2z}{2ze^{x+2y}}
\]
Consider a particle moving from \((a, b)\) to \((c, d)\) with constant speed. Assume it starts at time \(t = 0\) and arrives at time \(t = 1\). Then a formula for the position at time \(t\) is \((a + t(c - a), b + t(d - b))\).

Given a function \(f(x, y)\) define \(g(t) = f(a + t(c - a), b + t(d - b))\). Hence \(g(0) = f(a, b)\) and \(g(1) = f(c, d)\). Compute \(g'(t)\).

\[
g'(t) = f_x(a + t(c - a), b + t(d - b)) \cdot (c - a) + f_y(a + t(c - a), b + t(d - b)) \cdot (d - b)
\]

The linear approximation formula says

\[
g(1) \approx g(0) + g'(0)
\]

or

\[
f(c, d) \approx f(a, b) + f_x(a, b) \cdot (c - a) + f_y(a, b) \cdot (d - b)
\]

Letting \(c\) and \(d\) vary we get the linearization of \(f(x, y)\) at \((a, b)\).

\[
f(x, y) \approx f(a, b) + f_x(a, b) \cdot (x - a) + f_y(a, b) \cdot (y - b)
\]
\[ f(x, y) \approx f(a, b) + f_x(a, b) \cdot (x - a) + f_y(a, b) \cdot (y - b) \]

Using linear approximation, estimate the change in \( g(x, y) = xe^{x^2y} \) when \((x, y)\) changes from \((2, 0)\) to \((1.9, 0.2)\).
\[ f(x, y) \approx f(a, b) + f_x(a, b) \cdot (x - a) + f_y(a, b) \cdot (y - b) \]

Using linear approximation, estimate the change in \( g(x, y) = xe^{x^2 y} \) when \((x, y)\) changes from \((2, 0)\) to \((1.9, 0.2)\).

**Step 1:** Find the linearization of \( g(x, y) \) at \((2, 0)\).
- \( g_x(x, y) = e^{x^2 y} + x(2xy)e^{x^2 y} = (1 + 2x^2 y)e^{x^2 y}; \ g_x(2, 0) = 1. \)
- \( g_y(x, y) = 0 \cdot e^{x^2 y} + x(x^2)e^{x^2 y} = (x^3)e^{x^2 y}; \ g_y(2, 0) = 8. \)

**Step 2:** Apply the linear approximation estimate,
\[
g(1.9, 0.2) - g(2, 0) \approx g_x(2, 0) \cdot (1.9 - 2) + g_y(2, 0) \cdot (0.2 - 0) = 1 \cdot (-0.1) + 8 \cdot 0.2 = 1.5
\]
\[ F(T, W) \]

<table>
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<tr>
<th>**</th>
<th>25</th>
<th>-31</th>
<th>-24</th>
<th>-17</th>
<th>-11</th>
<th>-4</th>
<th>3</th>
<th>9</th>
<th>16</th>
<th>23</th>
</tr>
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<tbody>
<tr>
<td>Speed</td>
<td>15</td>
<td>-26</td>
<td>-19</td>
<td>-13</td>
<td>-7</td>
<td>0</td>
<td>6</td>
<td>13</td>
<td>19</td>
<td>25</td>
</tr>
<tr>
<td>(W mph)</td>
<td>10</td>
<td>-22</td>
<td>-16</td>
<td>-10</td>
<td>-4</td>
<td>3</td>
<td>9</td>
<td>15</td>
<td>21</td>
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<tr>
<td></td>
<td>5</td>
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<td>7</td>
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<td>19</td>
<td>25</td>
<td>31</td>
</tr>
</tbody>
</table>

** Temperature \( (T^\circ F) \)

Estimate \( F(-5.5, 25.2) \).

**Linear approximation estimate:**

\[
F(T + \Delta T, W + \Delta W) \approx F(T, W) + F_T(T, W) \Delta T + F_W(T, W) \Delta W
\]

Here \( T = -5, W = 25 \): values in table closest to the numbers we want. Then

\[
\Delta T = -5.5 - (-5) = -0.5 \\
\Delta W = 25.2 - 25 = 0.2
\]

Since we don’t know \( F_T(-5, 25) \) or \( F_W(-5, 25) \) we need to use our work last lecture to estimate them.
We can only estimate \( F_T(-5, 25) \) using the forward difference estimate.

\[
F_T(-5, 25) \approx \frac{F(0, 25) - F(-5, 25)}{0 - (-5)} = \frac{-24 - (-31)}{5} = \frac{7}{5} = 1.4
\]

We can only estimate \( F_W(-5, 25) \) using the backward difference estimate.

\[
F_W(-5, 25) \approx \frac{F(-5, 25) - F(-5, 20)}{25 - 20} = \frac{-31 - (-29)}{5} = \frac{-2}{5} = -0.4
\]

**Some assembly required:**

\[
F(-5.5, 25.2) \approx F(-5, 25) + F_T(-5, 25) \Delta T + F_W(-5, 25) \Delta W
\]

\[
F(-5.5, 25.2) \approx -31 + (1.4)(-0.5) + (-0.4)(0.2) = -31 - 0.7 - 0.08 = -31.78
\]
A General Formula

For those of you having trouble keeping track of when $\frac{\partial x_i}{\partial x_j} = 0$ or is the partial for which you are trying to derive a formula, you might prefer just remembering a formula.

Suppose $F(x_1, \cdots, x_n) = 0$ and you have decided to define $x_i$ implicitly as a function of the other variables. Furthermore, suppose you want to find $\frac{\partial x_i}{\partial x_j}$ for any $j \neq i$.

The Chain Rule says $0 = \frac{\partial F}{\partial x_j} \frac{\partial x_j}{\partial x_j} + \frac{\partial F}{\partial x_i} \frac{\partial x_i}{\partial x_j} = \frac{\partial F}{\partial x_j} + \frac{\partial F}{\partial x_i} \frac{\partial x_i}{\partial x_j}$ so

$$\frac{\partial x_i}{\partial x_j} = -\frac{\frac{\partial F}{\partial x_j}}{\frac{\partial F}{\partial x_i}}$$

The way I remember which way the fraction on the right goes is to notice that no matter which partial you are trying to compute, the denominator is always the same, the partial of $F$ with respect to the variable which is the implicit function.

The advantage of this method is that there is less algebra to go wrong. The disadvantage is that you have to remember another formula. You will still need to understand the Chain Rule.

If you wish to go this route, rework the implicit differentiation problems using this method and see that you get the same answers.